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A COMPUTATIONAL APPROACH TO FIND POSITIVE SOLUTIONS OF A NONLINEAR ELLIPTIC PROBLEM

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Abstract

In this work, we consider numerical positive solutions of the equation $-\Delta u = \lambda f(u)$ with Dirichlet boundary condition in a bounded domain Ω, where $\lambda > 0$ and $f(u)$ is a nonlinear function of u. We study the behavior of the branches of numerical positive solutions for varying λ.

1. Introduction

We are interested in the positive solutions of the problem

$$\begin{cases} -\Delta u(x) = \lambda f(x, u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

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where Ω is a bounded domain in $\mathbf{R}^N$ $(N \geq 3)$ with boundary $\partial\Omega$, and $f(u) = u(1 - \sin u) + u^2$.

The problems involving Laplace equation arise quite frequently in the biological, social and physical sciences. For example, solutions of $-\Delta u = \lambda f(u)$ correspond to steady states for time motion, with *f* corresponding to external driving forces. The Laplace equation also plays an important role in field theories in which a field (e.g., electric, magnetic gravitational forces, or fluid velocity field) is given as the gradient of a potential function *u* [8].

On the other hand, the Dirichlet boundary value condition has an important physical significance. In electrostatics, for example, this condition specified the value of the potential function *u* on $\partial\Omega$ which induces the electric field $\vec{E} = -\nabla u$ in Ω. If we can show that the equation $-\Delta u = \lambda f(u)$ with Dirichlet condition is well-posed, then this means that the electric field is completely determined by the charge distribution inside Ω together with the value of the potential function *u* on $\partial\Omega$.

In this paper, we study numerical solutions of equation (1) that arise in various fields of physics, and have been studied by several authors. Among others, it describes the problems of thermal self-ignition [3], diffusion phenomena induced by nonlinear sources [5], or a ball of isothermal gas in gravitational equilibrium as proposed by Kelvin [1]. We also refer to [4, 9], where different models and further references may be found. In this paper, we concentrate on the numerical positive solutions of temperature distribution in an object heated by the application of a uniform electric current suggested in [6]. In fact, we show that the first eigenvalue of the problem

$$\begin{cases} -\Delta u(x) = \lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{2}$$

is a bifurcation point of the branch of numerical solutions that tends to right, also there is a positive $\lambda^* > \lambda_1$ such that for any $\lambda_1 < \lambda < \lambda^*$, we have two different numerical positive solutions, in λ^*, and a unique solution and for $\lambda > \lambda^*$.

The outline of this paper is as follows: In the next section, we present a useful numerical method and introduce the framework of the procedure to find numerical solutions. Section 3 contains some numerical results of the problem (1) for varying λ.

2. Finite Difference Method

Numerical techniques based on finite difference schemes can lead us to obtain approximate solutions for any PFEs [2, 7, 10]. In particular, for an elliptic partial differential equation of second order such as

$$Lu = \sum\sum\left(-a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\right) + \sum b_i \frac{\partial u}{\partial x_i} + cu = f(\lambda, u),$$

we can use this technique. In fact, the main idea is to find a numerical solution for (1) in a bounded domain Ω for special points with exact solution, i.e., we seek a solution array $\mathbf{u}$ in a point of a discrete grid $\mathbf{\Omega} \subset \Omega$, $\mathbf{u}(x) = u(x)$, where u is the exact solution of (1). It is often advantageous to study derivatives using the limit of difference quotients, and for this reason, we define "forward difference quotient" as follows:

$$\Delta_i^+ u(x) = \frac{u(x + he_i) - u(x)}{h},$$

where $h \neq 0$, e_i denotes the unit vector in the x_i-direction, h is small enough such that $0 < |h| < dist(x, \partial\Omega)$. Moreover, we can define "backward difference quotient":

$$\Delta_i^- u(x) = \frac{u(x) - u(x - he_i)}{h}.$$

If we put together these quotients, then we find a better approximation for $\frac{\partial u(x)}{\partial x_i}$ as follows:

$$\delta_i u(x) = \frac{u(x + he_i) - u(x - he_i)}{2h}.$$

It is proved in [8] that if we consider $u \in C^1(\Omega)$, then

$$\delta_i u(x) \to \frac{\partial u(x)}{\partial x_i} \quad \text{as } h \to 0.$$

We continue this procedure to gain higher derivatives such as

$$\delta_{ii}u(x) = \frac{u(x + he_i) - 2u(x) + u(x - he_i)}{h^2}$$

and

$$\delta_{ij}u(x) = \frac{u(x + he_i + ke_j) - u(x + he_i - ke_j) - u(x - he_i + ke_j) + u(x - he_i - ke_j)}{4hk}$$

that tend to

$$\frac{\partial^2 u(x)}{\partial x_i^2}, \quad \frac{\partial^2 u(x)}{\partial x_i \partial x_j}$$

as $h \to 0$ and $|(h, k)| \to 0$, respectively.

