# ON THE COMPLETENESS OF PSEUDO-RIEMANNIAN METRICS 

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#### Abstract

The completeness of several pseudo-Riemannian geometries is examined from a Hamiltonian point of view.


This paper stresses the role of compactness in determining the completeness of the geodesic flow on some pseudo-Riemannian manifolds. We remind the reader that the situation for indefinite metrics is very different from that of Riemannian geometry, where we have the following corollaries of the Hopf-Rinow theorem.

Corollary. A compact Riemannian manifold is complete.
Corollary. A left-invariant Riemannian metric on a Lie group is complete.

Here, completeness is defined as geodesic completeness, which means that all geodesics may be extended for all values of the affine parameter. This is equivalent, via the Hopf-Rinow theorem, to the manifold being complete as a metric space under the distance that the Riemannian metric defines. In the indefinite case, we have the striking example.

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Example 1. The Clifton-Pohl torus. On $R^{2}$, define a Lorentz metric by

$$
k=(\cos x d x+\sin x d y) \otimes(-\sin x d x+\cos x d y)
$$

Now $k$ is invariant by a $2 \pi$ translation in $x$ or $y$, so $k$ descends to a Lorentz metric on the torus $R^{2} /(2 \pi Z)^{2}$. It is straightforward to check that all non-periodic null geodesics are incomplete. This is surprising because there is


Figure 1. Momenta and energy level intersection in $T_{m}^{*} M$.
even some symmetry in the problem: observe that $\partial_{y}$ is a Killing field. More information may be found in the book by O'Neill [7].

Example 2. The affine group $A(1, R)$. This is the group of transformations $x \rightarrow a x+b$ with $a \neq 0$. Any indefinite quadratic form on the Lie algebra $\mathfrak{a}(1, R)$ induces a left-invariant Lorentz metric on $A(1, R)$ whose geodesic flow is incomplete.

In light of these examples, it is perhaps even more striking that the following theorem holds.

Theorem. A compact pseudo-Riemannian homogeneous space is geodesically complete.

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We wish to generalize this theorem, viewing it as a compactness result. This will give us a geometric point of view that is fruitful for examining several examples.

To begin, let ( $M, k$ ) be a compact pseudo-Riemannian manifold. Let $G$ be a Lie group of isometries of $(M, k)$. Let $X_{1}, \ldots, X_{r}$ be the infinitesimal generators of the action of the Lie algebra $\mathfrak{g}$ on $M$, and let $j_{a}=\left\langle\vartheta, X_{a}\right\rangle$ be the momentum corresponding to $X_{a}$. Here $\vartheta$ is the fundamental oneform on $T^{*} M$. Now the momenta and the Hamiltonian $h=\frac{1}{2} k(p, p)$ are conserved quantities of the geodesic flow, so in a cotangent space $T_{m}^{*} M$ we have the picture in Figure 1.

In Figure 1, the cone is a level set of the quadric $\frac{1}{2} k_{m}(p, p)$ (call it $Q$ ) and the plane is given by the level sets of the momenta $j_{a}=c_{a}$ (call it $P)$. The point of drawing the picture this way is that if we can follow the intersection around $M$, and knowing that the Hamiltonian and the momenta are conserved under the geodesic flow, we can show that under some conditions the whole intersection sits inside a compact set and hence the whole geodesic flow is complete. Now take a local trivialization of $T^{*} M$ about $m$. If we ask what happens to the intersection of $P$ and $Q$ as we vary the base point $m$, we see that if the intersection above $m$ is compact, then it is still compact for all points in some open neighbourhood of $m$. Furthermore, ignoring the base directions, the nearby intersections are in an open neighbourhood with compact closure of the one above $m$. This is seen by observing that the equation for the intersection, viewed in coordinates on $P$ is the equation for an ellipsoid (since it is compact and quadratic) and the stability of the intersection follows from the stability of the defining equation of the ellipsoid. This means that we can put an open ball with compact closure with centre $m$ about the intersection in $T_{m}^{*} M$, and crossed with a ball about $m$, get a polydisc in $T^{*} M$ that contains the intersection of the level sets. If the intersection is fiberwise compact, we can take a finite number of such polydiscs to produce a compact manifold with corners that contains the geodesic flow for the level set of momenta and energy that we are interested in. Since the
geodesic flow never intersects the boundary, we get completeness. More succinctly, we may say that the polydiscs allow us to construct a Riemannian metric on $M$ which uniformly bounds the geodesic velocity. This gives the completeness of the geodesic flow as $M$ is assumed compact. Formally, we may sum up this discussion as

