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ON THE COMPLEXITY OF THE RELATIVE INCLUSION STAR HEIGHT PROBLEM

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Abstract

Given a family of recognizable languages $L_1, ..., L_m$ and recognizable languages $K_1 \subseteq K_2$, the relative inclusion star height problem means to compute the minimal star height of some rational expression r over $L_1, ..., L_m$ satisfying $K_1 \subseteq L(r) \subseteq K_2$.

We show that this problem is of elementary complexity and give a detailed analysis of its complexity depending on the representation of K_1 and K_2 and whether $L_1, ..., L_m$ are singletons. We also consider the case $K_1 = K_2$.

1. Introduction

The star height problem was raised by L. C. Eggan in 1963 [5]: Is there an algorithm which computes the star height of recognizable languages? Like L.C.

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Eggan, we consider star height concerning rational expressions with union, concatenation, and iteration in contrast to extended star height which also allows intersection and complement. For several years, the star height problem was considered as the most difficult problem in the theory of recognizable languages, and it took 25 years until K. Hashiguchi showed the existence of such an algorithm which is one of the most important results in the theory of recognizable languages [11]. His solution to the star height problem relies on distance automata and yields an algorithm of non-elementary complexity, and it remained open to deduce any upper complexity bound from K. Hashiguchi's approach (cf. [17, Annexe B]).

Recently, the author presented another approach to the star height problem which relies on a generalization of distance automata, the distance desert automata. He showed that the star height of the language of a non-deterministic automaton is computable in double exponential space which is the first upper complexity bound to the star height problem [14, 16].

K. Hashiguchi also considered the relative star height problem: Given a finite family of recognizable languages $L_1, ..., L_m$ and some recognizable language K, compute the minimal star height over all rational expressions r over $L_1, ..., L_k$ satisfying $L(r) = K$ [11]. In 1991, he considered inclusion variants of these problems, as the inclusion star height problem: Given two recognizable languages $K_1 \subseteq K_2$, compute the minimal star height over all rational expressions r satisfying $K_1 \subseteq L(r) \subseteq K_2$ [12]. Finally, K. Hashiguchi considered the relative inclusion star height problem which is a joint generalization of the relative and the inclusion star height problem. In [12], K. Hashiguchi showed the decidability of all these variants of the star height problem. The proofs in [12] are complicated. Moreover, [12] is a continuation of the difficult series of papers [9-11]. As for the star height problem, it remained open to deduce upper complexity bounds from [12].

In this paper, we utilize distance desert automata and develop techniques from [14, 16] to give concise decidability proofs and upper complexity bounds to the relative inclusion star height problem and its particular cases. As one main result, we show that the relative inclusion star height problem, i.e., the most general variant, is of elementary complexity: it is decidable in triple exponential space.

We study in detail how the representation of K_1 and K_2 (resp. K) affects the complexity. In particular, we consider the case that K_2 resp. K is given as the complement language of some non-deterministic automaton. We also examine the particular case that the languages $L_1, \ldots, L_m$ are singletons. In this way, we achieve a large variety of results. We even obtain some new conclusions for the complexity of the star height problem: We can decide in $2^{h\mathcal{O}(n)}$ space whether the complement of the language of some n-state non-deterministic automaton is of star height h.

2. Preliminaries

2.1. Notations, Rational Expressions, and Automata

We denote by $\mathcal{P}(M)$ the power set of some set M. We let $\mathbb{N} := \{0, 1, 2, \ldots\}$.

Let Σ be some finite alphabet. We denote the empty word by ε. We denote by $|w|$ the length of some word $w \in \Sigma^*$.

We denote the set of rational expressions over Σ by $\mathsf{REX}(\Sigma)$ and define it as the least set of expressions which includes $\Sigma, \varepsilon, \emptyset$ and is closed such that for r, $s \in \mathsf{REX}(\Sigma)$, the expressions rs, $r \cup s$ and r^* belong to $\mathsf{REX}(\Sigma)$. We denote the language of some rational expression r by $L(r)$.

The star height of rational expressions is defined inductively: we set $\mathsf{sh}(\emptyset) := 0$, $\mathsf{sh}(\varepsilon) := 0$, and $\mathsf{sh}(a) := 0$ for every $a \in \Sigma$. For r, $s \in \mathsf{REX}(\Sigma)$, we set $\mathsf{sh}(rs) = \mathsf{sh}(r \cup s) := \max\{\mathsf{sh}(r), \mathsf{sh}(s)\}$, and $\mathsf{sh}(r^*) := \mathsf{sh}(r) + 1$.

For some language $L \subseteq \Sigma^*$, we define the star height of L by

$$\mathsf{sh}(L) := \min\{\mathsf{sh}(r) \mid L = L(r)\}.$$

We recall some standard terminology in automata theory. We assume that the reader is familiar with Kleene's theorem and basic operations as the complementation and determinization of automata. See, e.g., [3, 6, 19, 20, 22] for a survey.

A (non-deterministic) automaton is a quadruple $\mathcal{A} = [Q, E, I, F]$ where

1. Q is a finite set of states,
2. $E \subseteq Q \times \Sigma \times Q$ is a set of transitions, and
3. $I \subseteq Q$, $F \subseteq Q$ are sets called *initial resp. accepting states*.

Let $k \geq 1$. A path π in $\mathcal{A}$ of length k is a sequence $(q_0, a_1, q_1)(q_1, a_2, q_2) \cdots (q_{k-1}, a_k, q_k)$ of transitions in E. We say that π starts at q_0 and ends at q_k. We call the word $a_1 \cdots a_k$ the label of π. We denote $|\pi| := k$. As usual, we assume for every $q \in Q$ a path which starts and ends at q and is labeled with ε.