Now, by replacing these approximations in any elliptic equation of second order, we arrive at a finite system of equations that after solving give the desired array **u**.

The method of differences is especially suitable for the solution of boundary value problems, for instance, the problem of determining a function that satisfies the Laplace equation in the interior of a given field Ω and possesses given values at the boundary of the field; such problems arise in the exploration of stationary temperature distribution when the temperature at the boundary of the field is known, in investigating the tension in a twisted rod of prismatic section, etc. In this case, the procedure is as above.

It remains how we can choose the points of grid. Suppose Ω is a regular domain, for example, a square in plane, i.e., $\Omega = [a, b] \times [c, d]$. We can assume that the solution domain of the problem is covered by a mesh of grid-lines

$$x_i = a + ih, \quad i = 0, 1, 2, \dots, n_1,$$

$$y_j = c + jk, \quad j = 0, 1, 2, \dots, n_2$$

parallel to the axes and

$$x_0 = a, \quad x_{n_1} = b, \quad y_0 = c, \quad y_{n_2} = d.$$

Approximations u_{ij} to $u(x_i, y_j)$ are calculated at the point of intersection of these lines, namely, (x_i, y_j) which is referred to as the (i, j) grid-point. The constant spatial and temporal grid-spacing are

$$h = \frac{(b-a)}{n_1}, \quad k = \frac{(d-c)}{n_2},$$

respectively.

But a large number of physical problems have irregular boundary. For investigation, a point like P near bound of domain that does not have distance equal to h from the boundary can be used. A precise technique uses interpolation. In this section, we explain it briefly.

For simplicity, we confine ourselves in a space of dimension two. Suppose P_1 and P_2 are the points of a grid $\mathbf{\Omega}$ which are at distances h and k from P in the directions x and y, respectively, and that Q_1 and Q_2 are the nearest points to P on the boundary in the directions x and y respectively. Because of Dirichlet boundary condition, it follows that $u(Q_1) = u(Q_2) = 0$.

Let $dist(P, Q_1) = d_1$ and $dist(P, Q_2) = d_2$, where $0 \le d_1, d_2 \le 1$. We want to find $\frac{\partial u}{\partial x}(P)$ and $\frac{\partial u}{\partial y}(P)$ and $\frac{\partial^2 u}{\partial x^2}(P)$ and $\frac{\partial^2 u}{\partial y^2}(P)$ simultaneously.

By using Taylor expansion around P, we get

$$u(x, y) = u(P) + x\left(\frac{\partial u}{\partial y}\right)(P) + y\left(\frac{\partial u}{\partial y}\right)(P) + \frac{x^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)(P)$$

$$+ xy\left(\frac{\partial^2 u}{\partial x \partial y}\right)(P) + \frac{y^2}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)(P) + \cdots.$$

Without loss of generality, we can suppose P to be $(0, 0)$ and so we can write points P_1, P_2, Q_1 and Q_2 such as

$$(-h, 0), \quad (0, -k), \quad (d_1 h, 0), \quad (0, d_2 k),$$

respectively.

After substituting last formulas in Taylor expansion and omitting $o(h^3)$ and

$o(k^3)$, we arrive at

$$\begin{cases} d_2 k u_y(P) + \frac{1}{2}(d_2 k)^2 u_{yy}(P) & = u(Q_2) - u(P), \\ d_1 h u_x(P) + \frac{1}{2}(d_1 h)^2 u_{xx}(P) & = u(Q_1) - u(P), \\ -k u_y(P) + \frac{1}{2} k^2 u_{yy}(P) & = u(P_2) - u(P), \\ -h u_x(P) + \frac{1}{2} h^2 u_{xx}(P) & = u(P_1) - u(P) \end{cases} \tag{3}$$