Theorem. Let $(M, k)$ be a compact pseudo-Riemannian manifold. Let $G$ be a Lie group of isometries of ( $M, k$ ) with corresponding momentum $j$. Suppose that the level set $j=0$ contains no nonzero null directions. Then the geodesic flow on $M$ is complete.

Corollary [4]. Let $(M, k)$ be a pseudo-Riemannian compact manifold of signature $(n-s, s)$, where $2 s \leq n$. Suppose that there exist $s$ Killing fields on $M$, everywhere negative and linearly independent. Then ( $M, k$ ) is geodesically complete.

Corollary [5]. A compact pseudo-Riemannian homogeneous space is geodesically complete.

Example 3. Let $P$ be a principal bundle over a compact base $B$ with compact group $G$. Endow $G$ with a left-invariant Riemannian metric, and $B$ with a Riemannian metric. Choosing a principal connection on $P$ allows us to equip $P$ with a $G$-invariant pseudo-Riemannian metric $k$ by declaring the vertical part of a vector to agree with the metric on $G$, and the horizontal part to agree with the opposite metric on $B$. Since $G$ is a symmetry group of $k$, we get momenta $j_{a}=\left\langle\vartheta, X_{a}\right\rangle$ from the fundamental vector fields $X_{a}$. The space of vectors with all momenta zero is the horizontal space given by the connection, as is seen by the splitting of the fundamental one-form $\vartheta$ that the connection induces. Since the horizontal distribution contains no non-zero null vectors by design, we see that the pseudo-Riemannian metric $k$ on $P$ is complete.

This point of view may also be usefully applied to examine the completeness of left-invariant indefinite metrics on a Lie group $G$. Recall that the geodesic flow for a left-invariant metric may be formulated as a $G$-invariant Hamiltonian system on $T^{*} G$. Pulling this system back via
the left trivialization

$$
G \times \mathfrak{g}^{*} \rightarrow T^{*} G:(g, p) \rightarrow p_{g}
$$

we get the Euler-Arnol'd equations

$$
\begin{aligned}
& \dot{g}=T_{e} L_{g}\left(D_{2} h(g, p)\right) \\
& \dot{p}=a d_{D_{2} h(g, p)}^{*} p,
\end{aligned}
$$

where $h$ is the Hamiltonian. Left invariance means that $h(g, p)$ does not depend on $g$. We can study completeness by observing that it is determined solely by whether or not the Euler equation for $\dot{p}$ has a complete flow. Recall that $\mathfrak{g}^{*}$ is foliated by the co-adjoint orbits, and these are invariant manifolds for the Euler equation. Furthermore, in the left trivialization the Hamiltonian is a quadratic form $h=\frac{1}{2} k(p, p)$. Choosing coordinates $p_{1}, \ldots, p_{n}$ for $\mathfrak{g}^{*}$ gives us the Euler equations as

$$
\dot{p}_{j}=\left\{p_{j}, h\right\}=c_{j l}{ }^{m} k^{l n} p_{m} p_{n},
$$

(i.e., constant coefficient homogeneous quadratic differential equations) where the $c_{j l}{ }^{m}$ are the structure constants of the Lie algebra, and the co-adjoint orbits as the integral manifolds of the distribution spanned by the vector fields

$$
X_{k}=c_{j l}{ }^{m} p_{m} \partial^{l} .
$$

A more thorough discussion of this can be found in [1]. We now examine left invariant pseudo-Riemannian metrics on three dimensional Lie groups.

