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ON THE NICHOLS ALGEBRA ASSOCIATED TO $(q_{ij}) = \begin{pmatrix} -1 & \omega & 1 \\ 1 & -1 & \omega \\ 1 & \omega & \omega \end{pmatrix}$, OF RANK 3

TADAYOSHI TAKEBAYASHI

Department of Mathematics
School of Science and Engineering
Waseda University
Ohkubo Shinjuku-ku, Tokyo 169-8555, Japan
e-mail: takeba@aoni.waseda.jp

Abstract

We examine the defining relations of the Nichols algebra associated to $(q_{ij}) = \begin{pmatrix} -1 & \omega & 1 \\ 1 & -1 & \omega \\ 1 & \omega & \omega \end{pmatrix}$, of rank 3 by using the results by Angiono [2] and the method by Nichols [1].

1. Introduction

Nichols algebras are graded braided Hopf algebras with the base field in degree 0 and which are coradically graded and generated by its primitive elements [3-7]. Let V be a vector space and $c : V \otimes V \to V \otimes V$ be a linear isomorphism. Then (V, c) is called a *braided vector space*, if c is a solution of the braided equation, that is $(c \otimes id)(id \otimes c)(c \otimes id) = (id \otimes c)(c \otimes id)(id \otimes c)$. The pair (V, c) determines the Nichols algebras up to isomorphism. Let G be a group. Then a Yetter-Drinfeld

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module V over $\mathbb{K}G$ is a G-graded vector space $V = \bigoplus_{g\in G} V_g$, which is a G-module such that $g \cdot V_h \subset V_{ghg^{-1}}$ for all $g, h \in G$. The category ${}^G_G YD$ of $\mathbb{K}G$-Yetter-Drinfeld module is braided. For $V, W \in {}^G_G YD$, the braiding $c : V \otimes W \to W \otimes V$ is defined by $c(v \otimes w) = (g \cdot w) \otimes v$, $v \in V_g$, $w \in W$. Let V be a Yetter-Drinfeld module over G and let $T(V) = \bigoplus_{n\geq 0} T(V)(n)$ denote the tensor algebra of the vector space V. Let S be the set of all ideals and coideals I of $T(V)$ which are generated as ideals by $\mathbb{N}$-homogeneous elements of degree ≥ 2, and which are Yetter-Drinfeld submodules of $T(V)$. Let $I(V) = \sum_{I\in S} I$. Then $B(V) := T(V)/I(V)$ is called the *Nichols algebra* of $V \in {}^G_G YD$. In this article, we examine the defining relations of the Nichols algebra $B(V)$ associated to $(q_{ij}) = \begin{pmatrix} -1 & \omega & 1 \\ 1 & -1 & \omega \\ 1 & \omega & \omega \end{pmatrix}$, of rank 3.

2. Nichols Algebras of Cartan Type

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. Let G be an abelian group and V be a finite dimensional Yetter-Drinfeld module. Then the braiding is given by a nonzero scalar $q_{ij} \in \mathbb{K}$, $1 \leq i, j \leq \theta$, in the form $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, where $x_1, \ldots, x_\theta$, is a basis of V. If there is a basis such that $g \cdot x_i = \chi_i(g) x_i$ and $x_i \in V_{g_i}$, then V is called *diagonal type*. For the braiding, we have $c(x_i \otimes x_j) = \chi_j(g_i) x_j \otimes x_i$ for $1 \leq i, j \leq \theta$. Hence we have $(q_{ij})_{1\leq i, j\leq\theta} = (\chi_j(g_i))_{1\leq i, j\leq\theta}$. Let $B(V)$ be the Nichols algebra of V. Then we can construct the Nichols algebra by $B(V) \cong T(V)/I$, where I denote the sum of all ideals of $T(V)$ that are generated by homogeneous elements of degree ≥ 2 and that are coideals. If $B(V)$ is finite-dimensional, then the matrix (a_{ij}) defined for all $1 \leq i \neq j \leq \theta$ by $a_{ii} := 2$ and $a_{ij} := -\min\{r \in \mathbb{N} \mid q_{ij} q_{ji} q_{ii}^r = 1 \text{ or } (r+1)_{q_{ii}} = 0\}$ is a generalized Cartan matrix fulfilling $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ or $ord\ q_{ii} = 1 - a_{ij}$. (a_{ij}) is called the *Cartan matrix* associated to $B(V)$. To examine the defining relations of $B(V)$, we use the results by [3] and [2].

Proposition 2.1 [3]. (1) *For all* $1 \leq i \leq \theta$, *there exists a uniquely determined* (id, σ)*-derivation* $D_i : B(V) \to B(V)$ *with* $D_i(x_j) = \delta_{ij}$ (*Kronecker* δ) *for all* j.

