Far East Journal of Applied Mathematics
Volume 48, Number 1, 2010, Pages 17-23
Published Online: January 28, 2011
This paper is available online at http://pphmj.com/journals/fjam.htm

A STANDARD LYAPUNOV EQUATION FOR DESCRIPTOR SYSTEMS

ASADOLLAH AASARAAI and KAMELEH NASIRI

Department of Mathematics
Faculty of Science
University of Guilan
P. O. Box 41335-1914
Rasht, Iran
e-mail: k-nasiri@guilan.ac.ir

Abstract

It is known that the asymptotical stability of the linear time invariant descriptor system $E\dot{x}(t) = Ax(t)$ is related to the solution of the generalized Lyapunov equation $E^* XA + A^* XE = -Q.$ If E is singular, the generalized Lyapunov equation may have no solution even if all the finite eigenvalues of $\lambda E - A$ have a negative real part, and a solution, if it does exist, is not unique. This paper attempts to introduce a matrix G through which to introduce a standard Lyapunov equation, and then proves a relation between the asymptotical stability of the linear time invariant descriptor system with E singular and the solution of the standard Lyapunov equation. Meanwhile, the matrix G would be described as a contour integral and in terms of the coefficient of Laurent series of $(\lambda E - A)^{-1}$ at infinity.

2010 Mathematics Subject Classification: 93D05, 15A22, 34A09.

Keywords and phrases: descriptor system, matrix pencil, Weierstrass canonical normal form, Lyapunov equations.

*Corresponding author

Received June 15, 2010; Revised August 7, 2010

1. Introduction

We consider the linear time invariant descriptor system

$$E\dot{x}(t) = Ax(t), \tag{1}$$

where E and A are $n \times n$ matrices and $rank(E) = r < n.$ It is assumed that system (1) is regular, i.e., there exists a $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) \neq 0.$ Matrix $\lambda E - A$ is called a *matrix pencil*. We also show a matrix pencil by the matrix pair $(E, A).$ Eigenvalues of the matrix pencil (E, A) defined to be the roots of the characteristic polynomial $\det(\lambda E - A) = 0.$ $SP(E, A)$ is the set of eigenvalues of the pencil $(E, A).$ The value $\lambda_0 \in \mathbb{C}$ is called a *finite eigenvalue* of $\lambda E - A,$ if $\det(\lambda_0 E - A) = 0.$ If matrix E is singular, then $\lambda E - A$ is said to have an eigenvalue at infinity. For a regular pencil $(E, A),$ there exist two nonsingular matrices $T, S \in \mathbb{C}^{n \times n}$ such that

$$TES = \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix}, \qquad TAS = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix},$$

where matrices J and N are in the Jordan canonical form and matrix N is a nilpotent matrix; that is, there exists an integer v such that $N^v = 0$ but $N^{v-1} \neq 0.$

The representation (TES, TAS) of the pencil (E, A) is called the *Weierstrass canonical normal form*. See [6].

2. Asymptotical Stability of Linear Time Invariant State Space Systems

This section is intended to describe the asymptotical stability of the system

$$\dot{x}(t) = Ax(t) \tag{2}$$

and the relation of this problem with a standard Lyapunov equation.

System (2) is called *asymptotically stable* if $\lim_{t \to \infty} x(t) = 0.$

The following theorems describe the relation between the asymptotical stability of system (2) with the eigenvalues of matrix A and the solution of a standard Lyapunov equation. See [1].

Theorem 2.1. *System* (2) *is asymptotically stable if and only if all the eigenvalues of matrix A have a negative real part.*

Theorem 2.2. *System* (2) *is asymptotically stable if and only if for any Hermitian, positive definite matrix Q, there exists a unique Hermitian, positive definite matrix X, satisfying the Lyapunov equation* $XA + A^{*}X = -Q$.

3. Asymptotical Stability of Linear Time Invariant Descriptor Systems

The results described in the previous section can be extended to descriptor systems. The following theorems show these generalizations.

Theorem 3.1. *Let* $\lambda E - A$ *be a regular pencil. System* (1) *is asymptotically stable if and only if all the finite eigenvalues of* $\lambda E - A$ *have a negative real part.* See ([2], [6] and [8]).

