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DIFFEOMORPHISMS WITH INVERSE SHADOWING ON CLOSED SURFACES

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Abstract

It is proved that for every C^1-diffeomorphism f on a closed surface satisfying Axiom A, if f has the inverse shadowing property, then it satisfies the C^0-transversality condition. Using this fact, we can see that the inverse shadowing property is not C^1-generic. Also, we introduce the notion of C^2-stable inverse shadowing and show that if f is a C^2-stable inverse shadowing diffeomorphism on a closed surface, then (i) f is Kupka-Smale, (ii) f satisfies both Axiom A and the strong transversality condition if $\overline{P(f)} = \Omega(f)$, $\overline{P_s(f)} = \Omega_0(f)$, and there exists a dominated splitting on $\overline{P_s(f)}$.

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1. Introduction

Shadowing properties play important role in the study of dynamical systems. Up until now, various types of shadowing properties have been introduced and studied. The weak shadowing property, which was introduced by Corless and Pilyugin in [6] has been studied by various authors from viewpoints in dynamics for diffeomorphisms. The inverse shadowing property which is a "dual" notion of shadowing property was introduced by Corless and Pilyugin in [6], and the qualitative theory of dynamical systems with the property was developed by various authors. More precisely,

- Stability and hyperbolicity via shadowing properties have been studied : see [2, 4, 5, 7, 11, 18, 22, 24, 25, 26, 27, 28].
- The genericity problems related to shadowing properties under C^r-topology have been considered : see [1, 5, 6, 8, 9, 13, 20, 21].

In a remarkable paper of Pujals and Sambarino [22], they proved Palis conjecture in case of a closed surface as the starting point for solving it. This represents a step towards the global understanding of dynamics of surface diffeomorphisms. They have given in Theorem B [22], an information of hyperbolicity of a compact *f*-invariant set having a dominant splitting. Recently, they proved in [23], a Spectral Decomposition Theorem for the limit set $L(f)$ of a C^2-diffeomorphism f of a compact surface under the assumption of dominated splitting.

Sakai in [25] has introduced the notion C^0-transversality so as to characterize the shadowing property of f of a closed surface, and then in [26], for every diffeomorphism f satisfying Axiom A, a C^2-stable shadowing diffeomorphism f characterize the strong transversality condition. Finally, he has considered in [28] structural stability via shadowing property under the C^2-robust point of view. In fact, he has shown that a C^2-stable shadowing diffeomorphism f of a closed surface is Kupka-Smale and, under some additional assumptions, f is structurally stable.

In this paper, we have considered "dual" problem as in the point of inverse shadowing property which is similar in nature to stability related to the C^2-stable shadowing diffeomorphism f of a closed surface. In Section 2, we introduce the notion of inverse shadowing property and show that the inverse shadowing property is sufficient for C^0-transversality at an intersecting point of stable manifold and unstable manifold, whence the inverse shadowing property is not C^1-generic. In Section 3, we introduce the notion of C^r-stable inverse shadowing diffeomorphism f and prove that a C^2-stable inverse shadowing diffeomorphism f of closed surface is Kupka-Smale and, under some additional assumptions, f satisfies both Axiom A and the strong transversality condition.

2. C^1-Nondensity of Inverse Shadowing

The genericity problems related to shadowing properties under C^r-topology $(r \geq 0)$ have been studied (see [1, 6, 8, 19, 21]). More precisely, Corless and Pilyugin [6] have shown that there is a C^0-generic set of homeomorphisms having the weak shadowing property on a compact C^∞-manifold M. Moreover, Pilyugin and Plamenevskaya [19] have proved that the usual shadowing property is C^0-generic in the space of homeomorphisms of topological manifold M endowed with the C^0-topology. But in case of C^1-genericity, the same results do not hold any more. At first Pilyugin and Sakai [21] have shown that the shadowing property is not C^1-dense in $\mathrm{Diff}^1(\mathbb{T}^2)$. In higher dimension, Bonatti et al. [1] proved that the shadowing property is not C^1-dense in $\mathrm{Diff}^1(M)$ with $\dim M = 3$. Gan [8] showed in more weaker situation that the weak shadowing property is C^1-generic.

