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# THE CATEGORY NFuz OF THE *n*-DIMENSIONAL FUZZY SETS

**YOU-GUANG SHANG and XUE-HAI YUAN***

School of Mathematical Sciences
Dalian University of Technology
Dalian 116024, P. R. China

Faculty of Electronic Information and Electrical Engineering
Dalian University of Technology
Dalian 116024, P. R. China
e-mail: yuanxuehai@yahoo.com.cn

**Abstract**

In this paper, the connections between the *n*-dimensional fuzzy sets and topos theory are discussed. The category **NFuz** of the *n*-dimensional fuzzy sets is constructed. It is shown that the category **NFuz** has all topos properties except subobject classifiers. Furthermore, it is proved that the category **NFuz** has the middle object, and consequently, it forms a weak topos.

## 1. Introduction

Although the theory of fuzzy sets and the theory of topos are developed separately, they should be connected since both of them are dealing with vagueness through logic. Recently, this has been verified by various investigators [3-4, 6, 8,

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*Corresponding author

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10-12]. It is well known that the category **Set** of the classical sets is a topos. By the use of topos **Set**, the logic operators of classical sets such as negation, conjunction, implication and disjunction, are obtained both logically and naturally [5]. In [10-12], the category **Fuz** of fuzzy sets is redefined and the conceptions of middle object and weak topos are introduced. By the use of weak topos, the logic operators of fuzzy sets as defined by Zadeh are described naturally, and consequently, the category **Fuz** has a similar function to topos **Set**.

It is well known that some special $L$-fuzzy sets such as the interval-valued fuzzy sets [13], the intuitionistic fuzzy sets [1], the interval-valued intuitionistic fuzzy sets [2], and the type 2 fuzzy sets [7] et al., play an important role in fuzzy systems. In [9], the $n$-dimensional fuzzy set, a special $L$-fuzzy set and a generalization of some $L$-fuzzy sets such as the interval-valued fuzzy sets, the intuitionistic fuzzy sets and the interval-valued intuitionistic fuzzy sets et al., is proposed and the cut sets, the decomposition theorems and the representation theorems of $n$-dimensional fuzzy sets are presented. In order to study the $n$-dimensional fuzzy sets from the view of category, the category **NFuz** of the $n$-dimensional fuzzy sets is constructed and topoi properties of the category **NFuz** are discussed in this paper.

The rest of this paper is organized as follows: In Section 2, some preliminaries are provided. In Section 3, the category **NFuz** of the $n$-dimensional fuzzy sets is constructed and it is shown that the category **NFuz** has all topoi properties except subobject classifier. In Section 4, it is proved that the category **NFuz** has middle object and consequently, the category **NFuz** forms a weak topos.

## 2. Preliminary

**Definition 2.1** [9]**.** Let $I_n = \{(a_1, a_2, ..., a_n) | 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq 1\}$ and $X$ be a set. Then the mapping $A : X \to I_n$ is called an *n-dimensional fuzzy set*.

**Definition 2.2** [5]**.** A topos is a category $\mathscr{C}$ satisfying the five conditions as follows:

(1) Finite products exist in $\mathscr{C}$, i.e., for any objects $A$, $B$ in $\mathscr{C}$, there exists an object $C$ and morphisms $p_1 : C \to A$, $p_2 : C \to B$ satisfying for any morphisms $f : D \to A$ and $g : D \to B$, there exists an unique morphism $h : D \to C$ such that $p_1 \circ h = f$ and $p_2 \circ h = g$, i.e., Figure 2.1 is commutative. $C$ is denoted as $A \times B$.

$$\begin{tikzcd} & A \\ D \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & C \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & B \end{tikzcd}$$

**Figure 2.1**

$$\begin{tikzcd} E' \arrow[d, "\overline{e}"'] \arrow[dr, "e'"] & & \\ E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

**Figure 2.2**

$$\begin{tikzcd} B^A \times A \arrow[dr, "ev"] & \\ & B \\ D \times A \arrow[uu, "\overline{F} \times Id_A"] \arrow[ur, "F"'] & \end{tikzcd}$$

**Figure 2.3**

(2) Equalizer exists in $\mathscr{C}$, i.e., for any morphisms $f, g : A \to B$, there exists an object $E$ and a morphism $e : E \to A$ satisfying

(a) $f \circ e = g \circ e$.

(b) For any $e' : E' \to A$ satisfying $f \circ e' = g \circ e'$, there exists an unique morphism $\overline{e} : E' \to E$ such that $e \circ \overline{e} = e'$, i.e., Figure 2.2 is commutative.

