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ON q-SAALSCHÜTZ'S SUMMATION THEOREM

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Abstract

The aim of this research note is to obtain two results closely related to the q-Saalschütz's summation theorem. When $q \to 1$, we get two results closely related to the Saalschütz theorem for the series ${}_3F_2$ obtained earlier by Arora and Rathie.

1. Introduction and Results Required

The ${}_r\phi_s$ basic hypergeometric (or q-) series [3] is defined by

$$
{}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, x\right)
$$

$$
= \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n (q; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} x^n, \qquad (1.1)
$$

where $q \neq 0, \binom{n}{2} = \frac{n(n-1)}{2}$ and $r > s + 1$. For $|q| < 1$, let us define

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$$(a)_n = (a; q)_n = \begin{cases} 1, & n = 0, \\ \prod_{m=0}^{n-1} (1 - aq^m), & n = 1, 2, 3, \ldots, \end{cases}$$

$$(a)_\infty = (a; q)_\infty = \prod_{m=0}^{\infty} (1 - aq^m).$$

q-Saalschütz theorem [3]:

$$ {}_3\phi_2\left(\begin{matrix} a, b, q^{-n} \\ c, abc^{-1}q^{1-n} \end{matrix} ; q, q \right) = \frac{\left(\frac{c}{a}; q\right)_n \left(\frac{c}{b}; q\right)_n}{(c; q)_n \left(\frac{c}{ab}; q\right)_n}. \tag{1.2}$$

When $q \to 1$, we get the following Saalschütz theorem [2]:

$$ {}_3F_2\left(\begin{matrix} a, b, -n \\ c, 1 + a + b - c - n \end{matrix} \Big| 1 \right) = \frac{(c)_n (c - a - b)_n}{(c - a)_n (c - b)_n}. \tag{1.3}$$

The aim of this research note is to derive two results closely related to (1.2).

2. Main Results

The results to be proved are

$$ {}_3\phi_2\left(\begin{matrix} a, b, q^{-n} \\ c, abc^{-1}q^{2-n} \end{matrix} ; q, q \right)$$

$$= \frac{1}{(b - a)(c; q)_n \left(\frac{c}{abq}; q\right)_n}$$

$$\times \left[b(1 - a)\left(\frac{c}{aq}; q\right)_n \left(\frac{c}{b}; q\right)_n - a(1 - b)\left(\frac{c}{a}; q\right)_n \left(\frac{c}{bq}; q\right)_n \right], \tag{2.1}$$

$$ {}_3\phi_2\left(\begin{matrix} a, b, q^{-n} \\ c, abc^{-1}q^{3-n} \end{matrix} ; q, q \right)$$

$$= \frac{1}{(b-a)(c;q)_n\left(\frac{c}{abq^2};q\right)_n}$$

$$\times\left[\left\{\frac{b^2(1-a)(1-aq)}{b-aq}\right\}\left(\frac{c}{aq^2};q\right)_n\left(\frac{c}{b};q\right)_n\right.$$

$$+\left\{\frac{a^2(1-b)(1-bq)}{(bq-a)}\right\}\left(\frac{c}{a};q\right)_n\left(\frac{c}{bq^2};q\right)_n$$

$$\left.-\left\{\frac{abq(1-a)(1-b)\ (b-a)(1+q)}{(b-aq)(bq-a)}\right\}\left(\frac{c}{aq};q\right)_n\left(\frac{c}{bq};q\right)_n\right]. \tag{2.2}$$

3. Proofs

In order to derive our main results, we shall use the following result [4, Eq. (2.2)], which is also hold for the given ${}_3\phi_2$:

$$(b-a)\phi = b(1-a)\phi(aq) - a(1-b)\phi(bq), \tag{3.1}$$

where

$$\phi \equiv {}_3\phi_2\left(\begin{matrix} a,\ b,\ q^{-n} \\ c,\ abc^{-1}q^{2-n}\end{matrix};q,q\right).$$

It is easy to see that the two ϕ on the right-hand-side of (3.1) can be evaluated by (1.2) by simply changing a by aq in the first ϕ and b by bq in the second ϕ, and after simplification we get our first result (2.1).

In exactly the same manner, the result (2.2) can also be obtained with the help of the relation (3.1) by taking

$$\phi \equiv {}_3\phi_2\left(\begin{matrix} a,\ b,\ q^{-n} \\ c,\ abc^{-1}q^{3-n}\end{matrix};q,q\right)$$

and using the result (2.1).

4. Special Cases

In (2.1) and (2.2), if we take $q \to 1$, then we get the following results due to Arora and Rathie [1]:

$$
{}_3F_2\left(\begin{matrix} a,\ b,\ -n \\ c,\ 2+a+b-c-n \end{matrix} \,\Big|\, 1\right)
$$

$$
= \frac{1}{(a-b)(c)_n(c-a-b-1)_n}\left[a(c-a-1)_n(c-b)_n - b(c-a)_n(c-b-1)_n\right] \quad (4.1)
$$

and

$$
{}_3F_2\left(\begin{matrix} a,\ \ b,\ \ -n \\ c,\ 3+a+b-n \end{matrix} \,\Big|\, 1\right)
$$

$$
= \sum_{a\leftrightarrow b} \frac{a(a+1)(c-a-2)_n(c-b)_n}{(a-b)(a+1-b)(c)_n(c-a-2)_n} - \frac{2ab(c-a-1)_n(c-b-1)_n}{(c)_n(c-a-b-2)_n}, \quad (4.2)
$$

where $\sum_{a\leftrightarrow b} f(a, b) = f(a, b) + f(b, a)$.

Clearly, these results are closely related to the Saalschütz theorem (1.3).

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