(2) $\bigcap_{i=1}^{\theta} ker(D_i) = \mathbb{K}1$.

Proposition 2.2 [2]. *Let* (V, c) *be a braided vector space such that* $\dim V = 3$, *and let the corresponding generalized Dynkin diagram be* $\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-1}{\circ}\overset{r^{-1}}{\text{———}}\overset{r}{\circ}$. *Then* $B(V)$ *is presented by generators* x_1, x_2, x_3, *and relations*:

(2.2.1) $x_1^M = x_2^2 = x_3^N = x_{\alpha_1+2\alpha_2+\alpha_3}^P = 0$,

(2.2.2) $(ad_c x_1)^2 x_2 = (ad_c x_3)^2 x_2 = (ad_c x_1) x_3 = 0$,

(2.2.3) $[x_{\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2+\alpha_3}]_c = [x_{\alpha_1+\alpha_2+\alpha_3}, x_{\alpha_2+\alpha_3}]_c = 0$.

Using this, we obtain the following:

Proposition 2.3. *Let* $(q_{ij}) = \begin{pmatrix} -1 & \omega & 1 \\ 1 & -1 & \omega \\ 1 & \omega & \omega \end{pmatrix}$, ($\overset{-1}{\circ}\overset{\omega}{\text{———}}\overset{-1}{\circ}\overset{\omega^2}{\text{———}}\overset{\omega}{\circ}$) (*where* ω *is a primitive cube root of unity.*) *Then the Nichols algebra* $B(V)$ *is described as follows*:

Generators : x_1, x_2, x_3.

Relations : $x_1^2 = 0,\ x_2^2 = 0,\ x_3^3 = 0$,

$x_3^2 x_2 + x_3 x_2 x_3 + x_2 x_3^2 = 0,\ (x_1 x_2)^3 = -(x_2 x_1)^3,\ (x_2 x_3)^2 = (x_3 x_2)^2,\ x_1 x_3 = x_3 x_1$,

$x_2 x_1 x_2 x_3 + x_2 x_1 x_3 x_2 + \omega^2 x_2 x_3 x_2 x_1 + \omega x_1 x_2 x_3 x_2 + x_3 x_2 x_1 x_2 = 0$.

Its basis is given as follows:

$\{1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_1, x_2x_3, x_3x_2, x_3^2, x_1x_2x_1,$

$x_1x_2x_3, x_1x_3x_2, x_2x_1x_2, x_2x_1x_3, x_2x_3x_2, x_3x_2x_1, x_3x_2x_3,$

$x_1x_3^2, x_2x_3^2, (x_1x_2)^2, x_1x_2x_1x_3, x_1x_2x_3x_2, x_1x_3x_2x_1, x_1x_3x_2x_3,$

$(x_2x_1)^2$, $x_2x_1x_2x_3$, $x_2x_1x_3x_2$, $(x_2x_3)^2$, $x_2x_3x_2x_1$, $x_3x_2x_3x_1$, $x_2x_1x_3^2$,

$x_1x_2x_3^2$, $x_3x_2x_3^2$, $(x_1x_2)^2x_1$, $(x_1x_2)^2x_3$, $x_1x_2x_3x_2x_1$, $x_1(x_2x_3)^2$, $x_1x_3x_2x_3x_1$,

$(x_2x_1)^2x_2$, $(x_2x_1)^2x_3$, $x_2x_1x_2x_3x_2$, $(x_2x_3)^2x_1$, $x_3x_2x_3x_1x_2$, $x_1x_2x_1x_3^2$,

$x_3x_2x_1x_3^2$, $x_2x_1x_2x_3^2$, $x_3x_1x_2x_3^2$, $x_2x_3x_2x_3^2$, $x_3(x_2x_1)^2$, $x_3(x_1x_2)^2$,

$x_3x_2x_1x_2x_3$, $(x_1x_2)^3$, $(x_1x_2)^2x_1x_3$, $(x_1x_2)^2x_3x_2$, $x_1(x_2x_3)^2x_1$, $x_1x_3x_2x_3x_1x_2$,

$(x_2x_1)^2x_2x_3$, $(x_2x_1)^2x_3x_2$, $x_2x_1(x_2x_3)^2$, $x_2x_1x_2x_3x_2x_1$, $(x_2x_3)^2x_1x_2$,

$x_3x_2x_3x_1x_2x_1$, $x_3x_2x_3x_1x_2x_3$, $(x_2x_1)^2x_3^2$, $x_3x_1x_2x_1x_3^2$, $x_2x_3x_2x_1x_3^2$,