(3) There is a terminal object $U$ in $\mathscr{C}$. This means that for each object $A$, there is one and only one morphism from $A$ to $U$, which is denoted as !.

(4) Exponentials exist in $\mathscr{C}$, i.e., for any objects $A$, $B$ in $\mathscr{C}$, there exists an object $B^A$ in $\mathscr{C}$ and a morphism $ev : B^A \times A \to B$ satisfying, for any morphism $F : D \times A \to B$, there exists an unique morphism $\overline{F} : D \to B^A$ such that $ev \circ (\overline{F} \times Id_A) = F$, i.e., Figure 2.3 is commutative.

(5) There is a subobject classifier in $\mathscr{C}$, i.e., there is an object $\Omega$ and a morphism $\top$ from $U$ to $\Omega$ such that for each monomorphism $f$ from $A'$ to $A$, there exists an unique morphism $\chi_f$ from $A$ to $\Omega$ such that Figure 2.4(a) is a pullback. This means

(a) Figure 2.4(a) is commutative, i.e., $\chi_f \circ f = \top \circ !$.

(b) For each object $B$ and a morphism $g$ from $B$ to $A$ such that $\chi_f \circ g = \top \circ !$, then there exists an unique morphism $\overline{g}$ from $B$ to $A'$ such that $g = f \circ \overline{g}$ (see Figure 2.4(b)), where $\chi_f$ is called *CH*(*character*) of $(A', f)$.

For example, the category **Set** of classical sets is a topos, $U = \{0\}$ is a terminal object; $\Omega = 2 = \{0, 1\}$, $\top : U \to \Omega$, $\top(0) = 1$ is a subobject classifier. Then for any monomorphism $f : A' \to A$,

$$\chi_f(a) = \begin{cases} 1, & a \in f(A'), \\ 0, & a \notin f(A'). \end{cases}$$

The $\chi_f$ can be seen as the CH of $A'$.

$$\begin{tikzcd} A' \arrow[r, "!"] \arrow[d, "f"'] & U \arrow[d, "\top"] \\ A \arrow[r, "\chi_f"] & \Omega \end{tikzcd} \qquad \begin{tikzcd} B \arrow[dr, "\overline{g}"] \arrow[drr, "!"] \arrow[ddr, "g"'] & & \\ & A' \arrow[r, "!"] \arrow[d, "f"'] & U \arrow[d, "\top"] \\ & A \arrow[r, "\chi_f"] & \Omega \end{tikzcd}$$

(a) (b)

**Figure 2.4**

**Definition 2.3** [10]**.** A weak topos is a category $\mathscr{C}$ satisfying the five conditions as follows:

(1) Finite products exist in $\mathscr{C}$. (2) Equalizers exist in $\mathscr{C}$. (3) There is a terminal object in $\mathscr{C}$. (4) Exponentials exist in $\mathscr{C}$. (5) There is a middle object in $\mathscr{C}$, i.e., there exists a monomorphism $m : \Lambda \to \Delta$ such that

(a) $\mathrm{Hom}(A, \Delta) = \{f \mid f : A \to \Delta \text{ is a morphism}\}$ is partially ordered for all object $A$ in $\mathscr{C}$.

(b) There is a unique smallest morphism $\alpha$ so that the square (Figure 2.5(a)) is a pullback.

(c) For any monomorphism $f : A' \to A$, there is a unique morphism $\chi_f : A \to \Delta$ such that $\chi_f \leq \alpha$ and the square (Figure 2.5(b)) is a pullback. $\chi_f$ is called a *CH* of *f*.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{Id}_A"'] & \Lambda \arrow[d, "m"] \\ A \arrow[r, "\alpha"] & \Delta \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r] \arrow[d, "f"'] & \Lambda \arrow[d, "m"] \\ A \arrow[r, "\chi_f"] & \Delta \end{tikzcd}$$

(a) (b)

**Figure 2.5**

For example, the category **Fuz** of fuzzy sets is a weak topos, where middle object is

$$m : (0, 1] \to [0, 1], \quad x \mapsto x$$

and for any monomorphism $f : (A', \alpha') \to (A, \alpha)$,

$$\chi_f(a) = \begin{cases} \alpha'(a'), & \text{if } a = f(a') \in f(A'), \\ 0, & \text{else.} \end{cases}$$

The $\chi_f$ can be seen as the membership function of fuzzy set $(A', \alpha')$.