We call π successful if $q_0 \in I$ and $q_k \in F$. For every $0 \leq i \leq j \leq k$, we denote $\pi(i, j) := (q_i, a_i, q_{i+1}) \cdots (q_{j-1}, a_{j-1}, q_j)$ and call $\pi(i, j)$ a factor of π. For every $p, q \in Q$ and every $w \in \Sigma^*$, we denote by $q \overset{w}{\rightsquigarrow} p$ the set of all paths with the label w which start at p and end at q.

We denote the language of $\mathcal{A}$ by $L(\mathcal{A})$ and define it as the set of all words in Σ^* which are labels of successful paths. We call some $L \subseteq \Sigma^*$ recognizable, if L is the language of some automaton. We denote by $\mathsf{REC}(\Sigma^*)$ the class of all recognizable languages over Σ^*.

Let $\mathcal{A} = [Q, E, I, F]$ be an automaton. We call $\mathcal{A}$ normalized if there are states $q_I, q_F \in Q$ such that $I = \{q_I\}, \{q_F\} \subseteq F \subseteq \{q_I, q_F\}$, and $E \subseteq (Q \setminus \{q_F\}) \times \Sigma \times (Q \setminus I)$. It is well known that each automaton can be transformed in an equivalent normalized automaton by adding at most two states.

2.2. Distance Desert Automata

Distance desert automata were introduced by the author in [14, 16]. They include K. Hashiguchi's distance automata [8] and S. Bala's and the author's desert automata [1, 2, 13, 15] as particular cases. In the recent years, several authors developed more general automata models, e.g., R-, S- and B-automata. See [23, 24, 25, 26, 27] for recent developments.

Let $h \geq 0$ and $V_h := \{\angle_0, \curlyvee_0, \angle_1, \curlyvee_1, ..., \curlyvee_{h-1}, \angle_h\}$. We define a mapping $\Delta : V_h^* \to \mathbb{N}$. An intuitive approach to understand the mapping Δ is given in [14, 16]. Let $\pi \in V_h^*$. For every $0 \leq g \leq h$, we consider every factor π' of π satisfying $\pi' \in \{\angle_0, \curlyvee_0, ..., \angle_g\}^* = V_g^*$, count the number of occurrences of $\angle_g$, and choose the maximum of these values.

More precisely, for $0 \leq g \leq h$ and $\pi' \in V_h^*$, let $|\pi'|_g$ be the number of occurrences of the letter $\angle_g$ in π'. Let

1. $\Delta_g(\pi) : \max_{\substack{\pi' \in V_h^* \\ \pi' \text{ is a factor of } \pi}} |\pi'|_g$ and

2. $\Delta(\pi) := \max_{0 \le g \le h} \Delta_g(\pi)$.

It is easy to see that $0 \le \Delta(\pi) \le |\pi|$.

An h-nested distance desert automaton (for short distance desert automaton) is a tuple $\mathcal{A} = [Q, E, I, F, \theta]$ where $[Q, E, I, F]$ is an automaton and $\theta : E \to V_h$.

Let $\mathcal{A} = [Q, E, I, F, \theta]$ be an h-nested distance desert automaton. The notions of a path, a successful path, the language of $\mathcal{A}$, . . . are understood with respect to $[Q, E, I, F]$. For every transition $e \in E$, we say that e is marked by $\theta(e)$. We extend θ to a homomorphism $\theta : E^* \to V_h^*$. We define the semantics of $\mathcal{A}$ as follows. For $w \in \Sigma^*$, let

$$\Delta_{\mathcal{A}}(w) := \min_{p \in I, q \in F, \pi \in q \overset{w}{\rightsquigarrow} p} \Delta(\theta(\pi)).$$

We have $\Delta_{\mathcal{A}}(w) = \infty$ iff $w \notin L(\mathcal{A})$. Hence, $\Delta_{\mathcal{A}}$ is a mapping $\Delta_{\mathcal{A}} : \Sigma^* \to \mathbb{A} \cup \{\infty\}$.

If there is a bound $d \in \mathbb{N}$ such that $\Delta_{\mathcal{A}}(w) \le d$ for every $w \in L(\mathcal{A})$, then we say that $w \in L(\mathcal{A})$, is limited by d or for short that $\mathcal{A}$ is limited. Otherwise, we call $\mathcal{A}$ unlimited.

We need the following result.

Theorem 2.1 ([14, 16])**.** *Limitedness of distance desert automata is PSPACE-complete*.

3. Overview

3.1 The Star Height Problem and Some Variants of it

The star height problem was raised by L. C. Eggan in 1963 [5]: Given some recognizable language K, compute the star height of K. Or equivalently, given some recognizable language K and some integer h, decide whether $\mathsf{sh}(K) \le h$. For several years, in particular after R. McNaughton refuted some promising ideas in 1967 [18], the star height problem was considered as the most difficult problem in the theory of recognizable languages, and it took 25 years until K. Hashiguchi showed its decidability [11]. The complexity of Hashiguchi's algorithm is extremely large, and it remained open to deduce an upper complexity bound (cf. [17, Annexe B]). However, the author showed the following result:

Theorem 3.1 ([14, 16]). *Let $h \in \mathbb{N}$ and K be the language accepted by an n-state non-deterministic automaton. It is decidable in $2^{2^{\mathcal{O}(n)}}$ space whether $\mathsf{sh}(K) \leq h$.*

In the present paper, we consider some generalizations of the star height problem.