As a dual approach, the genericity problems via inverse shadowing properties under C^r-topology $(r \geq 0)$ have been considered (see [4, 6, 9, 13, 20, 21]). Mazur [13] has established the C^0-genericity of strong tolerance stability in the space of homeomorphisms of a compact smooth manifold M endowed with the C^0-topology. The strong tolerance stability implies the $\mathcal{T}_h$-inverse shadowing property, hence the $\mathcal{T}_h$-inverse shadowing property is C^0-generic in the space of homeomorphisms of compact C^∞-manifold M. The notion of $\mathcal{T}_h$-inverse shadowing property is the same as that of persistency appeared in [26]. Later Kościelniak and Mazur [9] have proved in more stronger condition that the $\mathcal{T}_C$-inverse shadowing property is C^0-generic in the space of homeomorphisms of compact C^∞-manifold M endowed with the C^0-topology under the condition that $\dim M \leq 3$. Recently, Choi et al. [4] have shown in more general setting that the first weak inverse shadowing is C^0-generic in the space of homeomorphisms of a compact metric space X with respect to the class of continuous methods induced by homeomorphisms.

In this section, we show that the inverse shadowing property is sufficient for C^0-transversality at an intersecting point of stable manifold and unstable manifold, whence the inverse shadowing property is not C^1-generic in $\mathrm{Diff}^1(\mathbb{T}^2)$.

Let M be a closed C^∞-manifold, and $\mathrm{Diff}^r(M)(r \geq 1)$ be the space of C^r-diffeomorphisms of M endowed with C^r-topology. Let Y be a closed, f-invariant subset of M. A C^1-diffeomorphism f has the *inverse shadowing property* on Y if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any diffeomorphism g with $d_{C^1}(g, f) < \delta$ and any point $x \in Y$, there exists a point $y \in Y$ for which $d(f^n(x), g^n(y)) < \varepsilon$, $n \in \mathbb{Z}$. When Y is the whole space M, f has the *inverse shadowing property* (ISP).

Remark 2.1. Choi et al. [5] have introduced the ISP as the notion of the inverse shadowing property with respect to the class $\mathcal{T}_d$. Since the $\mathcal{T}_C$-inverse shadowing property implies ISP [5], it follows from the result of Kościelniak and Mazur [9] that the ISP is C^0-generic in the space of homeomorphisms of a compact C^∞-manifold M endowed with the C^0-topology under the condition that $\dim M \leq 3$. Note that the inverse shadowing property is different from the shadowing property, in general [3].

Let $f \in \mathrm{Diff}^1(M)$ satisfy Axiom A. A hyperbolic set Λ is called a *basic set* if there is a compact neighborhood U of Λ in M such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$, and $f|_\Lambda$ has a dense orbit. The *non-wandering set* of f, $\Omega(f)$, is a disjoint union of basic sets $\Lambda_1 \cup \cdots \cup \Lambda_l$, and the *local stable* and *unstable manifold* are denoted by $W^s_{\varepsilon_0}(x)$ and $W^u_{\varepsilon_0}(x)\,(x \in \Omega(f))$, respectively for some $\varepsilon_0 > 0$. The *stable* *manifold*, $W^s(x)$, and the *unstable manifold*, $W^u(x)$, of $x \in \Omega(f)$ are defined in a usual way, and we put $W^\sigma(\Lambda) = \bigcup_{x\in\Lambda} W^\sigma(x)(\sigma = s, u)$.

In the following results let M be a surface. The notion of C^0-transversality between stable and unstable manifolds of basic sets Λ_i and Λ_j was introduced in [25] as follows. If there exists $x \in W^s(\Lambda_i) \cap W^u(\Lambda_j) \backslash \Lambda_i \cup \Lambda_j$, then for $\varepsilon > 0$ we denote by $C_\varepsilon^\sigma(x)$ the connected component of x in $W^\sigma(x) \cap B_\varepsilon(x)(\sigma = s, u)$, and let $B_\varepsilon^+(x)$ and $B_\varepsilon^-(x)$ be the components of $B_\varepsilon \backslash C_\varepsilon^s(x)$. Here $B_\varepsilon(x) = \{y \in M \mid d(x, y) \le \varepsilon\}$. We say that $W^s(x)$ and $W^u(x)$ *meet* C^0*-transversely* at x if $\dim W^\sigma(x) = 1$ $(\sigma = s, u)$, $B_\varepsilon^+(x) \cap C_\varepsilon^u(x) \ne \varnothing$ and $B_\varepsilon^-(x) \cap C_\varepsilon^u(x) \ne \varnothing$ for every $\varepsilon > 0$.

Theorem 2.2. *Let* $\Lambda_i (i = 1, 2)$ *be basic sets of* C^1*-diffeomorphism* f *on a closed surface satisfying Axiom A*, *and suppose that* $x \in W^s(p) \cap W^u(q) \backslash \Lambda_1 \cup \Lambda_2$ $(p \in \Lambda_1, q \in \Lambda_2)$. *If* f *has the inverse shadowing property*, *then* $W^s(p)$ *and* $W^u(q)$ *meet* C^0*-transversely at* x.