Theorem 3.2. *Let* $\lambda E - A$ *be a regular pencil. If all eigenvalues of* $\lambda E - A$ *are finite and lie in the left half-plane, then for every Hermitian, positive* (*semi*) *definite matrix Q the equation*

$$E^{*}XA + A^{*}XE = -Q \tag{3}$$

has a unique Hermitian, positive (*semi*) *definite solution X. Conversely, if there exist Hermitian, positive definite matrices X and Q satisfying* (3), *then all eigenvalues of the pencil* $\lambda E - A$ *are finite and lie in the left half-plane.* See [4]. □

In fact, equation (3) has a unique solution for every Q if matrix E is nonsingular and all the eigenvalues of pencil $\lambda E - A$ have a negative real part. But the disadvantage of equation (3) is that when E is singular it may have no solution even if all the finite eigenvalues of $\lambda E - A$ lie in the open half-plane, and a solution, if it does exist, is not unique. It is easy to see that if X is a solution of (3) and $u \in \ker E^{*}$, $X + uu^{*}$ is also a solution of (3).

The following example shows that equation (3) may have no solution when E is singular.

Example 3.1. Let

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to show that equation (3) with above E, A, Q, has no solution, although $\lambda = -1$ is the eigenvalue of $\lambda E - A$ with a negative real part.

For the asymptotical stability behavior of system (1) with E singular, some methods have been proposed, but all of them are restricted to special cases. For example, the results of [5] are suitable for systems of index $v = 1$ or the modified Lyapunov matrix equation methods are restricted to systems with an index of at most 3, see [7]. The methods described in [9], [10] require some ill-conditioned calculations for transformation matrices.

In the next section, we introduce a matrix G and prove a relation between G and the coefficients of Laurent series of $(\lambda E - A)^{-1}$ at infinity, then by using G a standard Lyapunov equation, which can be applied for system (1) with E singular, will be introduced.

4. A Standard Lyapunov Equation for Descriptor Systems

In this section, we use the Weierstrass canonical normal form and make a matrix G such that the nonzero eigenvalues of GA are equal to the finite nonzero eigenvalues of pencil $\lambda E - A$. Then we prove a relation between G and the coefficients of Laurent series of $(\lambda E - A)^{-1}$ at infinity.

Finally, we introduce a standard Lyapunov equation using this matrix and describe the relation between this Lyapunov equation with the asymptotical stability of system (1).

Theorem 4.1. *There exists a matrix G such that*

$$SP(GA) = SP(J) \cup \{0\}.$$

Proof. If T, S and I_{n_f} are the matrices in Weierstrass canonical normal form, then we define matrix G as

$$G = S diag(I_{n_f}, 0)T$$

and consider matrix GA. In fact, we want to show that the finite eigenvalues of $\lambda E - A$ belong to $SP(GA)$. This can be proved by the use of the definition of G as follows:

$$\begin{aligned} GA &= S diag(I_{n_f}, 0)TA = S diag(I_{n_f}, 0)TASS^{-1} \\ &= S diag(I_{n_f}, 0) diag(J, I_{n_\infty})S^{-1} \\ &= S diag(J, 0)S^{-1}. \end{aligned}$$

Therefore,

$$GA = Sdiag(J,\ 0)S^{-1}.$$

Since S is nonsingular

$$SP(GA) = SP(J) \cup \{0\}.$$

But according to the definition of Weierstrass canonical normal form, the finite eigenvalues of $\lambda E - A$ are the eigenvalues of the matrix J, so the finite and nonzero eigenvalues of $\lambda E - A$ and nonzero eigenvalues of GA are the same. □

In the next step, the relation of G with coefficients of Laurent series of $(\lambda E - A)^{-1}$ at infinity is shown. It is well known that Laurent series of $(\lambda E - A)^{-1}$ at infinity is in the following form:

$$(\lambda E - A)^{-1} = \sum_{n=-\infty}^{+\infty} h_n \lambda^{-n-1},$$

where

$$h_n = Sdiag(J^n,\ 0)T, \qquad n = 0, 1, 2, \ldots$$

and

$$h_n = Sdiag(0,\ -N^{-n-1})T, \qquad n = -1, -2, \ldots.$$

See [3].

Theorem 4.2. *Let c be a closed simple curve such that the finite eigenvalues of $\lambda E - A$ lie inside c. Then*

$$h_n = \frac{1}{2\pi i}\oint_c \lambda^n (\lambda E - A)^{-1} d\lambda, \qquad n \geq 0.$$