An instance of the inclusion star height problem is a pair (K_1, K_2) of recognizable languages K_1 and K_2 satisfying $K_1 \subseteq K_2$. The inclusion star height of (K_1, K_2) is defined by

$$\mathsf{sh}(K_1, K_2) := \min\{\mathsf{sh}(r) \mid K_1 \subseteq L(r) \subseteq K_2\}.$$

Clearly, $\mathsf{sh}(K_1, K_2) \leq \min\{\mathsf{sh}(K_1), \mathsf{sh}(K_2)\}$.

For every recognizable language K, we have $\mathsf{sh}(K) = \mathsf{sh}(K, K)$, and hence, Eggan's star height problem is a particular case of the inclusion star height problem.

An instance of the relative star height problem is a triple (K, m, σ) where

1. K is a recognizable language,
2. $m \geq 1$,
3. $\sigma : \Gamma \to \mathsf{REC}(\Sigma^*)$ where $\Gamma = \{b_1, ..., b_m\}$.

We call σ singular, if $|\sigma(b)| = 1$ for every $b \in \Gamma$.

The mapping σ extends to a homomorphism $\sigma : (\mathcal{P}(\Gamma^*), \bigcup, \cdot) \to (\mathcal{P}(\Sigma^*), \bigcup, \cdot)$.

For every $r \in \mathsf{REX}(\Gamma)$, we denote $\sigma(L(r))$ by $\sigma(r)$.

The relative star height of (K, m, σ) is defined by

$$\mathsf{sh}(K, m, \sigma) := \min\{\mathsf{sh}(r) \mid r \in \mathsf{REX}(\Gamma), \sigma(r) = K\},$$

where the minimum of the empty set is defined as ∞.

Assume $m = |\Sigma|$, $\Sigma = \{a_1, ..., a_m\}$, and $\sigma(b_i) = \{a_i\}$ for $i \in \{1, ..., m\}$. Clearly, we have $\mathsf{sh}(K) = \mathsf{sh}(K, m, \sigma)$ for every $K \in \mathsf{REC}(\Sigma^*)$. Hence, Eggan's star height problem is a particular case of the relative star height problem.

The finite power problem (FPP) means to decide whether some given recognizable language L has the finite power property, i.e., whether there exists some integer k such that $L^* = \bigcup_{i=0}^{k} L^i$. It was raised by J. A. Brzozowski in 1966, and it took more than 10 years until I. Simon and K. Hashiguchi independently showed its decidability [21, 7].

Let $L \subseteq \Sigma^*$ be a recognizable language and set $m := 1$ and $\sigma(b_1) := L$. We have $\sigma(b_1^k) = L^k$ for every $k \in \mathbb{N}$ and $\sigma(b_1^*) = L^*$. Hence, $\mathsf{sh}(L^*, m, \sigma) \leq 1$. The following assertions are equivalent:

1. $\mathsf{sh}(L^*, m, \sigma) = 0$.

2. There is a finite language $G \subseteq b_1^*$ such that $\sigma(G) = L^*$.

3. There exists some $g \in \mathbb{N}$ such that $\sigma(\{\varepsilon, b_1, b_1^2, ..., b_1^g\}) = L^*$.

4. The language L has the finite power property.

Hence, $\mathsf{sh}(L^*, m, \sigma) = 0$ iff L has the finite power property. Consequently, the finite power problem is a particular case of the relative star height problem.

An instance of the relative inclusion star height problem is a quadruple (K_1, K_2, m, σ) where

1. K_1, K_2 are recognizable languages satisfying $K_1 \subseteq K_2$,

2. m and σ are defined as for the relative star height problem.

The relative inclusion star height of (K_1, K_2, m, σ) is defined by

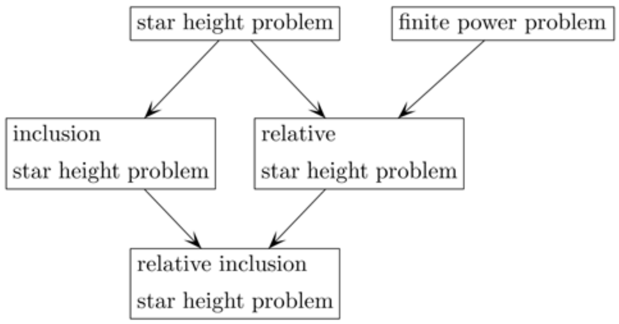
$$\mathsf{sh}(K_1, K_2, m, \sigma) := \min\{\mathsf{sh}(r) \mid r \in \mathsf{REX}(\Gamma), K_1 \subseteq \sigma(r) \subseteq K_2\}.$$

Given some instance (K_1, K_2, m, σ) of the relative inclusion star height problem, we call some $r \in \mathsf{REX}(\Gamma)$ a solution of (K_1, K_2, m, σ) if $\mathsf{sh}(r) = \mathsf{sh}(K_1, K_2, m, \sigma)$ and $K_1 \subseteq \sigma(r) \subseteq K_2$.

For some instance (K, m, σ) of the relative star height problem, the quadruple (K, K, m, σ) is an instance of the relative inclusion star height problem, and we have $\mathsf{sh}(K, m, \sigma) = \mathsf{sh}(K, K, m, \sigma)$. Hence, the relative star height problem is a particular case of the relative inclusion star height problem.

As above, the inclusion star height problem is particular case of the relative inclusion star height problem.

The following figure shows the relations between the five above problems. The arrows go from particular to more general problems.