Proof. Let $\Lambda_i (i = 1, 2)$ be basic sets of $f \in \mathrm{Diff}^1(M)$, and suppose that $x \in W^s(p) \cap W^u(q) \backslash \Lambda_1 \cup \Lambda_2 \, (p \in \Lambda_1, q \in \Lambda_2)$. To obtain a contradiction, let us assume that f has the inverse shadowing property, but the C^0-transversality condition at x is not satisfied. Since $W^s(x)$ and $W^u(x)$ do not meet C^0-transversely at x, $\dim W^s(x) = \dim W^u(x) = 1$, and there exists $\delta_0 > 0$ such that

$$B_{\delta_0}^+(x) \cap C_{\delta_0}^u(x) = \varnothing \quad \text{or} \quad B_{\delta_0}^-(x) \cap C_{\delta_0}^u(x) = \varnothing,$$

where $C_{\delta_0}^u(x) = W^u(x) \cap B_{\delta_0}(x)$. By using the local chart ([26, Lemma 2]) there exists $\delta_1 > 0$ and a C^1-diffeomorphism $\widetilde{\varphi}_x : B_{\delta_1}(x) \to \mathbb{R}^2$ such that

$$\widetilde{\varphi}_x(C_{\delta_1}^s(x)) \subset \mathbb{R} \times \{0\} \quad \text{and} \quad \widetilde{\varphi}_x(x) = (0, 0),$$

where $C_{\delta_1}^s(x) = W^s(x) \cap B_{\delta_1}(x)$.

We deal with only the case $B_\delta^-(x) \cap C_\delta^u(x) = \emptyset$ for $0 < \delta \le \min\{\delta_0, \delta_1\}$ satisfying $f^{-1}(\tilde{\varphi}_x^{-1}(\mathrm{graph}(\gamma(-\varepsilon_0, \varepsilon_0)))) \subset C_\delta^u(f^{-1}(x))$ (the other case follows in a similar way). Thus $B_\delta^+(x) \cap C_\delta^u(x) \ne \emptyset$. It is easy to see that there exists $\varepsilon_0 > 0$ and C^1-function $\gamma : [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}$ such that

$$\mathrm{graph}(\gamma) \subset \tilde{\varphi}_x(C_\delta^u(x)) \quad \text{and} \quad (0, \gamma(0)) = \tilde{\varphi}_x(x) = (0, 0).$$

Moreover, C^0-tangency implies $\gamma'(0) = 0$. It follows from the stable manifold theorem ([17, Lemma 2.2.20]) that there exists $\varepsilon_1 > 0$ satisfying the following properties:

$$d(f^k(z), f^k(x)) < \varepsilon_1, \quad \text{for } k \ge 0 \Rightarrow z \in C_\delta^s(x), \tag{1}$$

$$d(f^k(z), f^k(x)) < \varepsilon_1, \quad \text{for } k \le 0 \Rightarrow z \in C_\delta^u(x). \tag{2}$$

We assume that $0 < \varepsilon < \varepsilon_1$ is so small that

- there exists a neighborhood U of x such that $f^k(U)$ are disjoint for different $k\,(k \ne 0)$,
- $B_\varepsilon(x) \subset U$.

Then we find a number $d > 0$ corresponding to this $\varepsilon > 0$ in the definition of inverse shadowing of f. It is easy to construct a diffeomorphism g coinciding with f outside the set $f^{-1}(U)$ having $d_{C^1}(f, g) < d$, and such that

$$g(f^{-1}(\tilde{\varphi}_x^{-1}(\mathrm{graph}(\gamma(-\varepsilon_0, \varepsilon_0))))) \cap C_\delta^s(x) = \emptyset.$$

Assume that there exists $z \in M$ such that

$$d(g^k(z), f^k(x)) < \varepsilon, \quad \text{for } k \in \mathbb{Z}. \tag{3}$$

Since $f^k(z) = g^k(z)$ for $k \ge 0$,

$$d(f^k(z), f^k(x)) < \varepsilon, \quad \text{for } k \ge 0. \tag{4}$$

From (4) and (1), it follows that $z \in C_\delta^s(x)$. On the other hand, if $z' = g^{-1}(z)$, then

$$f^k(z') = g^k(z') \quad \text{for } k \le 0.$$

If follows from (3) that

$$d(f^k(g^{-1}(z)), f^k(f^{-1}(x))) < \varepsilon, \quad \text{for } k \leq 0. \tag{5}$$