Example 4. Lorentz metrics on $\operatorname{SL}(2, R) . S L(2, R)$ is the group of $2 \times 2$ matrices with determinant +1 . A basis for the Lie algebra may be taken as

$$
e_{1}=\frac{1}{2}\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right), e_{2}=\frac{1}{2}\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), e_{3}=\frac{1}{2}\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right),
$$



Figure 2. Typical intersections of $C_{0}$ and $Q_{0}$.
and with the associated coordinates $p_{1}, p_{2}, p_{3}$ on $\mathfrak{g}^{*}$ we have the Poisson brackets

$$
\left\{p_{1}, p_{2}\right\}=-p_{3}, \quad\left\{p_{2}, p_{3}\right\}=p_{1}, \quad\left\{p_{3}, p_{1}\right\}=p_{2} .
$$

Now $C=p_{1}^{2}+p_{2}^{2}-p_{3}^{2}$ is a Casimir for the Poisson bracket, and the co-adjoint orbits are the level sets of $C(C=0$ is the union of three orbits, the cones $p_{3}>0, p_{3}<0$, and the origin). The Hamiltonian may be represented as an indefinite quadratic form $Q$ on $\mathfrak{g}^{*}$. The zero level sets of $C$ and $Q$ typically look like one of the cases in Figure 2 (there is the possibility of $C_{0}$ and $Q_{0}$ intersecting in four lines, but in order to reduce clutter in the picture, we have omitted it). Define $C_{a}=C^{-1}(a), Q_{b}=Q^{-1}(b)$.

Lemma. If the level sets $C_{0}$ and $Q_{0}$ intersect only at the origin, then the intersection $C_{a} \cap Q_{b}$ is compact for any choice of $a$ and $b$.

Proof. Let us add the coordinate $p_{4}$ and homogenize the equations for the intersection $C_{a} \cap Q_{b}$,

$$
\begin{aligned}
& p_{1}^{2}+p_{2}^{2}-p_{3}^{2}=a p_{4}^{2}, \\
& Q=b p_{4}^{2} .
\end{aligned}
$$

Examining these equations in a neighbourhood of the hyperplane at infinity ( $p_{4}=0$ ) we get

$$
\begin{aligned}
& p_{1}^{2}+p_{2}^{2}-p_{3}^{2}=0 \\
& Q=0
\end{aligned}
$$

By hypothesis, the only solution of these equations is $\left(p_{1}, p_{2}, p_{3}\right)=$ $(0,0,0)$, but $\left(p_{1}, p_{2}, p_{3}, p_{4}\right)=(0,0,0,0)$ is not a point in projective space, so the quadrics do not intersect in a neighbourhood of infinity. Hence the intersection is bounded, and since it is closed, compact.

This shows that if $C_{0} \cap Q_{0}=\{0\}$, then the Lorentz metric is complete. This is because the remaining equation of the Euler-Arnol'd equations for the group variable is a non-autonomous linear differential equation, which for matrix groups looks like $\dot{A}=A(t) p(t)$.

For the case when $C_{0} \cap Q_{0}$ is two lines, choose coordinates so that one of the lines is the $p_{1}$ axis. On this line, which is an invariant manifold for the flow, the differential equation is (up to scale)

$$
\dot{p}_{1}= \pm p_{1}^{2}
$$

which is incomplete. Now any reasonable topology on the set of left invariant Lorentz metrics would preserve the intersection condition $C_{0} \cap Q_{0}$ for small perturbations of the metric, showing that we have two open sets of complete left-invariant metrics separated by open sets of incomplete ones, corresponding to the intersections in Figure 2.