In 1991, K. Hashiguchi showed that the relative inclusion star height problem is decidable:

Theorem 3.2 ([12]). *Given some instance* (K_1, K_2, m, σ) *of the relative inclusion star height problem,* $\mathsf{sh}(K_1, K_2, m, \sigma)$ *is.effectively computable.*

3.2 Main Results

In the paper, we examine the complexity of the above variants of the star height problem. As one main result, we show that the most general variant, the relative inclusion star height problem, is of elementary complexity.

We consider the complexity of the variants of the star height problem under various aspects. We distinguish the cases that either K_2 or its complement $\Sigma^* \setminus K_2$ or both K_2 and its complement $\Sigma^* \setminus K_2$ are given by non-deterministic automata with at most n_2 states. Note that we have the latter case if K_2 is given by a deterministic automaton with n_2 states.

Moreover, we distinguish the cases that Σ is singular or arbitrary.

3.2.1. The Relative Inclusion Star Height Problem

Let (K_1, K_2, m, σ) be an instance of the relative inclusion star height problem. By n_1 we denote the number of states of some non-deterministic automaton which recognizes K_1.

We assume that for $i \in \{1, ..., m\}$ the language $\sigma(b_i)$ is given by some normalized nondeterministic automaton $\mathcal{B}_i$. We denote by n_σ the sum of the number of states of $\mathcal{B}_i$ for $i \in \{1, ..., m\}$.

We achieve the following bounds on the space complexity of the relative inclusion star height problem:

Table 1. Complexities for the relative inclusion star height problem

	σ	bound	existence	$\mathsf{sh}(K_1, K_2, m, \sigma) \le h$	$\mathsf{sh}(K_1, K_2, m, \sigma) = ?$
K_2	sing.	n_2	$\mathcal{O}(n_1 n_\sigma n_2)$	$n_1 n_\sigma 2^{2^{\mathcal{O}(n_2)}}$	$n_1 n_\sigma 2^{2^{\mathcal{O}(n_2)}}$
	arb.	$2^{2^{n_2}}$	$n_1 n_\sigma 2^{2^{\mathcal{O}(n_2)}}$	$n_1 n_\sigma 2^{h 2^{\mathcal{O}(n_2)}}$	$n_1 n_\sigma 2^{2^{2^{\mathcal{O}(n_2)}}}$
$\Sigma^* \setminus K_2$	arb.	2^{n_2}	$n_1 n_\sigma 2^{\mathcal{O}(n_2)}$	$n_1 n_\sigma 2^{h\mathcal{O}(n_2)}$	$n_1 n_\sigma 2^{2^{\mathcal{O}(n_2)}}$
both	sing.	n_2	$\mathcal{O}(n_1 n_\sigma n_2)$	$n_1 n_\sigma 2^{h\mathcal{O}(n_2)}$	$n_1 n_\sigma 2^{\mathcal{O}(n_2^2)}$

We will prove the entries of Table 1 in Section 5.8.1. In the lines of the table we consider four cases: In the first two cases, K_2 is given by a non-deterministic automaton with n_2 states and σ is singular resp. not necessarily singular. In the third case, $\Sigma^* \setminus K_2$ is given by a non-deterministic automaton with n_2 states and σ is not necessarily singular. In the fourth case, both K_2 and $\Sigma^* \setminus K_2$ are given by non-deterministic automata with at most n_2 states and σ is singular.

There are no lines "$\Sigma^* \setminus K_2$ sing." and "both arb." in the table, since in these cases, we achieve just the same complexity results as in the more general case "$\Sigma^* \setminus K_2$ arb.".

In the column "bound" we give a bound on the relative star height of (K_1, K_2, m, σ) provided that (K_1, K_2, m, σ) has a solution. In the column "existence", we give an upper bound on the space complexity for the problem to decide the existence of a solution. The values in this column are essentially the values in the column "bound" multiplied by $n_1 n_\sigma$. Indeed, both the problem to decide the existence of a solution and the upper bound on $\mathsf{sh}(K_1, K_2, m, \sigma)$ are closely related to an automaton $\mathcal{A}_L$ which recognizes the language $L = \{w \in \Gamma^* \mid \sigma(w) \subseteq K_2\}$. In particular, the bound on $\mathsf{sh}(K_1, K_2, m, \sigma)$ is the star height of L which is at most as large as the number of states of $\mathcal{A}_L$. In Section 5.4, we will see that the number of states of $\mathcal{A}_L$ crucially depends on whether σ is singular.

In the column "$\mathsf{sh}(K_1, K_2, m, \sigma) \le h$" we give a space complexity for deciding whether or not $\mathsf{sh}(K_1, K_2, m, \sigma) \le h$. In the first line, this complexity does not depend on h. We will discuss this fact in Section 5.8.1.

If we want to decide whether $\mathsf{sh}(K_1, K_2, m, \sigma) \le h$ for some h which exceeds the value given in column "bound", then the problem to decide whether $\mathsf{sh}(K_1, K_2, m, \sigma) \le h$ is equivalent to the problem whether (K_1, K_2, m, σ) has a solution. Hence, if h is larger than the value in the column "bound", then we can decide $\mathsf{sh}(K_1, K_2, m, \sigma) \le h$ in the complexity shown in the column "existence".