$(x_1x_2)^2x_3^2$, $x_2x_3x_1x_2x_3^2$, $x_1x_2x_3x_2x_3^2$, $x_3(x_2x_1)^2x_2$, $x_3(x_2x_1)^2x_3$, $x_3(x_1x_2)^2x_3$,

$x_3(x_2x_1)^2x_1$, $(x_1x_2)^3x_3$, $(x_2x_1)^2x_3x_2x_1$, $x_3(x_1x_2)^3$, $(x_1x_2)^2x_3x_2x_3$,

$x_1(x_2x_3)^2x_1x_2$, $x_2x_1(x_2x_3)^2x_1$, $(x_2x_3)^2x_1x_2x_1$, $x_3x_2x_3(x_1x_2)^2$,

$x_2(x_1x_2)^2x_3^2$, $x_3(x_2x_1)^2x_3x_2$, $x_1(x_3x_2x_1)^2$, $x_1(x_2x_1)^2x_3^2$,

$x_3(x_1x_2)^2x_1x_3$, $x_1x_3x_2x_3x_1x_2x_3$, $(x_3x_2x_1)^2x_3$, $x_3(x_2x_1)^2x_3^2$,

$x_1x_2x_3x_2x_1x_3^2$, $x_3(x_1x_2)^2x_3^2$, $(x_2x_1)^2x_2x_3x_2$, $(x_2x_3)^2x_1x_2x_3$,

$x_2x_1x_2x_3x_2x_3^2$, $x_3x_2x_3x_1x_2x_3^2$, $(x_1x_2)^3x_3x_2$, $(x_2x_1)^2x_3x_2x_1x_2$,

$x_3(x_1x_2)^3x_3$, $(x_1x_2)^2x_3x_2x_3x_1$, $x_1(x_2x_3)^2x_1x_2x_1$, $x_3x_2x_3(x_1x_2)^2x_1$,

$(x_1x_2)^3x_3^2$, $x_1x_3x_2x_3(x_1x_2)^2$, $x_1(x_2x_3)^2x_1x_2x_3$, $(x_2x_3)^2x_1x_2x_1x_3$,

$x_3x_2x_3(x_1x_2)^2x_3$, $x_3x_2(x_1x_2)^2x_3^2$, $(x_3x_2x_1)^2x_3x_2$, $x_2x_1x_2x_3x_2x_1x_3^2$,

$x_2x_3(x_1x_2)^2x_3^2$, $(x_1x_2)^2x_3x_2x_3^2$, $(x_2x_3)^2(x_1x_2)^2$, $(x_2x_1)^2(x_2x_3)^2$,

$(x_2x_3)^2x_1x_2x_3x_2$, $x_3x_2x_1x_2x_3x_2x_3^2$, $x_1x_3x_2x_3x_1x_2x_3^2$, $x_3x_1(x_2x_1)^2x_3^2$,

$(x_1x_2)^3x_3x_2x_1$, $(x_1x_2)^3x_3x_2x_3$, $x_1(x_2x_3)^2(x_1x_2)^2$, $(x_2x_3)^2(x_1x_2)^2x_1$,

$(x_2x_1)^2(x_2x_3)^2x_1,\ x_3x_2x_3(x_1x_2)^3,\ x_3x_2x_3(x_1x_2)^2x_1x_3,\ x_3(x_1x_2)^3x_3^2,$

$x_1x_3x_2x_3(x_1x_2)^2x_3,\ x_1(x_2x_3)^2x_1x_2x_3x_1,\ (x_3x_2x_1)^3,\ x_1x_2x_3(x_2x_1)^2x_3^2,$

$x_1x_2x_3(x_1x_2)^2x_3^2,\ x_1x_3x_2x_3(x_1x_2)^2x_1,\ (x_2x_3)^2(x_1x_2)^2x_3,$

$(x_2x_3)^2x_1x_2x_1x_3x_2,\ x_2(x_1x_2)^2x_3x_2x_3^2,\ (x_3x_2x_1)^2x_3x_2x_3,$

$x_3x_2x_3(x_1x_2)^2x_3^2,\ x_3(x_1x_2)^2x_3x_2x_3^2,\ x_2x_1x_3x_2x_3x_1x_2x_3^2,$

$x_3x_2x_3(x_2x_1)^2x_3^2,\ x_1x_2x_3(x_1x_2)^3x_3,\ x_3(x_1x_2)^3x_3x_2x_1,\ x_3(x_1x_2)^3x_3x_2x_3,$

$x_3x_2x_3(x_1x_2)^3x_3,\ x_3x_2x_3(x_1x_2)^2x_1x_3x_2,\ x_2x_3(x_1x_2)^3x_3^2,\ (x_3x_2x_1)^3x_2,$

