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FACTORISATION OF GRAPHS

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Abstract

A graph G is decomposable into the subgraphs $G_1, G_2, G_3, ..., G_n$ of G if no G_i $(i = 1, 2, 3, ..., n)$ has isolated vertices and the edge set $E(G)$ can be partitioned into the subsets $E(G_1), E(G_2), ..., E(G_n)$. If $G_i \cong H$ for every i, we say that G is H-decomposable and we write $H \mid G$. A graph F without isolated vertices is a least common multiple of the graphs G_1 and G_2, if F is a graph of minimum size such that F is both G_1-decomposable and G_2-decomposable. The size (the number of edges) of a least common multiple of two graphs G_1 and G_2 is denoted by $\operatorname{lcm}(G_1, G_2)$. Chartrand et al. [Periodica Math. Hungar. 27(2) (1993), 95-104] found $\operatorname{lcm}(C_{2k}, K_{1,l})$ and $\operatorname{lcm}(C_3, K_{1,l})$. They also introduced a conjecture about $\operatorname{lcm}(C_n, K_{1,l})$, when n is an odd integer ≥ 5. Wang [Utilitas Math. 53 (1998), 231-242] proved the conjecture is true when $n = 5$. We proved that the conjecture is not true for some cases. For some cases we obtained a formula in [Far East J. Appl. Math. 6(2) (2002), 191-200]. In this paper, we show that the conjecture is true for the case when $(n, l) = 1$ and $\left\lceil \frac{2l+n}{n^2} \right\rceil$ is odd. When $1 < d < n$ and $\frac{n}{d} \cdot \frac{d+1}{2} \geq \frac{2l}{d} + 1$, where $d = \gcd(n, l)$, we establish a new formula.

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Introduction

In this paper we only consider simple graphs. A graph G is said to be H-decomposable, denoted by $H \mid G$, if $E(G)$ can be partitioned into subgraphs such that each subgraph is isomorphic to H. Such a factorization is called *isomorphic factorization*. The concept of isomorphic factorization was studied by Harary et al. [3]. Wilson [6] proved that for every graph F without isolated vertices, there can be found a complete graph K_n such that K_n is F-decomposable. Using this theorem Chartrand et al. [1] proved that any two graphs have a least common multiple. The size of a least common multiple of G_1 and G_2 is denoted by $\operatorname{lcm}(G_1, G_2)$. We follow standard notation in graph theory, the cardinality of the vertex set of a graph G, the order of G is denoted by $p(G)$ and the cardinality of the edge set of G, the size of G is denoted by $q(G)$.

Theorem 1 [1]**.** $\operatorname{lcm}(C_3, K_{1,l}) = \frac{3kl}{d}$, *where*

$$d = \gcd(3, l); \quad k = \left\lceil \frac{d(2l+3)}{9} \right\rceil.$$

For general odd integer n, they made the following conjecture [1]:

$$\operatorname{lcm}(C_n, K_{1,l}) = \frac{nkl}{d}, \text{ where } d = \gcd(n, l); \quad k = \left\lceil \frac{d(2l+n)}{n^2} \right\rceil.$$

The following theorem shows that the above conjecture does not hold good in general.

Theorem 2 [4]**.** *Let l be a multiple of n. Then*

$$\operatorname{lcm}(C_n, K_{1,l}) = \frac{n+1}{2} \cdot l; \text{ when } k < \frac{n+1}{2}$$

$$= kl \text{ when } k \geq \frac{n+1}{2}; \text{ where } k = 2 \cdot \frac{l}{n} + 1.$$

Theorem 3. *Let l and n be relatively prime positive integers with*

$n \geq 5$. *Suppose that* n *and* $k = \left\lceil \frac{2l + n}{n^2} \right\rceil$, *are odd. Then* $\operatorname{lcm}(C_n, K_{1,l}) = nkl$.

Proof.

$$\operatorname{lcm}(l, n) = nl$$

$$\operatorname{lcm}(C_n, K_{1,l}) = s \cdot nl,$$

where s is a positive integer.

Let F be a graph of size snl such that $C_n \mid F$ and $K_{1,l} \mid F$. Then F should be decomposable into sn stars $K_{1,l}$ and into sl cycles C_n. Every edge of a cycle should be incident with the centre of one of the stars. Therefore $sl \leq \binom{sn}{2}$,

$$sn - 1 \geq \frac{2l}{n}$$

$$s \geq \frac{2l + n}{n^2}$$

$$s \geq k.$$

Therefore, $\operatorname{lcm}(C_n, K_{1,l}) \geq nkl$.