that gives

$$\frac{\partial u}{\partial x}(P) = h^{-1}\left[\frac{u(Q_1)}{d_1(1+d_1)} - \frac{d_1 u(P_1)}{(1+d_1)} - \frac{(1-d_1)u(P)}{d_1}\right] + o(h^2),$$

$$\frac{\partial^2 u}{\partial x^2}(P) = 2h^{-2}\left[\frac{u(Q_1)}{d_1(1+d_1)} - \frac{u(P_1)}{(1+d_1)} - \frac{u(P)}{d_1}\right] + o(h),$$

$$\frac{\partial u}{\partial y}(P) = k^{-1}\left[\frac{u(Q_2)}{d_2(1+d_2)} - \frac{d_2 u(P_2)}{(1+d_2)} - \frac{(1-d_2)u(P)}{d_2}\right] + o(k^2),$$

$$\frac{\partial^2 u}{\partial y^2}(P) = 2k^{-2}\left[\frac{u(Q_2)}{d_1(1+d_1)} - \frac{u(P_2)}{(1+d_2)} - \frac{u(P)}{d_2}\right] + o(k).$$

If the points P_1 and P_2 lie irregularly, then we can use this procedure for them again. To find the value of *u* in the point *P*, we apply linear interpolation formula in direction *x* as follows:

$$u(P) = \left(\frac{d_1}{1+d_1}\right)u(P_1) + \left(\frac{1}{1+d_1}\right)u(Q_1)$$

or in direction *y* as follows:

$$u(P) = \left(\frac{d_2}{1+d_2}\right)u(P_2) + \left(\frac{1}{1+d_2}\right)u(Q_2).$$

3. Numerical Results

In this section, we consider problem (1) and use discussions presented in the previous section to find numerical solutions. At first, we note that to solve problem

(1), we consider $N \geq 3$. Let $N = 3$ and $\Omega = [0, 1] \times [0, 1] \times [0, 1]$ and the grid $\mathbf{\Omega} \subset \Omega$ be a division of Ω and $h = \frac{1}{4} (n_1 = n_2 = n_3 = 4)$.

We solve, numerically, the problem

$$\begin{cases} -(u_{xx} + u_{yy} + u_{zz}) = \lambda[u(x, y, z)(1 - \sin(u(x, y, z))) + u(x, y, z)^2], & (x, y, z) \in \Omega, \\ u(x, y, z) = 0, & (x, y, z) \in \partial\Omega. \end{cases} \tag{4}$$

Dirichlet boundary condition leads us to have

$$u_{0, j, k} = u_{i, 0, k} = u_{i, j, 0} = 0, \quad \forall 1 \leq i, j, k \leq 3,$$

where $u_{i, j, k} = u(x_i, y_j, z_k)$.

By using the approximations of u_{xx}, u_{yy} and u_{zz}, we have a system of equations of this type

$$-\frac{u_{i+1, j, k} + u_{i-1, j, k} + u_{i, j+1, k} + u_{i, j-1, k} + u_{i, j, k+1} + u_{i, j, k-1} - 6u_{i, j, k}}{h^2}$$

$$= \lambda(u_{i, j, k}(1 - \sin(u_{i, j, k})) + u_{i, j, k}^2).$$

Some of the equations of this system mentioned follows:

$$16(u_{211} + u_{121}u_{112} - 6u_{111}) + \lambda(u_{111}(1 - \sin(u_{111})) + u_{111}^2) = 0$$

for $i = j = k = 1$,

$$16(u_{212} + u_{122} + u_{113} + u_{121} - 6u_{112}) + \lambda(u_{112}(1 - \sin(u_{112})) + u_{112}^2) = 0$$

for $i = j = 1, \quad k = 2$,

$$\vdots$$

After solving this system, we can obtain **u** in grid $\mathbf{\Omega}$ that guides us to understand the behavior of solution branches. We express just some of values of u_{ijk}s in the following table. It is easy to see that λ^+ (the first eigenvalue of the problem (2)) in this case is 26.7 with decimal accuracy, also λ^* is around 50 and after that we have no positive solution.