From inequalities (5) and (2), $z' = g^{-1}(z) \in C^u_\delta(f^{-1}(x))$. Therefore, $g(z') = z \in g(C^u_\delta(f^{-1}(x)))$ and $z \in C^s_\delta(x)$. Since $g(C^u_\delta(f^{-1}(x))) \cap C^s_\delta(x) = \varnothing$, we obtain a contradiction. □

Pilyugin, Sakai and Tarakanov [20] showed that in $\mathrm{Diff}^1(\mathbb{T}^2)$, there exists an open subset W of Ω-stable diffeomorphisms such that every diffeomorphism $f \in W$ does not satisfy C^0-transversality condition. Thus, from Theorem 2.2, we have

Proposition 2.3. *ISP is not* C^1*-dense in* $\mathit{Diff}^1(\mathbb{T}^2)$.

Remark 2.4. Recently Pilyugin and Sakai [21] proved that for the Axiom A, C^1-diffeomorphism f on a closed surface, f has the inverse shadowing property with respect to a class of continuous method which is stronger than ISP, C^0-transversality condition and shadowing property are mutually equivalent.

Corollary 2.5. *ISP is not* C^2*-dense in* $\mathit{Diff}^2(M)$.

Proof. Let $\Lambda_i (i = 1, 2)$ be basic sets of $f \in \mathrm{Diff}^2(M)$, and suppose that $x \in W^s(p) \cap W^u(q) \backslash \Lambda_1 \cup \Lambda_2$ $(p \in \Lambda_1, q \in \Lambda_2)$. If f has the ISP, then $W^s(p)$ and $W^u(q)$ meet C^0-transversely at x. If $W^s(p)$ and $W^u(q)$ meet C^0-transversely at x, then they do not have a non-degenerate tangency at x. Thus, if Λ is a Newhouse wild hyperbolic set of $f \in \mathrm{Diff}^2(M)$ [14], and if we put $\Lambda_1 = \Lambda_2 = \Lambda$, then, by Newhouse's result and Theorem 2.2, there exists a non-empty C^2-open set $\mathcal{O}$ such that every $g \in \mathcal{O}$ does not have the ISP. This means that the ISP is not C^2-dense. □

3. C^2-Stable Inverse Shadowing

One of the interesting problems in differentiable dynamics in recent past have been to study the impact of C^r-robustly dynamical properties $(r \geq 1)$, especially shadowing and inverse shadowing properties on compact invariant sets. These kinds of researches have been used to characterize the stability of a given system and hyperbolicity on compact invariant subsets of the phase space in C^1-robust point of view (see [2, 5, 11, 12, 18, 20, 24, 27, 29]). On the other hand, hyperbolicity and stability for surface diffeomorphisms have been studied from C^2-robustly dynamical properties (see [22, 28]). Especially, Sakai [28] has considered structural stability via shadowing property under C^2-robust point of view. More precisely, he has proved that a C^2-stable shadowing surface diffeomorphism *f* is Kupka-Smale and, under some additional assumptions, *f* is structurally stable.

In this section, we consider "dual" problem as in the point of inverse shadowing property which is similar in nature to stability related to the C^2-stable shadowing surface diffeomorphism *f*. At first, we introduce the notion of C^2-stable inverse shadowing diffeomorphism *f* and prove that a C^2-stable inverse shadowing diffeomorphism *f* of closed surface is Kupka-Smale and, under some additional assumptions, *f* satisfies both Axiom A and the strong transversality condition.

An Axiom A diffeomorphism *f* satisfies the *strong transversality condition* if and only if the stable manifold $W^s(x, f)$ and the unstable manifold $W^u(x, f)$ are transversal for all $x \in M$ (i.e., $T_xW^s(x, f) + T_xW^u(x, f) = T_xM$).

Definition 3.1. Let $f \in \mathrm{Diff}^2(M)$. A diffeomorphism *f* is C^2-*stable inverse shadowing* if there is a C^2-neighborhood $\mathcal{U}(f)$ of *f* such that every $g \in \mathcal{U}(f)$ has ISP.