We show that there exists a graph G of size nkl such that $C_n \mid G$ and $K_{1,l} \mid G$.

Firstly we consider the case when $k = 1$. Though the proof for the general case is similar, the proof in this case helps us to make the idea behind the general proof clear.

Since $\left\lceil \frac{2l + n}{n^2} \right\rceil = k = 1$ and $l \equiv 0 \pmod{n}$, $\frac{2l + n}{n^2} < 1$. Hence $l < \frac{n \cdot (n - 1)}{2}$.

Construction of a Cycle C_n

We want to construct a graph of size nkl which is $K_{1,l}$-decomposable and C_n-decomposable. We give below two methods of forming cycles of length n. We use either of these or both depending on n and l, in our construction.

Let H be a set of $nk = n$ vertices $v_1, v_2, ..., v_n$. We can form $\frac{n-1}{2}$ edge-disjoint spanning cycles of the complete graph on these n vertices (which are also referred to as spanning cycles of H for simplicity).

Method 1. Form one of the spanning cycles of H say $C: v_1, v_2, v_3, ..., v_n, v_1$. Select an edge say $v_{i-1}v_i$. We add $\frac{n-1}{2}$ new vertices $x_1, x_2, ..., x_{\frac{n-1}{2}}$ and form a new path $v_i, x_1, v_{i+1}, x_2, v_{i+2}, x_{\frac{n-3}{2}}, v_{i+\frac{n-3}{2}}, x_{\frac{n-1}{2}}, v_{i-1}$. This path and the edge $v_{i-1}v_i$ form a cycle C_n. Similarly form cycles C_n for each edge of C.

Let the new graph be K. K can be decomposed into n cycles of length n.

For every v_i in H, $d_K(v_i) = 2 \cdot \frac{n-1}{2} + 2 = n + 1$. Of these $(n+1)$ edges, $(n-1)$ edges join v_i to vertices that are not in H. Two edges are incident with the vertices of H. They are edges of the spanning cycle C. In K, $d(v_i) = n + 1$, even. All vertices of K, not in H are of degree 2. Clearly K is connected. Hence K is Eulerian. Let L be an Eulerian circuit of K. Assuming an orientation for L, for every v_i in H, there is one edge of L that enters v_i on L, and one edge of L that exits v_i on L. Define H_i to be the subgraph induced by the edges in H that exit v_i on L.

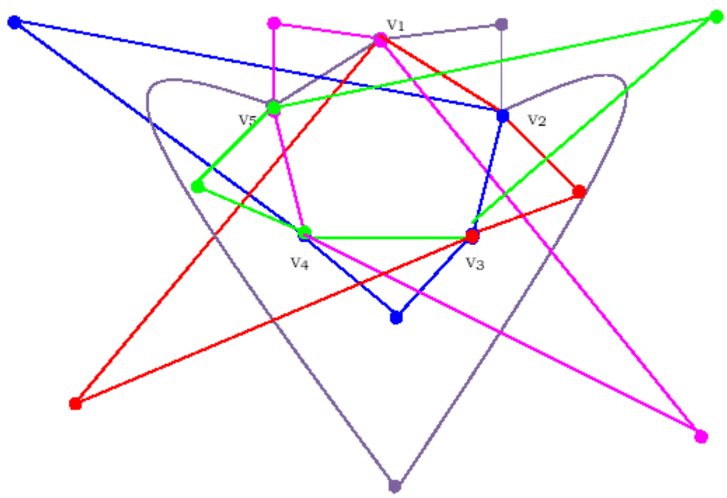
Number of such edges $= 1 + n - 1 = n$.

Hence $H_i \cong K_{1,n}$; $i = 1, 2, 3, ..., n$.

Hence K can be decomposed into n stars $K_{1,n}$ whose central vertices are the vertices $v_1, v_2, \ldots, v_n$ of H.

Conclusion. Using Method 1, for every spanning cycle of H, we can construct a graph which can be decomposed into n stars $K_{1,n}$ and n cycles of length n.

Illustration. $n = 5$.