Finally, the column "$\mathsf{sh}(K_1, K_2, m, \sigma) = ?$" gives the complexity of computing $\mathsf{sh}(K_1, K_2, m, \sigma)$. An algorithm which computes $\mathsf{sh}(K_1, K_2, m, \sigma)$ decides at first whether (K_1, K_2, m, σ) has a solution. If so, then the algorithm decides for $h = 0, 1, 2, \ldots$ whether $\mathsf{sh}(K_1, K_2, m, \sigma) \le h$. In this computation, h cannot exceed the value in the column "bound". Hence, the complexity in the column "$\mathsf{sh}(K_1, K_2, m, \sigma) = ?$" is essentially the complexity from the column "$\mathsf{sh}(K_1, K_2, m, \sigma) \le h$" where we use the value from the column "bound" as bound for h.

3.2.2. The relative star height problem

We consider the relative star height problem, i.e., we assume $K_1 = K_2$ and let $K := K_1 = K_2$. We distinguish the cases that K is given by a non-deterministic automaton with n states (lines 1 and 2 in Table 2), and the case that both K and

$\Sigma^* \setminus K$ are given by non-deterministic automata with at most n states (lines 3 and 4 in Table 2). We also distinguish the cases that σ is singular (lines 1 and 3 in Table 2) or not necessarily singular (lines 2 and 4 in Table 2). We achieve the following bounds on the space complexity:

Table 2. Complexities for the relative star height problem

	σ	bound	existence	$\mathsf{sh}(K_1, K_2, m, \sigma) \leq h$	$\mathsf{sh}(K_1, K_2, m, \sigma) = ?$
K_2	sing.	n	$\mathcal{O}(n_\sigma n^2)$	$n_\sigma 2^{2^{\mathcal{O}(n)}}$	$n_\sigma 2^{2^{\mathcal{O}(n)}}$
	arb.	2^{2^n}	$n_\sigma 2^{2^{\mathcal{O}(n)}}$	$n_\sigma 2^{h 2^{\mathcal{O}(n)}}$	$n_\sigma 2^{2^{2^{\mathcal{O}(n)}}}$
both	sing.	n	$\mathcal{O}(n_\sigma n^2)$	$n_\sigma 2^{h\mathcal{O}(n)}$	$n_\sigma 2^{\mathcal{O}(n^2)}$
	arb.	2^n	$n_\sigma 2^{\mathcal{O}(n)}$	$n_\sigma 2^{h\mathcal{O}(n)}$	$n_\sigma 2^{2^{\mathcal{O}(n)}}$

The entries are understood as for the relative inclusion star height problem and will be proved in Section 5.8.2.

3.2.3. The inclusion star height problem

We deal with the inclusion star height problem. Let (K_1, K_2) be an instance of the inclusion star height problem. We achieve the following complexity bounds:

In the lines, we distinguish the cases that either K_2, or $\Sigma^* \setminus K_2$, or both K_2 and $\Sigma^* \setminus K_2$ are given by non-deterministic automata with n_2 states.

Table 3. Complexities for the inclusion star height problem

	bound	$\mathsf{sh}(K_1, K_2) \leq h$	$\mathsf{sh}(K_1, K_2) = ?$
K_2	$\min\{n_1, n_2\}$	$n_1 2^{2^{\mathcal{O}(n_2)}}$	$n_1 2^{2^{\mathcal{O}(n_2)}}$
$\Sigma^* \setminus K_2$	$\min\{n_1, 2^{n_2}\}$	$n_1 2^{h\mathcal{O}(n_2)}$	$n_1 2^{\min\{n_1, 2^{n_2}\}\mathcal{O}(n_2)}$
both	$\min\{n_1, n_2\}$	$n_1 2^{h\mathcal{O}(n_2)}$	$n_1 2^{\min\{n_1, n_2\}\mathcal{O}(n_2)}$

Clearly, the column σ is irrelevant. Since (K_1, K_2) has always a solution, the column "existence" is irrelevant. The entries in the column "bound" arise due to the fact that $\mathsf{sh}(K_1, K_2)$ is less than $\mathsf{sh}(K_1)$ and less than $\mathsf{sh}(K_2)$.

3.2.4. The star height problem

Finally, we deal with the star height problem. Let K be a recognizable language. We achieve the following complexity bounds:

Table 4. Space complexity bounds for the star height problem

	bound	$\mathsf{sh}(K) \le h$	$\mathsf{sh}(K) = ?$
K	n	$2^{2^{\mathcal{O}(n)}}$	$2^{2^{\mathcal{O}(n)}}$
$\Sigma^* \setminus K$	2^n	$2^{h\mathcal{O}(n)}$	$2^{2^{\mathcal{O}(n)}}$
both	n	$2^{h\mathcal{O}(n)}$	$2^{\mathcal{O}(n)}$

In the lines, we distinguish the cases that either K, or $\Sigma^* \setminus K$, or both K and $\Sigma^* \setminus K$ are given by non-deterministic automata with at most n states. The entries are proved in Section 5.8.4.

For the computation of the star height of K (column "$\mathsf{sh}(K) = ?$"), we achieve the same double exponential space complexity bound regardless of whether K or its complement is given by some non-deterministic automaton with n states. However, the bound arises in two different ways. If K is given by some non-deterministic automaton, then the test "$\mathsf{sh}(K) \le h$" requires $2^{h2^{\mathcal{O}(n)}}$ space. Since $\mathsf{sh}(K) \le n$, the algorithm answers immediately "yes" if $h \ge n$. Hence, we can approximate $2^{h2^{\mathcal{O}(n)}}$ by $2^{n2^{\mathcal{O}(n)}}$ and absorb the factor n into $2^{\mathcal{O}(n)}$ which gives a complexity bound of $2^{2^{\mathcal{O}(n)}}$.