Theorem 3.2. *Let $f \in Diff^2(M)$ satisfy Axiom A. If f is C^2-stable inverse shadowing, then f satisfies the strong transversality condition.*

Proof. Suppose that *f* does not satisfy the strong transversality condition. By Lemma 2 in [25], there exist $\delta > 0$ and a C^2-diffeomorphism $\widetilde{\varphi}_x : B_\delta(x) \to \mathbb{R}^2$ such that

- $\widetilde{\varphi}_x(C^s_\delta(x)) \subset v - axis$,
- $\widetilde{\varphi}_x(x) = (0, 0)$,
- $T_{(0,0)}\widetilde{\varphi}_x(C^u_\delta(x)) = v - axis.$

It is easy to see that there are $\varepsilon > 0$ and a C^2-function $\gamma : [-\varepsilon, \varepsilon] \to \mathbb{R}$ such that

$$\operatorname{graph}(\gamma) \subset \widetilde{\varphi}_x(C_\delta^u(x)) \text{ and } (0, \gamma(0)) = \widetilde{\varphi}_x(x) = (0, 0).$$

Note that $\gamma'(0) = 0$. This implies $\gamma''(0) = 0$. Indeed, if $\gamma''(0) \neq 0$, $W^s(x)$ and $W^u(x)$ do not meet C^0-transversely at x. But this is not true by Theorem 2.2. So we can see that $\gamma'(0) = \gamma''(0) = 0$.

If we denote a C^2-metric as d_{C^2}, then for every δ', there exists $0 < \varepsilon < \varepsilon'$ such that

$$d_{C^2}(C_\varepsilon^u(x)), (C_\varepsilon^s(x)) < \delta',$$

where $\widetilde{\varphi}_x^{-1}(\operatorname{graph}(\gamma(-\varepsilon, \varepsilon))) = C_\varepsilon^u(x)$. Thus by using a standard procedure, for every $\nu > 0$ and every C^2-neighborhood $\mathcal{U}(f)$ of f such that every $g \in \mathcal{U}(f)$ has the inverse shadowing property, we can construct a C^2-diffeomorphism $\widetilde{f}$ C^2-near to f such that

- $\widetilde{f} = f$ on $M \setminus B_\nu(x)$,
- $W^s(x, \widetilde{f}) \cap B_{\nu'}(x) = W^u(x, \widetilde{f}) \cap B_{\nu'}(x)$,

where $0 < \nu' < \nu$ is sufficiently small. Then $\widetilde{f}$ has a C^0-tangential point x. By Theorem 2.2, this is a contradiction since $\widetilde{f}$ has the ISP. Therefore, the proof is completed. □

Denote C^r-int(IS(M)) [resp. C^r-int(SP(M))] $(r \geq 1)$ the C^r-interior of the set of all diffeomorphisms in $\operatorname{Diff}^r(M)$ having the inverse shadowing property [resp. the shadowing property]. Note that $f \in C^2$-int(IS(M)) means that f is C^2-stable inverse shadowing.

Remark 3.3. Let N be an arbitrary n-dimensional smooth manifold. Sakai [24] proved that every $f \in C^1$-int(SP(N)) satisfies strong transversality condition. Choi et al. [5] proved that f is in C^1-int(IS(N)) if and only if f satisfies both Axiom A and the strong transversality condition. So f is in $C^1\text{-int}(\mathrm{IS}(N))$ if and only if f is in $C^1\text{-int}(\mathrm{SP}(N))$. If f is in $C^1\text{-int}(\mathrm{IS}(N))$, then f is in $C^2\text{-int}(\mathrm{IS}(N))$. We do not know whether the converse does hold or not. But if the Axiom A condition is added, then the converse is true as in the following Corollary 3.4. Sakai [26] proved that for Axiom A C^2-diffeomorphism f, if f is in $C^2\text{-int}(\mathrm{SP}(N))$, then f satisfies the strong transversality condition.

Corollary 3.4. *Let $f \in Diff^2(M)$ satisfy Axiom A. Then f is C^2-stable inverse shadowing if and only if f satisfies the strong transversality condition.*

Proof. Let $f \in \mathrm{Diff}^2(M)$ satisfy Axiom A. If f satisfies the strong transversality condition, then f is in $C^1\text{-int}(\mathrm{IS}(M))$ by Remark 3.3 and so f is C^2-stable inverse shadowing. Next, since the non-wandering set of f is a disjoint union of basic sets, the converse holds from Theorem 3.2. □

Consider the ISP of rotation map f on the circle S^1 with coordinate $x \in [0, 1)$ which is necessary for following Theorems 3.6 and 3.7. We denote r to be the distance on S^1 induced by the usual distance on the real line.