Method 2. Consider a Hamilton path formed by the vertices of H, say $v_1, v_2, v_3, \ldots, v_n$. Form a path P_3 say $P^1 : v_1v_2v_3$. Add $\frac{n-1}{2}$ new vertices $x_1, x_2, \ldots, x_{\frac{n-1}{2}}$ and form the path $v_2, x_1, v_4, x_2, v_6, x_3, v_8, \ldots, x_{\frac{n-3}{2}}, v_{2+n-3}, x_{\frac{n-1}{2}}, v_1$. This path and the edge v_1v_2 form a cycle C_n. Again add $\frac{n-1}{2}$ new vertices $y_1, y_2, \ldots, y_{\frac{n-1}{2}}$ and form the path $v_3, y_1, v_5, y_2, v_7, \ldots, y_{\frac{n-3}{2}}, v_{3+n-3}, y_{\frac{n-1}{2}}, v_2$. This path and the edge v_2v_3 form another cycle C_n.

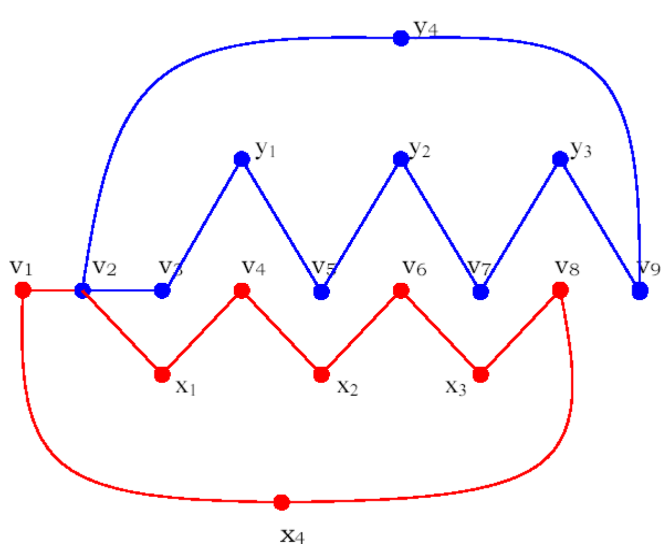
Let this graph of two cycles be denoted by F. Clearly F can be decomposed into two cycles C_n,

$$d(v_2) = 4 \text{ and } d(v_i) = 2; \quad i = 1, 3, 4, \ldots, n.$$

The two edges incident with v_i; $i = 1, 3, 4, \ldots, n$ forms a star $K_{1,2}$. Of the four edges incident with v_2, one edge is included in the star centered at v_1 and the other in the star centered at v_3. The remaining two edges incident with v_2 forms a star $K_{1,2}$. Thus F can be decomposed into n stars $K_{1,2}$ centered at $v_1, v_2, \ldots, v_n$.

Conclusion. Using Method 2, given any P_3 contained in H, we can construct a graph which can be decomposed into 2 cycles of length n and n stars $K_{1,2}$ centered at the vertices of H. Further, the only edges from H in this graph are the edges of P_3 that we started with.

Illustration. $n = 9$.



Similarly form another path P_3 say P^2 which is edge-disjoint from the path P^1. Form two cycles C_n as above, corresponding to the two edges of

P^2. These two new cycles can also be decomposed into n stars $K_{1,2}$.

In general, we form x edge-disjoint paths P_3 from a spanning path of n vertices of H, where $x \leq \frac{n-1}{2}$.

Let this graph be S. Clearly S can be decomposed into $2x$ cycles C_n. S can also be decomposed into n stars $K_{1,2x}$ centered at $v_1, v_2, \ldots, v_n$.

To Find $\text{lcm}(C_n, K_{1,l})$

By division algorithm, let $l = xn + y$; x, y are non-negative integers; $0 < y < n$.

Since $l < \frac{n \cdot (n-1)}{2}$, $x < \frac{n-1}{2}$.

Case 1. When y is even.

Form x edge-disjoint spanning cycles say $C_1, C_2, \ldots, C_x$ of H. This is possible as $x < \frac{n-1}{2}$. Then by using Method 1, corresponding to each of the spanning cycle C_i, we can form a graph G_i, which can be decomposed into n cycles of length n and also into n stars $K_{1,n}$ each centered at a vertex of H. We note that these graphs are edge-disjoint. Let K be the union of these graphs G_i, $1 \leq i \leq n$. We note that K is the edge-disjoint union of nx cycles of length n and n stars $K_{1,nx}$ each centered at a vertex of H.