## 3. The Category NFuz of the *n*-dimensional Fuzzy Sets

Suppose $\alpha : X \to I_n$ is an $n$-dimensional fuzzy set, $\alpha(x) = (A_1(x), A_2(x), \ldots, A_n(x))$ and $A_n(x) > 0$. Let $(X, \alpha)$ be an object; a mapping $f : X \to Y$ be a morphism from $(X, \alpha)$ to $(Y, \beta)$, which satisfies $\beta(f(x)) \geq \alpha(f(x))$ for all $x \in X$, i.e., $B_1(f(x)) \geq A_1(x), B_2(f(x)) \geq A_2(x), \ldots, B_n(f(x)) \geq A_n(x)$, where $\beta(y) = (B_1(y), B_2(y), \ldots, B_n(y))$. Then the composition of the mappings is the composition of the morphisms and the identical mapping is the identical morphism. In this way, we construct a category **NFuz**. Now we consider the properties of the category. We have

**Theorem 3.1.** *Category **NFuz** has all topoi properties except subobject classifier.*

**Proof.** (1) There exists a Finite Product for any two objects in the category **NFuz**.

Suppose $(X, \alpha)$ and $(Y, \beta)$ are two objects, where

$$\alpha(x) = (A_1(x), A_2(x), \ldots, A_n(x)), \quad \beta(x) = (B_1(x), B_2(x), \ldots, B_n(x)).$$

Let

$Z = X \times Y,$

$\gamma(x, y) = (\min\{A_1(x), B_1(y)\}, \min\{A_2(x), B_2(y)\}, \ldots, \min\{A_n(x), B_n(y)\}),$

$p_1 : Z \to X, \quad x \mapsto x; \quad p_2 : Z \to Y, \quad y \mapsto y.$

Obviously, $\{(Z, \gamma), p_1, p_2\}$ is a Finite Product of $(X, \alpha)$ and $(Y, \beta)$.

(2) There exists an Equalizer for any two morphisms in the category **NFuz**.

Suppose $(X, \alpha) \overset{f}{\underset{g}{\rightrightarrows}} (Y, \beta)$ are two morphisms. Let $Z = \{x \mid x \in X,\ f(x) = g(x)\}$, $\gamma(x) = \alpha(x) = (A_1(x), A_2(x), \dots, A_n(x)),\ e : Z \to X,\ \ x \mapsto x.$ Obviously, $\{(Z, \gamma), e\}$ is an Equalizer of $f$ and $g$.

(3) There exists a terminal object in category **NFuz**.

Suppose $I = (\{0\}, \delta)$, where $\delta(0) = (1, 1, \dots, 1)$. Then $I$ is a terminal object in category **NFuz**.

(4) There exists an Exponential for any two objects in category **NFuz**.

Suppose $(X, \alpha)$ and $(Y, \beta)$ are two objects. Let

$$Z^* = Y^X = \{f \mid f : X \to Y \text{ is a mapping}\},$$

$$\gamma^*(f) = \bigvee \{(\lambda_1, \lambda_2, \dots, \lambda_n) \mid (\lambda_1, \lambda_2, \dots, \lambda_n) \in I_n \text{ satisfying } (*)\}$$

$$\triangleq (C_1(f), C_2(f), \dots, C_n(f)),$$

$$Z = \{f \mid f \in Z^*,\ C_n(f) > 0\}; \quad \gamma(f) = \gamma^*(f), \quad \forall f \in Z,$$

$$\min\{\lambda_i,\ A_i(x)\} \le B_i(f(x)),\ \forall x \in X \ \ (i = 1, 2, \dots, n). \tag{$*$}$$

Then $Z \neq \varnothing$. In fact, let $f : X \to Y,\ \ x \mapsto y_0,$ where $y_0 \in Y$ satisfying $B_n(y_0) > 0$. Then $C_n(f) \ge B_n(y_0) > 0$. So $Z \neq \varnothing$ and consequently, $(z, \gamma)$ is an object in **NFuz**.

Let $ev : Z \times X \to Y,\ \ (f, x) \to f(x)$. According to the definition of $\gamma^*(f)$ above, we have that $\gamma(f) \wedge \alpha(x) \le \beta(f(x))$. So $ev$ is a morphism.

Suppose $F : (D, \delta) \times (X, \alpha) \to (Y, \beta)$ is a morphism. Then $F : D \times X \to Y$ is a mapping satisfying

$$\delta(d) \wedge \alpha(x) \le \beta(F(d, x)),$$

i.e.,

$$\min\{D_i(d),\ A_i(x)\} \le B_i(F(d, x))(i = 1, 2, \dots, n), \quad \forall (x, d) \in D \times X,$$

where $\delta(d) = (D_1(d), D_2(d), \dots, D_n(d))$.