λ	**u**		
10^{-3}	$u_{111} = 4.25 \times 10^{-15}$	$u_{121} = 9.85 \times 10^{-15}$	$u_{131} = 1.17 \times 10^{-14}$
	$u_{211} = 6.32 \times 10^{-15}$	$u_{221} = 1.53 \times 10^{-14}$	$u_{231} = 2.26 \times 10^{-14}$
	$u_{311} = 3.54 \times 10^{-15}$	$u_{321} = 7.73 \times 10^{-15}$	$u_{331} = 9.30 \times 10^{-15}$
10^{-3}	$u_{111} = 80449.8$	$u_{121} = 74839.6$	$u_{131} = 80932.7$
	$u_{211} = 74454.3$	$u_{221} = 69480.1$	$u_{231} = 74589.3$
	$u_{311} = 80889.7$	$u_{321} = 74411.0$	$u_{331} = 80509.0$
0.1	$u_{111} = 4.29 \times 10^{-13}$	$u_{121} = 9.92 \times 10^{-13}$	$u_{131} = 1.17 \times 10^{-12}$
	$u_{211} = 6.37 \times 10^{-13}$	$u_{221} = 1.54 \times 10^{-12}$	$u_{231} = 2.27 \times 10^{-12}$
	$u_{311} = 3.57 \times 10^{-13}$	$u_{321} = 7.78 \times 10^{-13}$	$u_{331} = 9.35 \times 10^{-13}$
0.1	$u_{111} = 602.65$	$u_{121} = 372.71$	$u_{131} = 351.31$
	$u_{211} = 72.231$	$u_{221} = 211.48$	$u_{231} = 773.04$
	$u_{311} = 254.09$	$u_{321} = 266.94$	$u_{331} = 561.31$
1	$u_{111} = 4.63 \times 10^{-12}$	$u_{121} = 1.05 \times 10^{-11}$	$u_{131} = 1.23 \times 10^{-11}$
	$u_{211} = 6.84 \times 10^{-12}$	$u_{221} = 1.63 \times 10^{-11}$	$u_{231} = 2.36 \times 10^{-11}$
	$u_{311} = 3.86 \times 10^{-12}$	$u_{321} = 8.31 \times 10^{-12}$	$u_{331} = 9.85 \times 10^{-12}$
1	$u_{111} = 18.985$	$u_{121} = 28.912$	$u_{131} = 17.173$
	$u_{211} = 23.418$	$u_{221} = 27.424$	$u_{231} = 18.457$
	$u_{311} = 10.330$	$u_{321} = 13.085$	$u_{331} = 8.961$
26.6	$u_{111} = 1.63 \times 10^{-19}$	$u_{121} = 2.46 \times 10^{-19}$	$u_{131} = 1.64 \times 10^{-19}$
	$u_{211} = 2.13 \times 10^{-19}$	$u_{221} = 3.09 \times 10^{-19}$	$u_{231} = 2.15 \times 10^{-19}$
	$u_{311} = 1.45 \times 10^{-19}$	$u_{321} = 2.08 \times 10^{-19}$	$u_{331} = 1.47 \times 10^{-19}$
26.6	$u_{111} = 2.860$	$u_{121} = 1.735$	$u_{131} = 0.785$
	$u_{211} = 0.960$	$u_{221} = 0.045$	$u_{231} = 0.189$
	$u_{311} = 0.804$	$u_{321} = 0.637$	$u_{331} = 0.358$
26.7	$u_{111} = 0.1246$	$u_{121} = 0.5945$	$u_{131} = 0.5133$
	$u_{211} = 0.5852$	$u_{221} = 1.0570$	$u_{231} = 0.7829$
	$u_{311} = 0.4398$	$u_{321} = 0.6573$	$u_{331} = 0.2708$