Since $x < \frac{n-1}{2}$, there is a spanning cycle C_{x+1} of H, edge-disjoint from the cycles $C_1, C_2, \ldots, C_x$. This cycle contains a path of length y since $y < n$. This path can be considered to be a union of $\frac{y}{2}$ paths of length 2. Corresponding to each of these $\frac{y}{2}$ paths, we can construct a graph which can be decomposed into n stars $K_{1,2}$ centered at the vertices of H and two cycles of length n. Let these graphs be $H_1, H_2, \ldots, H_{\frac{y}{2}}$.

Let F be the union of these graphs. Then F is the edge-disjoint union of $\frac{yn}{2}$ stars $K_{1,2}$ centered at vertices of H ($\frac{y}{2}$ copies of $K_{1,2}$ at each vertex of H) and y cycles of length n.

Then we let G to be the graph which is the union of F and K. Then, by our construction G is the edge-disjoint union $(nx + y) = l$ cycles of length n. Further, at each vertex of H, there are $\frac{y}{2}$ copies of $K_{1,2}$ and x copies of $K_{1,n}$, edge-disjoint. These together form a $K_{1,l}$ at each vertex of H. Thus G is decomposable into n copies of $K_{1,l}$.

Therefore G is a graph of size knl which is C_n-decomposable and also $K_{1,l}$-decomposable. (An illustration is provided at the end of the proof.)

Case 2. When y is odd.

Let $l = xn + y$ as in Case 1, with $0 < y < n$.

Let $l = (x-1)n + (n+y)$.

As $y < n$, $n + y < 2n$. Also $n + y$ is even, since n and y are odd.

As in Case 1, using the $(x-1)$ edge-disjoint spanning cycles of H, we can form a graph K which is the edge-disjoint union of $(x-1)n$ cycles of length n and n stars $K_{1,(x-1)n}$ each centered at a vertex of H.

Since, $x < \frac{n-1}{2}$, $x - 1 \leq \frac{n-1}{2} - 2$.

Hence there are at least 2 edge-disjoint spanning cycles of H, also edge-disjoint from K.

Since $y + n < 2n$, these two cycles will contain $\frac{y+n}{2}$ edge-disjoint copies of P_3. Then by using Method 2, as in Case 1, we can construct a graph F, edge-disjoint from K, which can be decomposed into $(n + y)$ cycles of length n and $\frac{y+n}{2} \cdot n$ copies of $K_{1,2}$ ($\frac{y+n}{2}$ copies of $K_{1,2}$ at each vertex of H).

As before, let G be the union of K and F. Then G is the edge-disjoint union of $(x-1)n + (n + y) = xn + y = l$ copies of C_n and n stars $K_{1,l}$, each centered at a vertex of H. This graph G of size knl is a least common multiple of C_n and $K_{1,l}$.

Here we have assumed $l > n$, since we take $x - 1 > 0$.

If $l < n$, then we form the graph of size $(l-1)n$ which is decomposable into C_n and $K_{1,l-1}$ by Case 1 and add an edge-disjoint spanning cycle. Thus G can be decomposed into $(l-1+1) = l$ cycles of length n and $K_{1,l-1} \cup K_{1,1} = K_{1,l}$ at each vertex of H.

Illustration 1. $\operatorname{lcm}(C_7, K_{1,20})$.

$$n = 7, \quad l = 20, \quad k = 1, \quad nk = 7, \quad kl = 20, \quad nkl = 140.$$

Let $l = 20 = 2 \times 7 + 6$, 6 is even.

Let H be a set of $nk = 7$ vertices say 1, 2, 3, 4, 5, 6, 7.

We can form 3 edge-disjoint spanning cycles from the vertices of H:

(1) 1, 2, 6, 3, 5, 4, 7, 1.

(2) 1, 3, 2, 7, 5, 6, 4, 1.

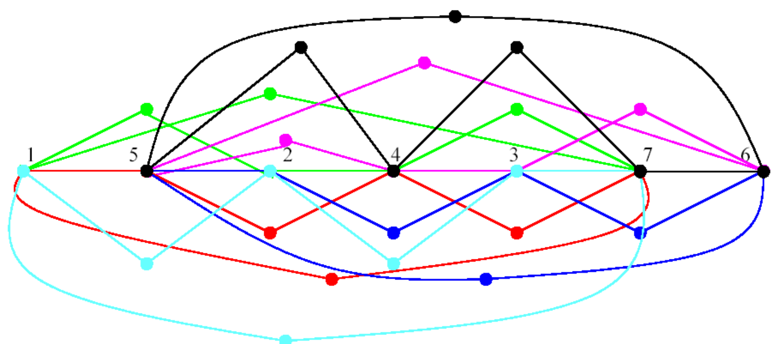
(3) 1, 5, 2, 4, 3, 7, 6, 1.