Example 5. The Euclidean group $E(2)$. This is the group of matrices of form

$$
\left(\begin{array}{ccc}
\cos \theta & -\sin \theta & a \\
\sin \theta & \cos \theta & b \\
0 & 0 & 1
\end{array}\right)
$$

The dual of the Lie algebra has coordinates $x, y, z$ with brackets

$$
\{x, y\}=0, \quad\{y, z\}=x, \quad\{z, x\}=y
$$

The co-adjoint orbits are the cylinders $x^{2}+y^{2}=$ constant, and the points on the $z$-axis. For a left-invariant Lorentz metric on $E(2)$, it is easy to see that the typical intersection of the null cone of the energy with a co-adjoint orbit is compact. With the same argument as in the case of $S L(2, R)$, this gives completeness of the geodesic flow for almost all leftinvariant Lorentz metrics. The only way that an energy surface of the metric can have a noncompact intersection with a co-adjoint orbit is if the $z$-axis is a null line. In this case it is easy to see that the Euler equation for $z$ is linear in $z$,

$$
\dot{z}=p(x, y) z+q(x, y)
$$

and since $x^{2}+y^{2}$ is constant, the solution $z(t)$ is defined for all time. This shows that the geodesic flow of any left invariant Lorentz metric on $E(2)$ is complete. This was previously observed in [3], but the approach is somewhat different.

There are some special cases of Lorentz metrics on $E(2)$ of interest. The first is the metric $x^{2}+y^{2}-z^{2}$, which was shown by Nomizu [6] to be flat. The second is the family of metrics $x^{2}+\left(1+k^{2}\right) y^{2}-z^{2}$ for which the Euler equations are

$$
\begin{aligned}
& \dot{x}=y z \\
& \dot{y}=-z x \\
& \dot{z}=-k^{2} x y,
\end{aligned}
$$

where $0<k^{2}<1$. The integral curve which passes through $(0,1,1)$ at $t=0$ defines the Jacobi elliptic functions

$$
t \rightarrow(x(t), y(t), z(t))=(\operatorname{sn}(t ; k), \operatorname{cn}(t ; k), \operatorname{dn}(t ; k)) .
$$

Example 6. The group $E(1,1)$, the rigid motions of two dimensional Minkowski space. This is the group of matrices of the form

$$
\left(\begin{array}{ccc}
\cosh \theta & -\sinh \theta & a \\
\sinh \theta & \cosh \theta & b \\
0 & 0 & 1
\end{array}\right)
$$

The dual of the Lie algebra has coordinates $x, y, z$ with brackets

$$
\{x, y\}=0,\{y, z\}=-x,\{z, x\}=y
$$

Co-adjoint orbits are given as the level sets of $C=x^{2}-y^{2}(C=0$ is the orbits $\{0,0, z\}$ and the half-planes $y= \pm x$ with $y \neq 0$ ). One sees from the argument for $S L(2, R)$ that a left-invariant indefinite metric on $E(1,1)$ will be complete if the null cone for the metric $Q_{0}$ only intersects the planes $y= \pm x$ at the origin. If the null cone intersects the planes transversally, then we are reduced to the following situation. On $\{(y, z) \mid y>0\},\{y, z\}=-y, h=Q(y, z)$, where $Q$ is an indefinite quadratic form. Then

$$
\dot{y}=\{y, h\}=\{y, z\} h_{z}=-y p(y, z),
$$

where $p(y, z)$ is linear in $z$. On the null line(s) for $Q$ the equation for $\dot{y}$ reduces to $\dot{y}=c y^{2}$, where $c \neq 0$. This is incomplete. We leave as an exercise for the reader the case where the null cone is tangent to the planes $y= \pm x$.

Example 7. The Heisenberg group $H$. This is the group of matrices of form

$$
\left(\begin{array}{lll}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{array}\right)
$$

The dual of the Lie algebra has coordinates $x, y, z$ with brackets

$$
\{x, y\}=z,\{y, z\}=0,\{z, x\}=0
$$

Co-adjoint orbits are given by the level sets of the Casimir $z$, except when $z=0$ they are points. It is clear that any left invariant metric on $H$ is complete if the null cone does not intersect the plane $z=0$ except at the origin, because then the intersection of any co-adjoint orbit and any energy level set is compact. In the case when the intersection is not compact, the Euler equations restricted to the co-adjoint orbits $z=$ constant are all linear differential equations and so complete. We note the special case of the metric $y^{2}-x z$, which is flat, a calculation done by Nomizu [6].