$(x_2x_3)^2(x_1x_2)^2x_1x_3,\ (x_1x_2)^3x_3x_2x_3^2,\ (x_3x_2x_1)^3x_3,\ x_1x_3x_2x_3(x_2x_1)^2x_3^2,$

$x_1x_3x_2x_3(x_1x_2)^2x_3^2,\ (x_1x_2x_3)^2x_1x_2x_3^2,\ x_1x_3x_2x_3(x_1x_2)^2x_3^2,$

$x_1x_3x_2x_3(x_1x_2)^2x_1x_3,\ (x_2x_3)^2(x_1x_2x_3)^2,\ (x_3x_2x_1)^2(x_3x_2)^2,$

$(x_2x_3)^2(x_1x_2)^2x_3^2,\ x_2x_3(x_1x_2)^2x_3x_2x_3^2,\ x_1x_2x_3(x_1x_2)^3x_3x_2,$

$x_3(x_1x_2)^3x_3x_2x_1x_3,\ x_1x_2x_3(x_1x_2)^3x_3^2,\ (x_3x_2x_1)^3x_2x_1,$

$x_1x_3x_2x_3(x_1x_2)^2x_3x_1x_2,\ x_3x_2x_3(x_1x_2)^3x_3x_2,\ x_3x_2x_3(x_1x_2)^3x_3^2,$

$(x_3x_2x_1)^3x_2x_3,\ x_3(x_1x_2)^3x_3x_2x_3^2,\ (x_3x_2x_1)^3x_3x_2,\ x_1(x_2x_3)^2(x_1x_2)^2x_3^2,$

$(x_3x_2x_1)^2(x_3x_2)^2x_1,\ x_3(x_1x_2x_3)^2x_1x_2x_3^2,\ x_3x_2x_3(x_1x_2)^2x_3x_2x_3^2,$

$x_1x_2x_3(x_1x_2)^3x_3x_2x_3,\ (x_3x_2x_1)^3x_2x_1x_2,\ (x_3x_2x_1)^3x_2x_1x_3,$

$x_3x_1x_2x_3(x_1x_2)^3x_3^2,\ (x_3x_2x_1)^4,\ (x_3x_2x_1)^3x_2x_3x_2,\ x_3x_2x_3(x_1x_2)^3x_3x_2x_3,$

$(x_3x_2x_1)^3x_3x_2x_3,\ x_1x_3x_2x_3(x_1x_2)^2x_3x_2x_3^2,\ (x_2x_3x_1)^3x_2x_3^2,$

$(x_3x_2x_1)^3x_2x_1x_2x_3,\ x_2x_3x_1x_2x_3(x_1x_2)^3x_3^2,\ (x_3x_2x_1)^4x_2,\ (x_3x_2x_1)^4x_3,$

$x_1(x_2x_3x_1)^3x_2x_3^2,\ x_2x_1x_3x_2x_3(x_1x_2)^2x_3x_2x_3^2,\ (x_3x_2x_1)^3x_2x_1x_2x_3x_2,$

$(x_3x_2x_1)^3x_2x_1x_2x_3^2,\ (x_3x_2x_1)^4x_2x_3,\ (x_3x_2x_1)^4x_2x_3x_2\}.$

Hence the Hilbert polynomial of $B(V)$ *is given as follows*:

$$P(t) = 1 + 3t + 6t^2 + 10t^3 + 14t^4 + 18t^5 + 22t^6 + 22t^7 + 22t^8$$

$$+ 22t^9 + 18t^{10} + 14t^{11} + 10t^{12} + 6t^{13} + 3t^{14} + t^{15}.$$

Proof. They are directly calculated. □

References

[1] W. D. Nichols, Bialgebras of type one, Commun. Alg. 6 (1978), 1521-1552.

[2] I. Angiono, A presentation by generators and relations of Nichols algebras of diagonal type and convex orders on root systems, preprint.

[3] N. Andruskiewitsch and H.-J. Schneider, Pointed Hopf algebras, New Directions in Hopi Algebras, MSRT Publications, Cambridge Univ. Press 43 (2002), 1-68.

[4] N. Andruskiewitsch and H.-J. Schneider, Finite quantum groups and Cartan matrices, Adv. Math. 154 (2000), 1-45.

[5] I. Heckenberger, Examples of finite-dimensional rank 2 Nichols algebras of diagonal type, Compos. Math. 143 (2007), 165-190.

[6] M. Graña, On Nichols algebras of low dimension, Contemp. Math. 267 (2000), 111-134.

[7] A. Milinski and H.-J. Schneider, Pointed indecomposable Hopf algebras over Coxeter groups, Contemp. Math. 267 (2000), 215-236.