Form two spanning cycles and form a graph which can be decomposed into $2n = 2 \times 7 = 14$ cycles C_7 as in Method 1.

This can also be decomposed into 7 stars $K_{1,7+7} = K_{1,14}$.

Consider the vertices in order of the next spanning cycle: 1, 5, 2, 4, 3, 7, 6, 1.

Form three edge-disjoint subpaths of length 2 each: 1, 5, 2; 2, 4, 3; 3, 7, 6 and form $2 \times 3 = 6$ cycles C_7 as described in Method 2.



These 6 cycles of length 7 can be decomposed into 7 stars $K_{1,6}$ centered at 1, 2, 3, 4, 5, 6, 7. Thus G can be decomposed into 20 cycles C_7 and G can also be decomposed into 7 stars $K_{1,20}$,

$$q(G) = 140.$$

Thus $\operatorname{lcm}(C_7, K_{1,20}) = 140 = nkl$.

Illustration 2. $\operatorname{lcm}(C_7, K_{1,19})$.

$$n = 7, \quad l = 19, \quad k = 1, \quad nk = 7, \quad kl = 19, \quad nkl = 133$$

$$l = 19 = 2 \times 7 + 5.$$

Therefore, let $l = 1 \times 7 + 12$.

Let H be a set of $nk = 7$ vertices say 1, 2, 3, 4, 5, 6, 7.

We can form 3 edge-disjoint spanning cycles from the vertices of H:

(1) 1, 2, 6, 3, 5, 4, 7, 1.

(2) 1, 3, 2, 7, 5, 6, 4, 1.

(3) 1, 5, 2, 4, 3, 7, 6, 1.

Form a spanning cycle and form a graph which can be decomposed into $n = 7$ cycles C_7 as in Method 1. This can also be decomposed into 7 stars $K_{1,7}$.

Consider the vertices in order of the next two spanning cycles and form three edge-disjoint subpaths of length 2 each corresponding to the vertices of each spanning cycle that can be formed.

As in illustration, corresponding to each set, form 6 cycles C_7. This can be decomposed into 7 stars $K_{1,6+6} = K_{1,12}$.

Thus G can be decomposed into 19 cycles C_7 and 7 stars $K_{1,19}$.

Thus $\text{lcm}(C_7, K_{1,19}) = 133 = 7 \times 19 = nkl$.

Now we shall explain the construction of cycles C_n from nk vertices by the following methods when $k \geq 2$.

Method 1. From n vertices, (n odd) we can form $\frac{n-1}{2}$ edge-disjoint Hamilton cycles. For each Hamilton cycle, we can form a graph which can be decomposed into n cycles C_n, as well as into n stars $K_{1,n}$ as in Method 1, of the case $k = 1$.

Method 2. n and k are odd positive integers.

Consider a null graph H on nk vertices: 1, 2, 3, ..., nk.

Arrange these nk vertices as k sets of n vertices say

(1) $\{1, 2, 3, ..., n\}$;

(2) $\{n+1, n+2, ..., 2n\}$;

(3) $\{2n+1, 2n+2, ..., 3n\}$;

....

(k) $\{(k-1)n+1, ..., nk\}$. (A)

Consider one such set of n vertices say $1, 2, 3, \ldots, n$.

Form a path $i, i+1, i+2,$ where $i \in \{1, 2, 3, \ldots, n-2\}$.

For the edge $i, i+1,$ we add $\frac{n-1}{2}$ new vertices $x_1, x_2, x_3, \ldots, x_{\frac{n-1}{2}}$ and form the path $i+1, x_1, i+3, x_2, i+5, x_3, i+7, \ldots, n-3, x_{\frac{n-3}{2}}, n-1, x_{\frac{n-1}{2}}, i.$ (Addition is performed modulo n.)

This path and the edge $i, i+1$ form a cycle C_n.