If $\Sigma^* \setminus K$ is given by a non-deterministic automaton with n states, then the test "$\mathsf{sh}(K) \le h$" requires just $2^{h\mathcal{O}(n)}$ space. Now, we do not necessarily have $\mathsf{sh}(K) \le n$, we just have $\mathsf{sh}(K) \le 2^n$. Thus, the algorithm can answer immediately "yes" if $h \le 2^n$. Hence, the computation of $\mathsf{sh}(K)$ requires $2^{2^{\mathcal{O}(n)}\mathcal{O}(n)}$, i.e., $2^{2^{\mathcal{O}(n)}}$ space.

3.2.5. Variants of the limitedness problem

To achieve the above results on the relative inclusion star height problem and its particular cases, we show some generalized variants of the limitedness problem of distance desert automata.

Let $\mathcal{A}$ be a distance desert automaton and let $L' \subseteq \Sigma^*$. We say that $\mathcal{A}$ is limited on L' iff there is some $d \in \mathbb{N}$ such that $\Delta_{\mathcal{A}}(w) \leq d$ for every $w \in L(\mathcal{A}) \cap L'$.

Theorem 3.3. *Let $\mathcal{A}$ be a distance desert automaton and let $\mathcal{A}'$ be an automaton. To decide whether $\mathcal{A}$ is limited on $L(\mathcal{A}')$ is PSPACE-complete in the number of states of $\mathcal{A}$ and $\mathcal{A}'$.*

We show that the mappings definable by distance desert automata are somehow closed under inverse homomorphisms.

Let $m \geq 1$ and $\Gamma = \{b_1, ..., b_m\}$. Moreover, let $\tau : \Gamma \to \mathsf{REC}(\Sigma^*)$ be a mapping. We extend σ to a homomorphism $\tau : \mathcal{P}(\Gamma^*) \to \mathcal{P}(\Sigma^*)$.

We assume that for every $i \in \{1, ..., m\}$, the language $\tau(b_i)$ is given by a normalized, nondeterministic automaton $\mathcal{B}_i$. We assume that $\varepsilon \notin \tau(b_i)$. We denote by n_τ the sum of the numbers of states of the automata $\mathcal{B}_i$ for $i \in \{1, ..., m\}$.

Let $h \geq 1$ and $\mathcal{A} = [Q, E, I, F, \theta]$ be an h-nested distance desert automaton over Γ.

We define a mapping $\Delta' : \Sigma^* \to \mathbb{N} \cup \{\infty\}$ by setting

$$\Delta'(w) := \min\{\Delta_{\mathcal{A}}(u) \mid u \in \Gamma^*, w \in \tau(u)\}$$

for every $w \in \Sigma^*$.

Proposition 3.4. *We can effectively construct an $(h+1)$-nested distance desert automaton $\mathcal{A}'$ over Σ with at most $|Q| \cdot (n_\tau - 2m + 1)$ states which computes Δ'.*

We show by Example 4.2 that the condition $\varepsilon \notin \tau(b_i)$ for $i \in \{1, ..., m\}$ is necessary for Proposition 3.4.

4. Variants of the Limitedness Problem

4.1. Limitedness on a Recognizable Language

In this section, we prove Theorem 3.3.

Let $\mathcal{A} = [Q, E, I, F, \theta]$ be a distance desert automaton and let $\mathcal{A}' = [Q', E', I', F']$ be an automaton. We denote $L' := L(\mathcal{A})$.

We define a distance desert automaton $\mathcal{A}''$ by a product construction. Let $Q'' := Q \times Q'$, $I'' := I \times I'$, and $F'' := F \times F'$. For every $a \in \Sigma$, $p, q \in Q$, and p', $q' \in Q'$, we put the transition $t := ((p, p'), a, (q, q'))$ in E'' iff $(p, a, q) \in E$ and $(p', a, q') \in E'$. If this is the case, then we set $\theta''(t) = \theta((p, a, q))$.

Lemma 4.1. *For every* $w \in \Sigma^*$, *we have*

$$\Delta_{\mathcal{A}''}(w) = \begin{cases} \Delta_{\mathcal{A}}(w) & \text{if } w \in L(\mathcal{A}) \cap L' \\ \infty & \text{if } w \notin L(\mathcal{A}) \cap L'. \end{cases}$$

In particular, $\mathcal{A}''$ *is limited iff* $\mathcal{A}$ *is limited on* L'.

Proof. Let $w \in \Sigma^*$. Assume $w \notin L(\mathcal{A}) \cap L'$. By the construction of $\mathcal{A}''$, there is no accepting path for w in $\mathcal{A}''$, and hence, $\Delta_{\mathcal{A}''}(w) = \infty$. We assume $w \in L(\mathcal{A}) \cap L'$ in the rest of the proof.

Given two accepting paths π(resp. π') for w in $\mathcal{A}$ (resp. $\mathcal{A}'$), we can construct an accepting path π'' for w in $\mathcal{A}''$ such that $\theta''(\pi'') = \theta(\pi)$. Consequently, $\Delta_{\mathcal{A}''}(w) \leq \Delta_{\mathcal{A}}(w)$, and in particular, $\Delta_{\mathcal{A}''}(w) \in \mathbb{N}$.