Lemma 3.5. *Let f be a rotation map on the unit circle S^1. Then f does not have the ISP.*

Proof. To prove this, we divide the following two cases. First we assume that f is a rational rotation map. Then there exists $n \in \mathbb{N}$ such that f^n is the identity map on the unit circle. So f^n does not have the ISP. Therefore, f does not have the ISP. Secondly, we assume that f is an irrational rotation map generated by the mapping

$$\phi(x) = x + \alpha(\mathrm{mod}\, 1), \quad \alpha \in \mathbb{Q}^c,$$

and g has the ISP. Take $\varepsilon = \frac{1}{4}$. Choose the corresponding constant $\delta > 0$ such that for any diffeomorphism g with $d_{C^1}(g, f) < \delta$ and any point $x \in S^1$, there exists a point $y \in S^1$ for which

$$r(f^n(x), g^n(y)) < \frac{1}{4}, \quad n \in \mathbb{Z}. \tag{6}$$

There exists a rational number $\beta = \frac{l}{m}$ such that $|\alpha - \beta| < \delta$. Consider the system g generated by the mapping

$$\psi(x) = x + \beta (\bmod 1),\ \beta \in \mathbb{Q}.$$

Then $d_{C^1}(g, f) < \delta$. It is easy to see that for any $x \in S^1$, $g^m(x) = x$ and $\{f^{km}(x) : k \in \mathbb{Z}\}$ is dense in S^1. For $x \in S^1$ and the corresponding point y satisfying (6), we can see that there exists $k_0 \in \mathbb{Z}$ satisfying the inequality

$$r(f^{k_0 m}(x),\ g^{k_0 m}(y)) \geq \frac{1}{4}.$$

This is a contradiction. □

Let $P(f)$ be the set of periodic points of f.

Theorem 3.6. *If f is C^2-stable inverse shadowing, then every $p \in P(f)$ is hyperbolic.*

Proof. Let f be a C^2-stable inverse shadowing diffeomorphism and fix any $p \in P(f)$. Assuming that $f^n(p) = p$ $(n > 0)$ is not hyperbolic, we shall derive a contradiction. For simplicity suppose $n = 1$ (the other case is similar). With a C^2-small perturbation, we can find g C^2-nearby f (we may suppose that $g \in C^2$-int(IS(M)) such that $g(p) = p$ and

$$D_p g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A is a constant $\rho \in \mathbb{R}$ or 2×2-matrix and B is a hyperbolic matrix (with respect to some coordinates), satisfying one of the following three possible cases:

(a) $\rho = 1$,

(b) $\rho = -1$,

(c) the eigenvalues of A are of the form $\rho_1 = e^{i\theta}$, $\rho_2 = e^{-i\theta}$ for some real $\theta \neq k\pi$ and $k \in \mathbb{Z}$.

In cases (a) and (b), following the proof of Sakai ([28, p. 239]), we approximate g by C^3-diffeomorphism $\hat{g}$ (with respect to the C^2-topology) such that:

- $\hat{g}(p) = p$,
- there is a C^3-$\hat{g}$-invariant curve $W^c(p, \hat{g})$, which is tangent to the eigenspace associated with $\rho = 1$ (case (a)) or $\rho = -1$ (case (b)),
- if we consider $W^c(p, \hat{g}) \subset \mathbb{R}$ and p is the origin 0 with respect to corresponding coordinates, then the restriction $\hat{g}_{|W^c(p, \hat{g})}$ has the following expressions:

$$\hat{g}_{|W^c(p, \hat{g})}(t) = t + at^2 + o(t^3) \text{ for the case (a),}$$

$$\hat{g}_{|W^c(p, \hat{g})}(t) = -t + at^3 + o(t^4) \text{ for the case (b),} \tag{7}$$

where $t \in W^c(p, \hat{g})(\subset \mathbb{R})$ for $|t| < \varepsilon$. Note that $\hat{g} \in C^2$ - $\mathrm{int}(\mathrm{IS}(M))$ and there exists a hyperbolic matrix $B_{\hat{g}}$ for $\hat{g}$ corresponding to B. Since the condition is satisfied generically, assume $a \neq 0$. We treat the case $a > 0$ in (7) (the other case is similar). In case (a), let

$$D_p(\hat{g}) = \begin{pmatrix} 1 & 0 \\ 0 & \rho' \end{pmatrix} \quad \text{for} \quad \rho' > 1.$$