Continuing in this manner one can examine all three dimensional Lie groups. If the commutation relations of the algebra are reduced to the form

$$
\begin{aligned}
& {\left[e_{1}, e_{2}\right]=a e_{2}+b^{3} e_{3},} \\
& {\left[e_{2}, e_{3}\right]=b^{1} e_{1},} \\
& {\left[e_{3}, e_{1}\right]=b^{2} e_{2}-a e_{3},}
\end{aligned}
$$

then according to [2], the isomorphism classes of three-dimensional Lie algebras are given by the 'Bianchi' classification of Table 1. The completeness column refers to the completeness of Lorentz metrics on the corresponding group. The table is divided in two to reflect the fact that the first section is the unimodular algebras and the second part is not. Note further that the algebra of type V is discussed more fully below in the discussion of the special class $\mathfrak{S}$. Also, the calculations involved for the nonunimodular algebras of types V, VI, and VII are a little more involved because these algebras have no nontrivial Casimirs, and consequently more work is required in order to find the co-adjoint orbits.

Example 8. A special class of solvable groups $\mathfrak{S}$. Groups $G$ in $\mathfrak{S}$ have the property that the bracket $[e, f]$ of any two elements in the Lie algebra $\mathfrak{g}$ of $G$ is a linear combination of $e$ and $f$. Nomizu [6] showed that $G$ belongs to $\mathfrak{S}$ if and only if the Lie algebra contains a codimension one commutative ideal $\mathfrak{u}$ and an element $b \notin \mathfrak{u}$ such that $[b, e]=e$ for all $e \in \mathfrak{u}$.

Picking a basis $b, e_{1}, \ldots, e_{n}$ for $\mathfrak{g}$ with $e_{1}, \ldots, e_{n}$ spanning $\mathfrak{u}$ we see that matrices for the co-adjoint action are of the form

$$
\left[a d_{e}^{*}\right]=\left(\begin{array}{cc}
0 & \beta \\
0 & \alpha I
\end{array}\right)
$$

where $\alpha \in R, \beta \in R^{n}$. From this we see that the co-adjoint orbit through $p \in \mathfrak{g}^{*}$ is the half-plane consisting of vectors of form $x b^{*}+y p$ with $y>0$.

Table 1. Completeness of 3-d Lie algebras

| Type | $a$ | $b^{1}$ | $b^{2}$ | $b^{3}$ | Completeness |
| :---: | :---: | :---: | :---: | :---: | :---: |
| I | 0 | 0 | 0 | 0 | Complete |
| II | 0 | 1 | 0 | 0 | Complete |
| VII | 0 | 1 | 1 | 0 | Complete |
| VI | 0 | 1 | -1 | 0 | Mixed |
| IX | 0 | 1 | 1 | 1 | Complete |
| VIII | 0 | 1 | 1 | -1 | Mixed |
| V | 1 | 0 | 0 | 0 | Incomplete |
| VI | 1 | 0 | 0 | 1 | Mixed |
| VII | $a$ | 0 | 1 | 1 | Mixed |
| III $(a=1)$ | $a$ | 0 | 1 | -1 | Mixed |
| $\mathrm{VI}(a \neq 1)$ | $a$ | 0 | 1 | -1 | Mixed |

If $p \in \operatorname{span}\left\{b^{*}\right\}$, then the co-adjoint orbit is just the point $\{p\}$. The Poisson bracket on the orbit is $\{x, y\}=c y$ with $c \neq 0$. If we take any left invariant pseudo-Riemannian metric on $G$, then there is a two-plane in $\mathfrak{g}^{*}$ containing $\operatorname{span}\left\{b^{*}\right\}$ on which the associated quadratic form restricts to an indefinite one. By the same argument we used for $E(1,1)$, there is a null line on which the Euler equation reduces to $\dot{x}=x^{2}$, which is incomplete. We summarize this discussion as

Proposition. Any left-invariant pseudo-Riemannian metric on a Lie group $G$ in the class $\mathfrak{S}$ is geodesically incomplete.