For the edge $i+1,$ $i+2,$ we add $\frac{n-1}{2}$ new vertices $y_1, y_2, y_3, \ldots, y_{\frac{n-1}{2}}$ and form the path $i+2, y_1, i+4, y_2, i+6, y_3, i+8, \ldots, n-2, y_{\frac{n-3}{2}}, n, y_{\frac{n-1}{2}}, i+1.$

This path and the edge $i+1, i+2$ forms a cycle C_n,

$$\deg(i+1) = 4.$$

Degree of all other vertices $= 2$. The two edges incident with the vertices: $1, 2, 3, \ldots, i, i+2, \ldots, n$ form a star $K_{1,2}$.

Of the four edges incident with $i+1$, two edges are accounted for the stars centered at i and $i+2$. The remaining two edges incident with $i+1$ form a star $K_{1,2}$.

Thus this graph can be decomposed into 2 cycles C_n and also into n stars $K_{1,2}$.

Similarly construct two cycles C_n for each of the k sets of n vertices in (A). This is a graph with k components. This graph can be decomposed into $2k$ cycles C_n. This graph can also be decomposed into nk stars $K_{1,2}$ whose central vertices are the vertices of H.

(Corresponding to each of the k sets of n vertices of (A), we can form $\frac{n-1}{2}$ edge-disjoint subpaths each of length 2. [For example say 12, 23; 34, 45; ...; $(n-2)(n-1)$, $(n-1)n$; for the first set n vertices of (A).] Corresponding to each such subpaths of length 2, we can form two cycles C_n, as described above. Consequently for each set of n vertices, there can be formed n stars $K_{1,2}$ for each subpath whose central vertices are the vertices of each set of H.)

In general, selecting x edge-disjoint subpaths $\left(x \leq \frac{n-1}{2}\right)$ of length 2, from each of the k sets of n vertices of (A), we can form $2x$ cycles C_n. This graph can be decomposed into nk stars $K_{1,2x}$, each centered at 1, 2, 3, ..., nk; the vertices of H. This graph can also be decomposed into $2xk$ cycles C_n.

This also proves that for any even integer $l \leq n$, $\operatorname{lcm}(C_n, K_{1,l}) \leq knl$.

Construction

Suppose k is an odd integer. Therefore nk is odd.

Let H be a null graph on nk vertices: 1, 2, 3, ..., nk.

Formation of kl cycles C_n:

$$k > \frac{2l+n}{n^2}$$

$$nk > 2 \cdot \frac{l}{n} + 1$$

$$nk - 1 > 2 \cdot \frac{l}{n}$$

$$nk(nk-1) > 2lk$$

$$kl < \frac{nk(nk-1)}{2}.$$

Therefore, $l < \frac{n(nk-1)}{2}$.

Let $l = x \cdot n + y$, where x and y are non-negative integers and $0 < y < n$, by division algorithm.

Case 1. y is even $= 2r$ (say),

$$x < \frac{nk-1}{2}.$$

Corresponding to each spanning cycle of the nk vertices we can form a graph which is decomposable into nk cycles C_n and nk stars $K_{1,n}$ centered at vertices of H as in Case 1 of $k = 1$. Let K be the union of such graphs formed from x edge-disjoint Hamilton cycles. This can be decomposed into xnk cycles C_n and nk stars $K_{1,nx}$ centered at the vertices of H.

Now $r \leq \frac{n-1}{2}$. Choose a Hamilton cycle of H edge-disjoint from x cycles already chosen. Now the cycle can be decomposed into k sets of n adjacent vertices in an obvious way. [Suppose the cycle is $v_1, v_2, \ldots, v_{nk}$, we can take $v_1, v_2, \ldots, v_n; v_{n+1}, \ldots, v_{2n}; v_{2n+1}, \ldots, v_{2n}; \ldots$.] Now, if we choose one copy of P_3 from each set of vertices and proceed as in Method 2, and take the union of graphs, we will get a graph decomposable into $2k$ cycles C_n and a $K_{1,2}$ at each vertex of H. Choosing r copies of P_3 from each set and taking the union of the graphs, we get a graph say F, which is decomposable into $2rk$ cycles of K_n and $K_{1,2r}$ at each vertex of H. So, if we let G to be the union of K and F, then G can be decomposed into $xnk + 2rk = k(xn + y) = kl$ cycles C_n and nk stars $K_{1,nx+2r} = K_{1,l}$.

Case 2. y is odd.

Here as $k > 1$, $l > n$ and hence $x \geq 1$.

Let $l = (x-1)n + (y+n) = tn + y^1$, where y^1 is even and $n < y^1 < 2n$.