Note that the other case $\rho' < -1$ can be dealt with similarly and $\hat{g} \in C^2$-int (IS(M)). To obtain a contradiction, it is enough to show that there exists $\varepsilon_0 > 0$ such that for any integer $n > 0$, we find a diffeomorphism g_n on M with $d_C^1(g_n, \hat{g}) < \frac{1}{n}$ and $x_n \in M$ satisfying the following : for all $y \in M$, there exists an integer m such that

$$d(\hat{g}^m(x_n), g_n^m(y)) \geq \varepsilon_0.$$

Take $\varepsilon_0 > 0$ satisfying $(W^c(p, \hat{g}) \cap B_{2\varepsilon_0}(p)) \subset (W^c(p, \hat{g}) \cap B_{\varepsilon}(p))$. Let $n > 0$ be an arbitrary natural number. Then we can obtain a C^2-diffeomorphism

g_n on M with $d^1_C(g_n, \hat{g}) < \frac{1}{n}$ satisfying

$$g_{n\,|W^c(p,\, g_n)}(t) = t + at^2 + \frac{1}{2n},$$

where $t \in W^c(p, g_n)(\subset \mathbb{R})$ for small enough $|t| < \varepsilon_0$ and $W^c(p, g_n) \cap B_{\varepsilon_0}(p)$ $= W^c(p, \hat{g}) \cap B_{\varepsilon_0}(p)$.

Choose a point $x_n = p$. Then x_n is a fixed point of $\hat{g}$. Since $\rho' > 1$, $g_n^m(y)$ is far away from $W^c(p, \hat{g}) \cap B_{\varepsilon}(p)$ as m approach to ∞. So it is easy to see that for all $y \in M$, there exists an integer m such that

$$d(x_n, g_n^m(y)) \geq \varepsilon_0.$$

This is a contradiction since $\hat{g} \in C^2$-int (IS(M)).

For simplicity, denote $\hat{g}$, $B_{\hat{g}}$ and $W^c(p, \hat{g})$ by g, B and $W^c(p)$, respectively. In case (b), since

$$\frac{d}{dt} g_{|W^c(p)}(t) = -1 + 3at^2 + o(t^3) \quad \text{and} \quad \frac{d^2}{dt^2} g_{|W^c(p)}(t) = 6at + o(t^2)$$

with respect to corresponding coordinates, we see that

$$\frac{d}{dt} g_{|W^c(p)}(0) = -1 \quad \text{and} \quad \frac{d^2}{dt^2} g_{|W^c(p)}(0) = 0.$$

Thus, perturbing g in a neighbourhood of p with respect to the C^2-topology, there exist $\tilde{g} \in C^2$-int(IS(M)) (C^2-nearby g) and $\varepsilon_0 > 0$ such that:

- there exists the centre manifold ($C^3 - \tilde{g}$-invariant curve) $W^c(p, \tilde{g})$ of p such that

$$W^c(p, \tilde{g}) \cap B_{\varepsilon_0}(p) = W^c(p) \cap B_{\varepsilon_0}(p),$$

- $\tilde{g}_{|W^c(p,\, \tilde{g})}(t) = -t$ for $t \in W^c(p, \tilde{g})$ if $|t|$ is small enough.

Clearly, $\widetilde{g}^2_{|W^c(p,\,\widetilde{g})\cap B_{\varepsilon_0}}$ is the identity map. Since $\widetilde{g}$ has the ISP, $\widetilde{g}^2$ has the ISP. Thus it is not hard to show that the restriction $\widetilde{g}^2_{|W^c(p,\,\widetilde{g})\cap B_{\varepsilon_0}}$ has the ISP since $B_{\widetilde{g}} = [\rho'] \, (\rho' > 1)$ is hyperbolic. However, we can see that any identity map does not have the ISP. This is a contradiction.

In case (c), following the proof of Newhouse et al. ([15, Theorem 5.2 and Remark 5.3]) and Sakai ([28, p. 240]), we shall derive a contradiction. Suppose that there exists a smooth arc $\{\varphi_\mu\}_{\mu\in\mathbb{R}}$ of diffeomorphisms on M (the corresponding map $\Phi : M \times \mathbb{R} \to M \times \mathbb{R}$ defined by $\Phi(x,\,\mu) = (\varphi_\mu(x),\,\mu)$ for $(x,\,\mu) \in M \times \mathbb{R}$ is C^∞) such that $\varphi_0 = g$ and $(p,\,0) \in M \times \mathbb{R}$ is a Hopf point unfolding generically. Then, we obtain the approximated arc $\{\widetilde{\varphi}_\mu\}_{\mu\in\mathbb{R}}$ satisfying the following assertions:

- the eigenvalues of $D_p\widetilde{\varphi}_0$ have the form $e^{2\pi i}$ with α irrational,
- the centre manifold $W^c(p,\,\widetilde{\varphi}_0)$ is C^∞,
- there is a $\widetilde{\varphi}_\mu$-invariant attracting (or repelling) circle $\mathcal{C}$ (in the manifold) near p for $\mu > 0$ small enough,
- the restriction $\widetilde{\varphi}_\mu|\mathcal{C}$ is conjugated to a rotation map.