26.7	$u_{111} = 1.2148$	$u_{121} = 2.0266$	$u_{131} = 1.6532$
	$u_{211} = 1.2040$	$u_{221} = 1.4454$	$u_{231} = 0.7634$
	$u_{311} = 1.0653$	$u_{321} = 1.6640$	$u_{331} = 2.6820$
30	$u_{111} = 1.2955$	$u_{121} = 2.2439$	$u_{131} = 1.7846$
	$u_{211} = 2.0192$	$u_{221} = 2.0925$	$u_{231} = 1.7774$
	$u_{311} = 2.3508$	$u_{321} = 0.9996$	$u_{331} = 2.3140$
30	$u_{111} = 0.5280$	$u_{121} = 1.4508$	$u_{131} = 2.4223$
	$u_{211} = 0.2535$	$u_{221} = 2.6319$	$u_{231} = 1.1280$
	$u_{311} = 2.4035$	$u_{321} = 1.4551$	$u_{331} = 2.2615$
50	$u_{111} = 1.1952$	$u_{121} = 1.5492$	$u_{131} = 1.9407$
	$u_{211} = 2.1985$	$u_{221} = 1.6464$	$u_{231} = 1.5396$
	$u_{311} = 0.6659$	$u_{321} = 0.3503$	$u_{331} = 1.5895$
50	$u_{111} = 1.1970$	$u_{121} = 1.5679$	$u_{131} = 1.8112$
	$u_{211} = 2.2326$	$u_{221} = 1.6447$	$u_{231} = 1.5609$
	$u_{311} = 0.6524$	$u_{321} = 0.3238$	$u_{331} = 1.5956$
51	$u_{111} = 1.5347$	$u_{121} = 1.422$	$u_{131} = 1.970$
	$u_{211} = 2.112$	$u_{221} = -0.110$	$u_{231} = -0.2909$
	$u_{311} = 1.947$	$u_{321} = 1.066$	$u_{331} = 1.609$
60	$u_{111} = 19.460$	$u_{121} = -7.040$	$u_{131} = 2.070$
	$u_{211} = -7.192$	$u_{221} = 2.459$	$u_{231} = 2.563$
	$u_{311} = 2.118$	$u_{321} = 0.9975$	$u_{331} = 1.287$
100	$u_{111} = 8.716$	$u_{121} = -4.885$	$u_{131} = 4.527$
	$u_{211} = 1.899$	$u_{221} = 2.020$	$u_{231} = -2.214$
	$u_{311} = 1.320$	$u_{321} = -1.848$	$u_{331} = -2.198$

4. Conclusion

In this paper, the finite difference method has been applied for solving nonlinear elliptic equations. For any bounded domain in any dimension, we recover the mentioned formulas that we use here. Further, we accommodate the finite difference method to deal with Dirichlet, Neumann, and mixed boundary conditions. By using the results in last section, we can draw the bifurcation diagram of the solutions in the plane $(\lambda, \| u \|)$, where

$$\| u \| = \| u \|_{\infty} = \sup_{(x, y, z) \in [0,1] \times [0,1] \times [0,1]} u(x, y, z).$$

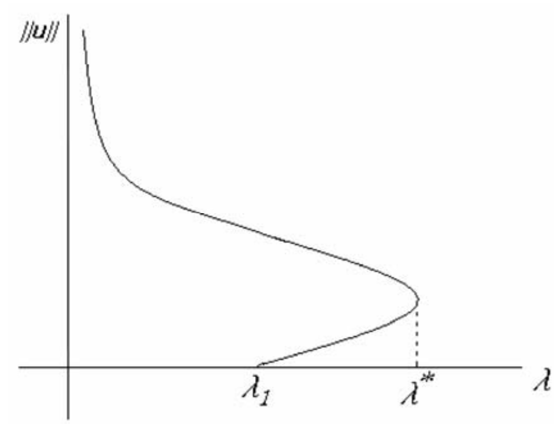

Figure 1. Bifurcation diagram.

References

[1] S. Chandrasekhar, An Introduction to the Study of Steller Structure, Dover Publications, Inc., New York, 1957, ii+509 pp.

[2] M. Dehghan, Numerical procedures for a boundary value problem with a non-linear boundary condition, Appl. Math. Comput. 147 (2004), 291-306.

[3] I. M. Gelfand, Some problems in the theory of quasi-linear equations, Amer. Math. Soc. Trans. 29(2) (1963), 295-381.

[4] D. D. Joseoh and T. S. Landgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Ration. Mech. Anal. 49 (1973), 241-269.

[5] D. D. Joseoh and E. M. Sparrow, Nonlinear diffusion induced by nonlinear sources, Quart. Appl. Math. 28 (1970), 327-342.

[6] H. B. Keller and D. S. Cohen, Some positive problems suggested by nonlinear heat generation, J. Math. Mech. 16 (1967), 1361-1376.

[7] O. A. Ladyzhenskaya, The Boundary Value Problems of Mathematical Physics, Springer-Verlag, New York, 1985.

[8] R. McOwen, Partial Differential Equations, Prentice-Hall, Inc., 1996.

[9] F. Mignot and J. P. Puel, Sur une classede problemes non lineairesavec nonlinearite positive, Croissante, Convexe. Comm. Part. Diff. Eq. 5(8) (1980), 791-836.

[10] I. G. Petrovsky, Lectures on Partial Differential Equations, Dover Publications, Inc., 1991.