Since $x \leq \frac{nk-1}{2} - 1$, $t \leq \frac{nk-1}{2} - 2$.

As in Case 1, let K be the union of such graphs formed from $x-1$ edge-disjoint Hamilton cycles. This can be decomposed into $(x-1)nk$ cycles C_n and nk stars $K_{1,(x-1)n}$ centered at the vertices of H.

Now, choose two Hamilton cycles edge-disjoint from $x-1$ cycles already chosen. Form a graph G which is decomposable into $(y+n)k$ cycles C_n and nk stars $K_{1,\, y+n}$ centered at the vertices of H.

Let G be the union of K and F. Then G can be decomposed into $(x-1)nk + (y+n)k = kl$ cycles C_n and nk stars $K_{1,(x-1)n+y+n} = K_{1,l}$.

Hence G is a least common multiple of C_n and $K_{1,l}$.

Remark. When k is even.

nk is even. Let H be a null graph on nk vertices.

There can be formed $\left(\frac{nk}{2} - 1\right)$ edge-disjoint Hamilton cycles by the nk vertices of H.

Number of edges in these $\left(\frac{nk}{2} - 1\right)$ cycles $= \left(\frac{nk}{2} - 1\right) \cdot nk$.

Therefore when $kl \leq \left(\frac{nk}{2} - 1\right) \cdot nk$, the same construction as in Case 2 holds. Hence, when $(l, n) = 1$, the only case where we are not able to prove that $\operatorname{lcm}(C_n, K_{1,l}) = nkl$ is when k is even and $\left(\frac{nk-2}{2}\right)n < l < \left(\frac{nk-1}{2}\right)n$.

Theorem 4. *Let* n *be an odd integer* ≥ 5 *and* $d = \gcd(n, l)$. *We assume* $1 < d < n$, *and* $\frac{n}{d}\frac{d+1}{2} \geq \frac{2l}{d} + 1$. *Then* $\operatorname{lcm}(C_n, K_{1,l}) = \frac{nl}{d}\frac{d+1}{2}$.

Proof. $\operatorname{lcm}(n, l) = \frac{nl}{d}$.

Therefore, $\operatorname{lcm}(C_n, K_{1,l}) = s \cdot \frac{nl}{d}$, where s is a positive integer. Let F be a graph of size $s \cdot \frac{nl}{d}$ such that $C_n \mid F$ and $K_{1,l} \mid F$. Then F can be decomposed into $\frac{ns}{d}$ stars $K_{1,l}$ and $\frac{ls}{d}$ cycles C_n. Maximum length of the cycle that can be formed by the vertices of F such that each edge of this cycle is incident with a centre of a star is $2 \cdot \frac{ns}{d}$.

Hence,

$$2 \cdot \frac{ns}{d} \geq n$$

$$s \geq \frac{d}{2}.$$

As d is odd, $s \geq \frac{d+1}{2}$.

We show that there exists a graph G of size $\frac{nl}{d}\frac{d+1}{2}$ such that $C_n \mid G$ and $K_{1,l} \mid G$.

For convenience, let $\frac{n}{d} \cdot \frac{d+1}{2} = p$.

Let H be a set of p vertices: $v_1, v_2, v_3, \ldots, v_p$.

Since $\frac{n}{d}\frac{d+1}{2} \geq \frac{2l}{d} + 1$, we can form $\frac{l}{d}$ spanning cycles by the vertices of H.

We form cycles C_n as follows.

Notation. By a spanning cycle of H, we mean a spanning cycle of the complete graph with H as set of vertices.

Form a spanning cycle C_p of H. Partition this cycle C_p into $\frac{d+1}{2}$ edge-disjoint paths each of length $\frac{n}{d}$ say:

$$1, 2, 3, \ldots \frac{n}{d}+1; \quad \frac{n}{d}+1, \ldots, 2\frac{n}{d}+1; \ldots.$$

Consider one such path say P. Excluding this path P, the number of edges remaining in the cycle $C_p = \frac{n}{d} \cdot \frac{d+1}{2} - \frac{n}{d} = \frac{n}{d} \cdot \frac{d-1}{2}$.

