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ON THE NICHOLS ALGEBRA ASSOCIATED TO $(q_{ij}) = \begin{pmatrix} \omega & \omega^2 & 2\omega - 1 \\ 1 & -1 & -1 \\ \frac{1}{2\omega - 1} & 1 & -1 \end{pmatrix}$, OF RANK 3

TADAYOSHI TAKEBAYASHI

Department of Mathematics
School of Science and Engineering
Waseda University
Ohkubo Shinjuku-ku, Tokyo, 169-8555, Japan
e-mail: takeba@aoni.waseda.jp

Abstract

We examine the defining relations of the Nichols algebra associated to $(q_{ij}) = \begin{pmatrix} \omega & \omega^2 & 2\omega - 1 \\ 1 & -1 & -1 \\ \frac{1}{2\omega - 1} & 1 & -1 \end{pmatrix}$, of rank 3 by using the results by Angiono [2] and the method by Nichols [1].

1. Introduction

Nichols algebras are graded braided Hopf algebras with the base field in degree 0 and which are coradically graded and generated by its primitive elements ([3], [4], [5], [6], [7]). Let V be a vector space and $c : V \otimes V \to V \otimes V$ be a linear isomorphism. Then (V, c) is called a *braided vector space*, if c is a solution of the braid equation, that is $(c \otimes id)(id \otimes c)(c \otimes id) = (id \otimes c)(c \otimes id)(id \otimes c)$. The

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pair (V, c) determines the Nichols algebras up to isomorphism. Let G be a group. A Yetter-Drinfeld module V over $\mathbb{K}G$ is a G-graded vector space $V = \oplus_{g \in G} V_g$, which is a G-module such that $g \cdot V_h \subset V_{ghg^{-1}}$, for all $g, h \in G$. The category ${}_G^G YD$ of $\mathbb{K}G$-Yetter-Drinfeld module is braided. For $V, W \in {}_G^G YD$, the braiding $c : V \otimes W \to W \otimes V$ is defined by $c(v \otimes w) = (g \cdot w) \otimes v$, $v \in V_g$, $w \in W$. Let V be a Yetter-Drinfeld module over G and let $T(V) = \oplus_{n \geq 0} T(V)(n)$ denote the tensor algebra of the vector space V. Let S be the set of all ideals and coideals I of $T(V)$ which are generated as ideals by $\mathbb{N}$-homogeneous elements of degree ≥ 2, and which are Yetter-Drinfeld submodules of $T(V)$. Let $I(V) = \sum_{I \in S} I$. Then $B(V) := T(V)/I(V)$ is called the *Nichols algebra* of $V \in {}_G^G YD$. In this article, we examine the defining relations of the Nichols algebra $B(V)$ associated to $(q_{ij}) =$

$$\begin{pmatrix} \omega & \omega^2 & 2\omega - 1 \\ 1 & -1 & -1 \\ \frac{1}{2\omega - 1} & 1 & -1 \end{pmatrix},$$ of rank 3.

2. Nichols Algebras of Cartan Type

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. Let G be an abelian group and V be a finite dimensional Yetter-Drinfeld module. Then the braiding is given by a nonzero scalar $q_{ij} \in \mathbb{K}$, $1 \leq i, j \leq \theta$, in the form $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, where $x_1, ..., x_\theta$, is a basis of V. If there is a basis such that $g \cdot x_i = \chi_i(g) x_i$ and $x_i \in V_{g_i}$, then V is called *diagonal type*. For the braiding, we have $c(x_i \otimes x_j) = \chi_j(g_i) x_j \otimes x_i$ for $1 \leq i, j \leq \theta$. Hence we have $(q_{ij})_{1 \leq i, j \leq \theta} = (\chi_j(g_i))_{1 \leq i, j \leq \theta}$. Let $B(V)$ be the Nichols algebra of V. We can construct the Nichols algebra by $B(V) \cong T(V)/I$, where I denotes the sum of all ideals of $T(V)$ that are generated by homogeneous elements of degree ≥ 2 and that are coideals. If $B(V)$ is finite-dimensional, then the matrix (a_{ij}) defined by for all $1 \leq i \neq j \leq \theta$ by $a_{ii} := 2$ and $a_{ij} := -\min\{r \in \mathbb{N} \mid q_{ij} q_{ji} q_{ii}^r = 1 \text{ or } (r+1)_{q_{ii}} = 0\}$ is a generalized