Since $\Delta_{\mathcal{A}''}(w) \in \mathbb{N}$, there is an accepting path π'' for w in $\mathcal{A}''$ such that $\theta(\theta''(\pi'')) = \Delta_{\mathcal{A}''}(w)$. By selecting the first components of the states in π'', we obtain an accepting path π for w in $\mathcal{A}$ such that $\theta''(\pi'') = \theta(\pi)$. Hence, $\Delta_{\mathcal{A}''}(w) \geq \Delta_{\mathcal{A}}(w)$. □

Proof of Theorem 3.3. Decidability in PSPACE follows immediately from Lemma 4.1 and Theorem 2.1. The problem is PSPACE-hard, since it is a generalization of the limitedness problem for distance desert automata. □

4.2. Limitedness and Substitutions

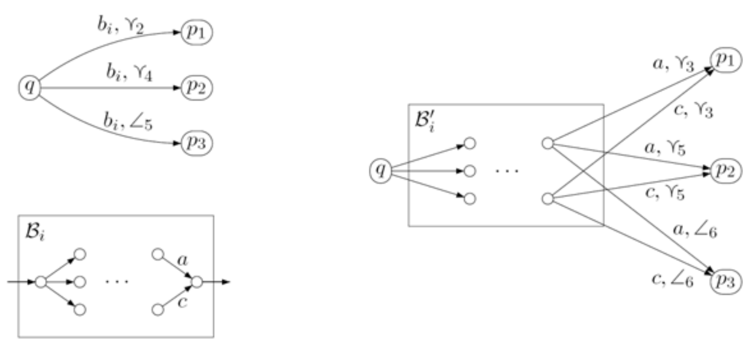
Let $m, \Gamma, \tau, \mathcal{B}_1, \ldots, \mathcal{B}_m$ be as in Section 3.2.5.

Proof of Proposition 3.4. At first, we deal with some preliminaries. We define a homomorphism lift $\ell : V_h^* \to V_{h+1}^*$ by setting for every $i \in \{0, \ldots, h+1\}$, $\ell(\angle_i) := \angle_{i+1}$ and for every $i \in \{0, \ldots, h\}$, $\ell(\curlyvee_i) := \curlyvee_{i+1}$. It is easy to verify that for every $\pi \in V_h^*$, we have $\Delta(\pi) = \Delta(\ell(\pi))$. Consequently, the nested distance desert automata $\mathcal{A}$ and $\mathcal{A} = [Q, E, I, F, \ell \circ \theta]$ are equivalent.

Let $\pi \in V_{h+1}^*$ be some word such that $\angle_0$ does not occur in π. We denote by $\overline{\pi} \in V_{h+1}^*$ the word obtained by erasing all letters $\curlyvee_0$ in π. We can easily verify that $\Delta(\pi) = \Delta(\overline{\pi})$. Note that the factors of π and the factors of $\overline{\pi}$ are essentially the same up to the occurrences of $\curlyvee_0$.

To construct $\mathcal{A}'$, we replace the transitions in $\mathcal{A}$ by copies of $\mathcal{B}_i$. Let $q \in Q$ and $i \in \{1, ..., m\}$ such that there exists at least one transition of the form $\{q\} \times \{b_i\} \times Q$ in E. Let P be the states $p \in Q$ which admit a transition $(q, b_i, p) \in E$. We create $|P|$ copies of the accepting state of $\mathcal{B}_i$. We insert the new automaton $\mathcal{B}_i'$ into $\mathcal{A}$ and merge q and the initial state of $\mathcal{B}_i'$ and we merge each state in P and one accepting state of $\mathcal{B}_i'$. □

The key idea of the transition marks in $\mathcal{A}'$ is the following: For every $(p, b_i, q) \in E$ and every word $w \in \tau(b_i)$ there is some path $\pi \in p \overset{w}{\rightsquigarrow} q$ in $\mathcal{A}'$ such that $\theta(\pi) = \curlyvee_0^{|w|-1}\ell(\theta((p, b_i, q)))$.



We proceed this insertion for every $q \in Q$, $i\{1, ..., m\}$ provided that there exists at least one transition of the form $\{q\} \times \{b_i\} \times Q$ in E. One can easily verify that the constructed automaton computes Δ'.

For every state of $\mathcal{A}$, we insert at most one copy of each $\mathcal{B}_i$. Since initial and accepting states are unified, we insert at most $n_\tau - 2m$ new states for each state of $\mathcal{A}$. Thus, $\mathcal{A}'$ has $|Q|$ states from $\mathcal{A}$ and at most $|Q|(n_\tau - 2m)$ states due to insertion of $\mathcal{B}_i$'s.

The reader should be aware that the above restriction $\varepsilon \notin \tau(b_i)$ is not just to simplify the proof as the following example shows.

Example 4.2. Assume $\Sigma = \{a\}$, $\Gamma = \{b_i\}$ and $\tau(b_i) = \{\varepsilon, a\}$. Let $\mathcal{A}$ be some nested distance desert automaton such that $\Delta_{\mathcal{A}}(b_1^{10}) = 10$ but $\Delta_{\mathcal{A}}(w) = \infty$ for $w \in \Gamma^* \setminus \{b_1^{10}\}$.

Let Δ' be as above. For every $w \in \{\varepsilon, a, ..., a^{10}\}$, we have $\Delta'(w) = 10$. However, for mappings of nested distance desert automata, we have either $0 \le \Delta'(w) \le |w|$ or $\Delta'(w) = \infty$.

One can probably generalize the concept of nested distance desert automata by marking transitions with words or even subsets of V_h^+ to achieve a concept of automata which allow us to compute mappings like Δ' from Example 4.2. However, such a generalization is not subject of the present paper.