Recall that $\widetilde{\varphi}_\mu$ has the ISP and B_μ is hyperbolic, where B_μ is the matrix for $\widetilde{\varphi}_\mu$ corresponding to B. We can see that $\widetilde{\varphi}_\mu|\mathcal{C}$ has the ISP, but this is a contradiction by Lemma 3.5. The proof is completed. □

Let $\|\cdot\|$ be a Riemannian metric on TM. For $f \in \mathrm{Diff}^2(M)$, an f-invariant set Λ is said to have a *dominant splitting* if we can decompose its tangent bundle into two Df-invariant subbundles $T_\Lambda M = E \oplus F$ such that there exist constants $C > 0$ and $0 < \lambda < 1$ with the following property:

$$\| Df^n_{|E(x)} \| \cdot \| Df^{-n}_{|F(f^n(x))} \| \le C\lambda^n, \quad \text{for all } x \in \Lambda,\ n \ge 0.$$

Let us denote by $\Omega_1(f)$ the set of sinks of f in $\Omega(f)$ and $\Omega_2(f)$ the set of sources of f in $\Omega(f)$. We define

$$\Omega_0(f) = \Omega(f) - (\Omega_1(f) \cup \Omega_2(f)).$$

Notice that $\Omega_0(f)$ is compact. We denote by $P_s(f)$ the set of periodic points of saddle type. Also, note that f satisfies *Kupka-Smale condition* if any periodic point of f is hyperbolic and the stable and unstable manifolds of periodic points are all transversal.

Theorem 3.7. *Let f be a C^2-stable inverse shadowing diffeomorphism. Then*

(i) *f satisfies Kupka-Smale condition.*

(ii) *if $\overline{P(f)} = \Omega(f)$, $\overline{P_s(f)} = \Omega_0(f)$ and there exists a dominated splitting on $\overline{P_s(f)}$, then f satisfies both Axiom A and the strong transversality condition.*

Proof. Let f be a C^2-stable inverse shadowing diffeomorphism. Then the assertion (i) follows from Theorems 3.2 and 3.6 quickly. To prove the assertion (ii), we suppose further that $\overline{P(f)} = \Omega(f)$ and there is a dominated splitting on $\overline{P_s(f)}$. By Theorem 3.6, every $p \in P(f)$ is hyperbolic. Since $\overline{P_s(f)}$ has a dominated splitting, it follows from Theorem *B* in [22] that $\overline{P_s(f)} = \Lambda_1 \cup \Lambda_2$, where Λ_1 is hyperbolic and Λ_2 consists of a finite union of normally hyperbolic periodic simple closed curves $\mathcal{C}_1, ..., \mathcal{C}_n$ such that $f^{mi} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugated to an irrational rotation. Here m denotes the smallest number such that $f^{mi}(\mathcal{C}_i) = \mathcal{C}_i$. Since f is a C^2-stable inverse shadowing diffeomorphism, f^{mi} also is C^2-stable inverse shadowing. Moreover, the normal hyperbolicity of $\mathcal{C}_i$ shows that the restriction $f^{mi}|_{\mathcal{C}_i}$ satisfies the ISP.

On the other hand, it follows from Lemma 3.5 that an irrational rotation does not have the ISP. Since the ISP is invariant under a topological conjugacy, $\Lambda_2 = \varnothing$ is concluded. Thus $\overline{P_s(f)}$ is hyperbolic.

To prove that f satisfies Axiom A, it suffices to show that the set of sinks and sources in $P(f)$ is finite. Assume that the set of sinks in $P(f)$ is infinite. Then $\sharp\Omega_1(f) = \infty$. Since $\overline{\Omega_1(f)} - \Omega_1(f) \subset \Omega_0(f) = \overline{P_s(f)}$ and $\Omega_0(f)$ is hyperbolic, there exists a neighborhood $\mathcal{U}$ of $\Omega_0(f)$ such that the maximal invariant set in this neighborhood is hyperbolic. This means that all the sinks except only a finite number of sinks in $P(f)$ are contained in $\mathcal{U}$. This is a contradiction to the hyperbolicity of the maximal invariant subset of $\mathcal{U}$. Thus the set of sinks in $P(f)$ is finite. Similarly the set of sources in $P(f)$ is finite. It follows from these facts that f satisfies Axiom A. The strong transversality condition follows directly from Theorem 3.2, and thus the proof is completed. □

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