Proof. We have

$$(\lambda E - A)^{-1} = [\lambda T^{-1} diag(I_{n_f},\ N)S^{-1} - T^{-1} diag(J,\ I_{n_\infty})S^{-1}]^{-1}$$

$$= Sdiag(\lambda I_{n_f} - J,\ \lambda N - I_{n_\infty})^{-1} T$$

$$= Sdiag[(\lambda I_{n_f} - J)^{-1},\ (\lambda N - I_{n_\infty})^{-1}]T,$$

so

$$\frac{1}{2\pi i}\oint_c \lambda^n(\lambda E - A)^{-1}d\lambda$$

$$= \frac{1}{2\pi i} diag\left[\oint_c \lambda^n(\lambda I_{n_f} - J)^{-1}d\lambda, \oint_c \lambda^n(\lambda N - I_{n_\infty})^{-1}d\lambda\right]T,$$

but c includes all finite eigenvalues of $\lambda E - A$, so

$$\frac{1}{2\pi i}\oint_c \lambda^n(\lambda I_{n_f} - J)^{-1}d\lambda = J^n$$

and since N is nilpotent

$$\frac{1}{2\pi i}\oint_c \lambda^n(\lambda N - I_{n_\infty})^{-1}d\lambda = 0$$

so

$$\frac{1}{2\pi i}\oint_c \lambda^n(\lambda E - A)^{-1}d\lambda = Sdiag(J^n, 0)T = h_n, \qquad n \geq 0. \qquad \square$$

Corollary 4.1. *G is the coefficient of λ^{-1} in the Laurent series of $(\lambda E - A)^{-1}$ at infinity*.

Proof. The coefficient of λ^{-1} in the Laurent series of $(\lambda E - A)^{-1}$ at infinity is equal to

$$h_0 = \frac{1}{2\pi i}\oint_c (\lambda E - A)^{-1}d\lambda = Sdiag(I_{n_f}, 0)T = G.$$

Theorem 4.3. *System* (1) *is asymptotically stable if and only if for any Hermitian*, *positive definite matrix Q*, *there exists a unique Hermitian*, *positive definite matrix X satisfying the Lyapunov equation $XGA + A^*G^*X = -Q$.*

Proof. According to Theorems 2.1 and 2.2 for any Hermitian, positive definite matrix Q, there exists a unique Hermitian, positive definite matrix X satisfying the Lyapunov equation $XGA + A^*G^*X = -Q$ if and only if all the eigenvalues of GA have a negative real part.

According to Theorem 4.1 all the eigenvalues of GA have a negative real part if and only if all the eigenvalues of J have a negative real part. But the eigenvalues of J

are the finite eigenvalues of pencil $\lambda E - A$. So the finite eigenvalues of $\lambda E - A$ have a negative real part, or system (1) is asymptotically stable if and only if for any Hermitian, positive definite matrix Q, there exists a unique Hermitian, positive definite matrix X satisfying the Lyapunov equation $XGA + A^* G^* X = -Q$. □

Conclusion

In this paper, we explored the asymptotical stability of descriptor systems which is usually described by means of a generalized Lyapunov equation, which is not suitable when E is singular. To overcome this problem, we introduced a matrix G and a standard Lyapunov equation applicable for descriptor systems even if E is singular. Also the matrix G has been described as a contour integral and in terms of the coefficients of Laurent series of $(\lambda E - A)^{-1}$ at infinity.

References

[1] B. N. Datta, Stability and inertia, Linear Algebra Appl. 302/303 (1999), 563-600.

[2] D. H. Owens and D. L. Debeljkovic, Consistency and Lyapunov stability of linear descriptor systems: a geometric analysis, IMA, J. Numer. Anal. 2 (1985), 139-151.

[3] F. L. Lewis, Fundamental, reachability and observability matrices for descriptor systems, IEEE Trans. Automat. Contr. AC-30 (1985), 502-505.

[4] F. R. Gantmacher, Theory of Matrices, Chelsea, New York, 1959.

[5] J. Y. Ishihara and M. H. Terra, On the Lyapunov theorem for singular systems, IEEE Trans. Automat. Contr. 47(11) (2002), 1926-1930.

[6] L. Dai, Singular Control Systems, Vol. 118, Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 1989.

[7] P. C. Muller, Eigenvalue bounds for matrix pencils, Annual Meeting of the GAMM, Dresden, 2004.

[8] R. Byers, C. He and V. Mehrmann, Where is the nearest non-regular pencil, Linear Algebra Appl. 285 (1998), 81-105.

[9] T. Stykel, On criteria for asymptotic stability of differential-algebraic equations, Z. Angew. Math. Mech. 82(3) (2002), 147-158.

[10] Q. Wang, Q. L. Zhang, G. S. Zhang and W. Q. Liu, Lyapunov equation with positive definite solution for descriptor systems, Proc. 4th Asian Control Conf. WA9-16, Singapore, 2002, pp. 385-389.