Let $\overline{F}: D \to Z,\ d \mapsto \overline{F}(d),$ where $\overline{F}(d)(x) = F(d)(x);$

$$\sum\nolimits^{f} = \{(\lambda_1, \lambda_2\ ..., \lambda_n) | (\lambda_1, \lambda_2\ ..., \lambda_n) \in I_n \text{ and satisfies } (*)\}.$$

Then $(D_1(d), D_2(d), ..., D_n(d)) \in \sum^{f}$. So $C_n(\overline{F}(d)) \geq D_n(d) > 0.$

Consequently, $\overline{F}(d) \in Z,$ and

$$\gamma(\overline{F}(d)) = (C_1(\overline{F}(d)), C_2(\overline{F}(d)), ..., C_n(\overline{F}(d)) \geq (D_1(d), D_2(d), ..., D_n(d)) = \delta(d).$$

Thus, $\overline{F}$ is a morphism. Obviously, $ev \circ (\overline{F} \times Id_X) = F$ and $\overline{F}$ is unique.

Therefore, $\{(Z, \gamma), ev\}$ is an Exponential of $(X, \alpha)$ and $(Y, \beta)$.

(5) There does not exist any subobject classifier in **NFuz**.

Assume that there exists a monomorphism $\top : U \to \Omega$ satisfying $\Omega$-axiom. Let $X = \{0\},\ \alpha(0) = \left(0, 0, ..., 0\frac{1}{2}\right);\ Y = \{0\},\ \beta(0) = (0, 0, ..., 0, 1).$ Then there exists a morphism $\chi_f$ such that Figure 3.1(a) is a pullback square.

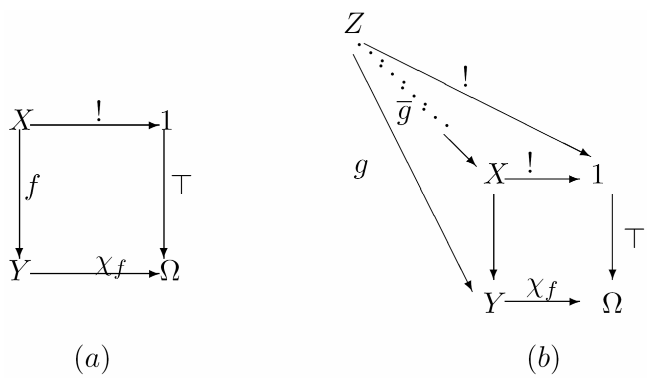


**Figure 3.1**

Let $f(0) = 0,\ \ g(0) = 0.$ Then $\chi_f \circ g = \top \circ !.$ Then there exists a unique morphism $\overline{g} : Y \to X$ such that Figure 3.1(b) is commutative. So $\alpha(\overline{g}(0)) \geq \beta(0),$ consequently, $1/2 \geq 1,$ which contradicts itself. Therefore, no subobject classifier exists in **NFuz**. □

## 4. The Category NFuz Forms a Weak Topos

**Theorem 4.1.** *The category **NFuz** has a middle object.*

**Proof.** Let $M = I_n - \{(0, 0, ..., 0)\}$, $\Omega = I_n$ and $m : M \to I_n$, $(a_1, a_2, ..., a_n) \mapsto (a_1, a_2, ..., a_n)$; $\omega : \Omega \to I_n$, $(a_1, a_2, ..., a_n) \mapsto (1, 1, ..., 1)$. Then $(M, m)$ and $(\Omega, \omega)$ are objects in **NFuz**. Let $\top : M \to \Omega$, $(a_1, a_2, ..., a_n) \mapsto (a_1, a_2, ..., a_n)$. Then $\top$ is a morphism.

(a) Suppose $(X, \alpha)$ is an object in **NFuz**. Clearly,

$$\hom((X, \alpha), (\Omega, \omega)) = \{f \mid f : X \to \Omega \text{ is a mapping}\} = I_n^X$$

is a poset.

(b) Suppose $(X, \alpha)$ is an object in **NFuz**. We consider Figure 4.1(a):

$$\begin{tikzcd} X \arrow[r, "\alpha'"] \arrow[d, "Id_X"'] & M \arrow[d, "\top"] \\ X \arrow[r, "\alpha_X"] & \Omega \end{tikzcd} \qquad\qquad \begin{tikzcd} X \arrow[r, "\beta'"] \arrow[d, "Id_X"'] & M \arrow[d, "\top"] \\ X \arrow[r, "\beta"] & \Omega \end{tikzcd}$$

$(a)$ $\qquad\qquad$ $(b)$

**Figure 4.1**

where $\alpha'(x) = \alpha(x)$, $\alpha_X(x) = \alpha(x)$. Obviously, Figure 4.1(a) is a pullback square.