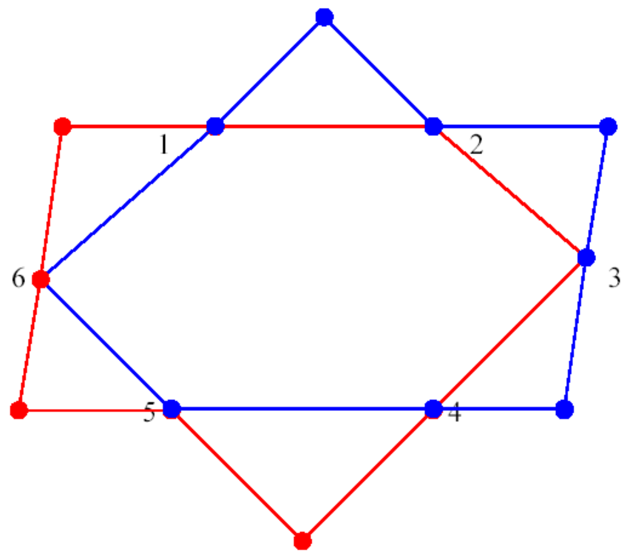
For each of these $\frac{n}{d} \cdot \frac{d-1}{2}$ edges, we add a new vertex and join it with the end vertices of the corresponding edges. These newly formed $2 \cdot \frac{n}{d} \cdot \frac{d-1}{2}$ edges and the path P form a cycle of length $2 \cdot \frac{n}{d} \cdot \frac{d-1}{2} + \frac{n}{d} = n$.

Similarly form cycles C_n for each of the subpaths of C_p.

(We illustrate for the case $\text{lcm}(C_9, K_{1,6})$),

$$n = 9, \quad l = 6, \quad d = 3, \quad \frac{n}{d} = 3, \quad \frac{n}{d} \cdot \frac{d+1}{2} = 6.$$

Let H be a set of 6 vertices 1, 2, 3, 4, 5, 6. Form a spanning cycle C say 1, 2, 3, 4, 5, 6, 1. Partition this cycle into $\frac{d+1}{2} = 2$ edge-disjoint subpaths of length 3 say 1, 2, 3, 4; 4, 5, 6, 1.



The construction outlined above applied to the path yields the graph shown above. Clearly, this graph can be decomposed into 2 cycles of length 9 and also into 6 stars $K_{1,3}$ centered at the vertices of H. A similar construction with respect to another edge-disjoint Hamilton cycle leads to a similar graph. Taking the union of these two graphs, we get a graph of size 36, which can be decomposed into 4 cycles of length 9 and also into 6 stars $K_{1,6}$ centered at the vertices of H.)

Proof of the theorem resumed.

This graph can be decomposed into $\frac{d+1}{2}$ cycles C_n.

For every vertex v_i in H, $d(v_i) = 2 + 2 \cdot \frac{d-1}{2} = d + 1$.

Of these, $(d-1)$ edges join v_i to vertices that are not in H. 2 edges join v_i to vertices of H (edges of the spanning cycle).

This graph can be decomposed into p stars $K_{1,d}$ centered at the vertices of H.

Similarly form $\frac{l}{d}$ spanning cycles by joining the vertices of H. For each of these cycles, construct $\frac{d+1}{2}$ cycles C_n as explained above. This completes the construction of G.

Number of edge-disjoint cycles C_n in $G = \frac{l}{d} \cdot \frac{d+1}{2}$.

Size of $G = \frac{nl}{d} \frac{d+1}{2}$.

G is C_n-decomposable.

For every v_i in H, $d(v_i) = \frac{l}{d} \cdot (d+1)$.

Of these, $\frac{l}{d} \cdot (d-1)$ edges join v_i to vertices that are not in H.

$\frac{l}{d} \cdot 2$ edges (edges of the spanning cycle) join v_i to vertices of H. They form an Eulerian circuit say C [assuming an orientation]. Thus for every v_i in H, there are $\frac{l}{d}$ edges of C that enter v_i on C and $\frac{l}{d}$ edges of C that exit v_i on C.

Let H_i be the subgraph induced by the edges that enter v_i on C and the edges that are not incident with the vertices of H.

Number of such edges $= \frac{l}{d} + \frac{l}{d} \cdot (d-1) = l$.

Hence, $H_i \cong K_{1,l}$.

The subgraphs H_i; $i = 1, 2, 3, \ldots, \frac{n}{d} \cdot \frac{d+1}{2}$ constitute a $K_{1,l}$-decomposition of G.

Thus G is $K_{1,l}$-decomposable.

Thus $\operatorname{lcm}(C_n, K_{1,l}) = \frac{nl}{d} \frac{d+1}{2}$.

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