Example 9. The Killing form on a semi-simple Lie group is nondegenerate and so may be used for a left-invariant indefinite metric on the group. Such a metric is always complete as the following calculation shows. If $k: \mathfrak{g} \rightarrow \mathfrak{g}^{*}$ is the Killing form, with inverse $k^{-1}$, then the Euler equation is

$$
\dot{p}=a d_{k^{-1}(p)}^{*} p .
$$

Letting $X=k^{-1}(p)$, and pairing the Euler equation with $Y$ in $\mathfrak{g}$ gives

$$
\begin{aligned}
\langle\dot{p}, Y\rangle & =\left\langle a d_{k^{-1}(p)}^{*} p, Y\right\rangle \\
& =\left\langle p,\left[k^{-1}(p), Y\right]\right\rangle \\
& =\langle k(X),[X, Y]\rangle \\
& =k(X,[X, Y]) .
\end{aligned}
$$

Furthermore, $\langle\dot{p}, Y\rangle=\langle k(\dot{X}), Y\rangle=k(\dot{X}, Y)$ or $k(\dot{X}, Y)=k(X,[X, Y])$. Adinvariance of the Killing form implies that $k([X, Y], Z)+k(Y,[X, Z])=0$, so that $k(X,[X, Y])=k([X, X], Y)=0$, which gives us that $\dot{p}=0$. By our previous comments, this implies that the geodesic flow is complete.

The reader will have noticed that in all the Lie group examples the incompleteness was shown by examining the null geodesics. If one has a solution to the Euler equations that blows up in finite time that represents an incomplete geodesic, it becomes asymptotically close to the null cone $Q=0$. It seems reasonable to make the following conjecture.

Conjecture. An incomplete left-invariant pseudo-Riemannian metric on a Lie group is necessarily null-incomplete.

## Notes:

1. If $X$ is a conformal Killing field, then $L_{X} g=f g$ for some function $f$, if $j=\langle\vartheta, X\rangle$ and $h=\frac{1}{2} g(p, p)$, then $d j / d t=\{j, h\}=-f h$. If $M$ is compact, then we get a uniform bound on $d j / d t$ under the geodesic flow, and hence a growth estimate on $j$. This can be used to extend our theorem in the spirit of a compactness argument. This extension of the result of Guediri and Lafontaine was observed by Romero and Sanchez [9].
2. The first example of the completeness of the bundle $P$ removes the restriction that $\operatorname{dim} G \leq \operatorname{dim} B$ in [4].
3. The discussion for $S L(2, R)$ corrects the assertion in [3] that a generic left-invariant Lorentz metric on $S L(2, R)$ is incomplete.

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4. The proposition about the incompleteness of groups in the class $\mathfrak{S}$ was also proven by Guediri [3].
5. The conjecture about null-incompleteness is similar to one described in Romero and Sanchez [8] that an incomplete compact Lorentz metric is null incomplete.

## References

[1] R. Cushman and L. Bates, Global aspects of classical integrable systems, Birkhauser, 1997.
[2] B. Dubrovin, A. Fomenko and S. Novikov, Modern Geometry-methods and Applications, Volume 1 of Graduate Texts in Mathematics 93, Springer, 1984.
[3] M. Guediri, On completeness of left-invariant Lorentz metrics on solvable groups, Revista Matemática de la Universidad complutense de Madrid 9(2) (1996), 337-350.
[4] M. Guediri and J. Lafontaine, Sur la complétude des variétés pseudoRiemanniennes, J. Geo. Phys. 15 (1995), 150-158.
[5] J. Marsden, On completeness of homogeneous pseudo-Riemannian manifolds, Indiana Univ. Math. J. 22 (1973), 1065-1066.
[6] K. Nomizu, Left-invariant Lorentz metrics on Lie groups, Osaka J. Math. 16 (1979), 143-150.
[7] B. O'Neill, Semi-Riemannian Geometry, Academic Press, 1983.
[8] A. Romero and M. Sanchez, On the completeness of geodesics obtained as a limit, J. Math. Phys. 34(8) (1993), 3768-3774.
[9] A. Romero and M. Sanchez, New properties and examples of incomplete Lorentzian tori, J. Math. Phys. 35(4) (1994), 1992-1997.

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