Let $\beta'$ be a morphism such that Figure 4.1(b) is a pullback square. Since $\beta \circ Id_X = \top \circ \beta'$, so we have that $\beta(x) = \top(\beta'(x)) = \beta'(x) = m(\beta'(x)) \geq \alpha(x)$. That is, $\alpha_X$ is the minimum among the morphisms such that Figure 4.1(a) form pullback squares, i.e., $\alpha_X$ is the minimum in $\hom((X, \alpha), (\Omega, \omega))$.

(c) Suppose $f : (X, \alpha) \to (Y, \beta)$ is a monomorphism. Then $f$ is a monomapping satisfying $\beta(f(x)) \geq \alpha(x)$. Let $\chi_f : Y \to \Omega$, $y \mapsto \chi_f(y)$, where

$$\chi_f(y) = \begin{cases} \alpha(x), & \text{if } y = f(x) \in f(X), \\ (0, 0, 0), & \text{else.} \end{cases}$$

Then $\chi_f$ is a morphism satisfying $\chi_f(y) \le \beta(y)$. Let $g(x) = \alpha(x)$. Then Figure 4.2(a) is a pullback square and $\chi_f \circ f = \top \circ g$.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & M \arrow[d, "\top"] \\ Y \arrow[r, "\chi_f"] & \Omega \end{tikzcd}$$

$(a)$

$$\begin{tikzcd} Z \arrow[dr, dotted, "h"] \arrow[drr, "t"] \arrow[ddr, "s"'] & & \\ & X \arrow[r, "g"] \arrow[d, "f"] & M \arrow[d, "\top"] \\ & Y \arrow[r, "\chi_f"] & \Omega \end{tikzcd}$$

$(b)$

**Figure 4.2**

Suppose $(Z, \gamma)$ is an object in **NFuz**. Let $s$ and $t$ be two morphisms such that $\chi_f \circ s = \top \circ t$. Then $\chi_f(s(z)) = t(z) \neq (0, 0, ..., 0)$. There exists an $x \in X$ such that $f(x) = s(z)$. Let

$$h : Z \to X, \quad z \mapsto x, \quad \text{if} \quad s(z) = f(x).$$

Because $f$ is a monomorphism, $x$ such that $h(x) = f(x)$ is unique, and consequently, $h$ is a mapping. Moreover,

$$\gamma(z) \le m(t(z)) = t(z) = \top(t(z)) = \chi_f(s(z)) = \chi_f(f(x)) = \alpha(x) = \alpha(h(x)),$$

so $h$ is a morphism. Because

$$t(z) = \alpha(x) = \alpha(h(z)) = g(h(z)), \quad f(h(z)) = f(x) = s(z),$$

Figure 4.2(b) is commutative. Obviously, $h$ is unique. Therefore, Figure 4.2(a) is a pullback square.

(d) Now we prove the uniqueness of $\chi_f$.

Suppose $\chi'_f : Y \to \Omega$ satisfying $\chi'_f(y) \le \beta(y)$ and substituting $\chi'_f$ for $\chi_f$ such that Figure 4.2(a) is still a pullback square. Because $\chi'_f(f(x)) = \top(g(x))$ $= g(x) = \alpha(x)$, $\chi'_f(x) = \chi_f(y)$, $\forall y \in f(X)$. Assume that there exists a $y \in Y \setminus f(X)$

such that $\chi'_f(y) \neq (0, 0, ..., 0)$. If $\chi'_f(y) = (a_1, a_2, ..., a_n)$, then $a_n > 0$. Let $Z = \{y \mid \gamma(y) = (a_1, a_2, ..., a_n)\}$. Consider Figure 4.2(b):

$$\chi'_f(s(y)) = \chi'_f(y) = (a_1, a_2, ..., a_n) = t(y),$$

where $s(y) = y$, $t(y) = \gamma(y)$. So there exists a unique $h : Z \to X$ such that Figure 4.2(b) is commutative. Consequently, $y = s(y) = f(h(x)) \in f(x)$. This contradicts that $y \notin f(X)$. Thus $\chi'_f(y) = (0, 0, ..., 0)$, $\forall y \in Y \setminus f(X)$, i.e., $\chi'_f = \chi_f$. Therefore, $\top : (M, m) \to (\Omega, \omega)$ is a middle object in **NFuz**. □

**Note.** From Theorem 3.1, we conclude that the category **NFuz** does not form a topos. From Theorem 3.1 and Theorem 4.1, we conclude that the category **NFuz** forms a weak topos.

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