Cartan matrix fulfilling $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$ or *ord* $q_{ii} = 1 - a_{ij}$. (a_{ij}) is called the *Cartan matrix* associated to $B(V)$. To examine the defining relations of $B(V)$, we use the results [3] and [2].

Proposition 2.1 ([3]). (1) *For all* $1 \le i \le \theta$, *there exists a uniquely determined* (id, σ)*-derivation* $D_i : B(V) \to B(V)$ *with* $D_i(x_j) = \delta_{ij}$ *(Kronecker* δ*) for all j.*

(2) $\bigcap_{i=1}^{\theta} ker(D_i) = \mathbb{K}1.$

Proposition 2.2 ([2]). *Let* (V, c) *be a braided vector space that* $\dim V = 3$, *and the corresponding generalized Dynkin diagram is* $\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-1}{\circ}\overset{r^{-1}}{\text{———}}\overset{r}{\circ}$. *Then* $B(V)$ *is presented by generators* x_1, x_2, x_3, *and relations*

(2.2.1) $x_1^M = x_2^2 = x_3^N = x_{\alpha_1+2\alpha_2+\alpha_3}^P = 0,$

(2.1.2) $(ad_c x_1)^2 x_2 = (ad_c x_3)^2 x_2 = (ad_c x_1)x_3 = 0,$

(2.1.3) $[x_{\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2+\alpha_3}]_c = [x_{\alpha_1+\alpha_2+\alpha_3}, x_{\alpha_2+\alpha_3}]_c = 0.$

If $M, N, P < \infty$, *then* $\dim B(V) = 16MNP$.

Using these, obtain the following.

Proposition 2.3. *Let* $(q_{ij}) = \begin{pmatrix} \omega & \omega^2 & 2\omega - 1 \\ 1 & -1 & -1 \\ \frac{1}{2\omega - 1} & 1 & -1 \end{pmatrix}$, $\Big(\overset{\omega}{\circ}\overset{\omega^2}{\text{———}}\overset{-1}{\circ}\overset{-1}{\text{———}}\overset{-1}{\circ}\Big)$ (*where* ω *is a primitive cube root of unity*). *Then the Nichols algebra* $B(V)$ *is described as follows*:

Generators : x_1, x_2, x_3.

Relations : $x_1^3 = 0,\ x_2^2 = 0,\ x_3^2 = 0,\ x_1^2 x_2 + \omega x_1 x_2 x_1 + \omega^2 x_2 x_1^2 = 0,$

$$(x_1x_2)^2 = \omega^2(x_2x_1)^2,\ x_1x_3 = (2\omega - 1)x_3x_1,\ (x_2x_3)^2 = -(x_3x_2)^2,$$

$$x_2x_1x_2x_3 + (\omega^2 - 1)x_2x_1x_3x_2 + (2 - \omega^2)x_2x_3x_2x_1 - x_1x_2x_3x_2$$

$$+ (\omega^2 - 2)x_3x_2x_1x_2 = 0.$$

Its basis is given as follows:

$$\{1, x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2x_1, x_2x_3, x_3x_2, x_2x_1^2, x_1x_2x_1, x_1x_2x_3,$$
$$x_1x_3x_2, x_2x_1x_2, x_2x_1x_3, x_2x_3x_2, x_3x_2x_3, x_3x_2x_1, x_1x_2x_1^2, (x_1x_2)^2,$$
$$x_1x_2x_1x_3, x_1x_2x_3x_2, x_1x_3x_2x_3, x_1x_3x_2x_1, x_2x_1x_2x_3, x_3x_2x_1^2, x_2x_3x_1^2$$
$$x_2x_3x_2x_1, (x_2x_3)^2, x_3x_2x_3x_1, x_2x_3x_1x_2, x_2x_1x_2x_1^2, x_3x_1x_2x_1^2, x_2x_3x_2x_1^2,$$
$$x_3x_2x_3x_1^2, x_1x_2x_3x_1^2, (x_1x_2)^2x_3, x_1x_2x_3x_2x_1, x_1(x_2x_3)^2, x_1x_3x_2x_3x_1,$$
$$x_3(x_1x_2)^2, x_2x_1x_2x_3x_2, (x_2x_3)^2x_1, x_2x_3x_1x_2x_1, x_2x_3x_1x_2x_3,$$
$$x_3x_2x_3x_1x_2, x_1x_2x_3x_1x_2, x_1x_2x_3x_2x_1^2, (x_2x_3)^2x_1^2, x_2x_1x_2x_3x_1^2,$$
$$x_3x_1x_2x_3x_1^2, (x_1x_2)^2x_3x_2, x_1(x_2x_3)^2x_1, x_2x_1(x_2x_3)^2, x_3(x_1x_2)^2x_3,$$
$$x_2x_1x_2x_3x_2x_1, x_2x_3(x_1x_2)^2, (x_2x_3x_1)^2, (x_2x_3)^2x_1x_2, (x_3x_2x_1)^2,$$
$$x_2x_1x_3x_2x_1^2, x_1x_2x_3x_1x_2x_1, (x_1x_2x_3)^2, x_1(x_2x_3)^2x_1^2, x_2x_1x_2x_3x_2x_1^2,$$
$$(x_1x_2)^2x_3x_2x_1, (x_1x_2)^2x_3x_2x_3, x_2x_1(x_2x_3)^2x_1, x_2x_3(x_1x_2)^2x_3, (x_2x_3x_1)^2x_2,$$
$$(x_2x_3)^2x_1x_2x_1, x_3x_2x_3x_1x_2x_1^2, (x_3x_2x_1)^2x_2, x_1x_2x_3(x_1x_2)^2, (x_1x_2x_3)^2x_1,$$
$$x_1(x_2x_3)^2x_1x_2, x_2x_1(x_2x_3)^2x_1^2, (x_1x_2)^2x_3x_2x_3x_1, x_3(x_1x_2)^2x_3x_2x_1,$$
$$(x_2x_3x_1)^2x_2x_1, (x_1x_2x_3)^2x_1x_2, (x_1x_2)^2x_3x_2x_1^2, x_2x_1(x_2x_3)^2x_1x_2,$$
$$(x_2x_3x_1)^2x_2x_3, x_1(x_2x_3)^2x_1x_2x_1, x_3x_2x_1(x_2x_3)^2x_1^2, x_1x_2x_1(x_2x_3)^2x_1^2,$$
$$(x_1x_2x_3)^2x_1x_2x_1, x_3(x_1x_2)^2x_3x_2x_1^2, x_2x_1(x_2x_3)^2x_1x_2x_1, (x_2x_3)^2x_1x_2x_1^2,$$
$$(x_1x_2x_3)^2(x_1x_2)^2, x_2x_3(x_1x_2)^2x_3x_2x_1^2, x_2x_1(x_2x_3)^2x_1x_2x_1^2, (x_1x_2x_3)^2(x_1x_2)^2x_1\}.$$

Hence the Hilbert polynomial of $B(V)$ *is given as follows*:

$$P(t) = 1 + 3t + 6t^2 + 9t^3 + 13t^4 + 16t^5 + 16t^6 + 13t^7 + 9t^8 + 6t^9 + 3t^{10} + t^{11}.$$

Proof. They are directly calculated. □

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