By arguing as for Proposition 3.4, we obtain:

Proposition 4.3. *We can effectively construct an automaton $\mathcal{A}'$ over Σ with at most $|Q| \cdot (n_\tau - 2m + 1)$ states which recognizes $\tau(L(\mathcal{A}))$.*

Proof. The proof is similar but simpler than the proof of Proposition 3.4. □

5. The Main Proofs

5.1. String Expressions

We recall the notion of a string expression from R. S. Cohen [4]. We define the notions of a string expression, a single string expression and the degree in a simultaneous induction.

Every word $w \in \Sigma^*$ is a single string expression of star height $\mathsf{sh}(w) = 0$ and degree $\mathrm{dg}(w) := |w|$. Let $n \ge 1$ and $r_1, ..., r_n$ be single string expressions. We call $r := r_1 \cup \cdots \cup r_n$ a string expression of star height $\mathsf{sh}(r) := \max\{\mathsf{sh}(r_i) | 1 \le i \le n\}$ and degree $\mathrm{dg}(r) := \max\{\mathrm{dg}(r_i) | 1 \le i \le n\}$. The empty set $\varnothing$ is a string expression of star height $\mathsf{sh}(\varnothing) = 0$ and degree $\mathrm{dg}(\varnothing) := 0$.

Let $n \geq 2$, $a_1, ..., a_n \in \Sigma$, and $s_1, ..., s_{n-1}$ be string expressions. We call the expression $s := a_1 s_1^* a_2 s_2^* \cdots s_{n-1}^* a_n$ a single string expression of star height $\mathsf{sh}(s) = 1 + \max\{\mathsf{sh}(s_i) | 1 \leq i \leq n\}$ and degree $\mathrm{dg}(s) := \max(\{n\} \cup \{\mathrm{dg}(s_i) | 1 \leq i \leq n\})$.

String expressions define languages because they are particular rational expressions.

The following lemma is due to R. S. Cohen [4].

Lemma 5.1 ([4, 14, 16])**.** *Let $L \subseteq \Sigma^*$ be a recognizable language. There is a string expression s such that we have $L = L(s)$ and $\mathsf{sh}(s) = \mathsf{sh}(L)$.*

We need another well-known lemma.

Lemma 5.2 ([14, 16])**.** *Let $L \subseteq \Sigma^*$ be recognizable. We have $\mathsf{sh}(L) = \mathsf{sh}(L \setminus \{\varepsilon\})$.*

5.2. We Fix an Instance

For the rest of Section 5, we fix an instance (K_1, K_2, m, σ) of the relative inclusion star height problem. We assume that K_1 is given by some non-deterministic finite automaton $\mathcal{A}_1 = [Q_1, E_1, I_1, F_1]$ and denote $n_1 := |Q_1|$. Below, we will show that we can freely assume $\varepsilon \notin K_1$.

In the rest of Section 5, we distinguish various cases concerning the representation of K_2. Sometimes, we assume that K_2 is given by some non-deterministic automaton $\mathcal{A}_2 = [Q_2, E_2, I_2, F_2]$ and denote $n_2 := |Q_2|$. We also deal with the case that $\Sigma^* \setminus K_2$ is given by some non-deterministic automaton $\mathcal{A}_2 = [Q_2, E_2, I_2, F_2]$ and again denote $n_2 := |Q_2|$.

For every $i \in \{1, ..., m\}$, we assume that $\sigma(b_i)$ is given by some normalized, non-deterministic automaton $\mathcal{B}_i$. We denote the sum of the number of states of all $\mathcal{B}_i$ for $i \in \{1, ..., m\}$ by n_σ.

The language $L := \{w \in \Gamma^* | \sigma(w) \subseteq K_2\}$ will be of particular interest. For every language $L' \subseteq \Gamma^*$ satisfying $\sigma(L') \subseteq K_2$, we have $L' \subseteq L$. In Section 5.4, we will construct automata which recognize L and its complement.

5.3. On the Empty Word

In this section, we deal with some notions to reduce the technical overhead caused by the empty word. The following lemma allows us to restrict our proof to the particular case $\varepsilon \notin K_1$.

Lemma 5.3. *We have* $\mathsf{sh}(K_1, K_2, m, \sigma) = \mathsf{sh}(K_1 \setminus \{\varepsilon\}, K_2, m, \sigma)$.

Proof. If $\varepsilon \notin K_1$, then the claim is obvious. Hence, we assume $\varepsilon \in K_1$. Thus, $\varepsilon \in K_2$.

$\cdots \geq \cdots$ If r is a solution of (K_1, K_2, m, σ), then r is also a solution of $(K_1 \setminus \{\varepsilon\}, K_2, m, \sigma)$. Hence, $\mathsf{sh}(K_1, K_2, m, \sigma) \geq \mathsf{sh}(K_1 \setminus \{\varepsilon\}, K_2, m, \sigma)$.

$\cdots \leq \cdots$ If r is a solution of $(K_1 \setminus \{\varepsilon\}, K_2, m, \sigma)$, then $r \cup \varepsilon$ is a solution of (K_1, K_2, m, σ). Hence, $\mathsf{sh}(K_1, K_2, m, \sigma) \leq \mathsf{sh}(K_1 \setminus \{\varepsilon\}, K_2, m, \sigma)$. □

If $\varepsilon \in K_1$, then we rather examine the instance $(K_1 \setminus \{\varepsilon\}, K_2, m, \sigma)$. Consequently, we assume $\varepsilon \in K_1$ for the rest of Section 5.

We define homomorphism $\sigma_\varepsilon : \mathcal{P}(\Gamma^*) \to \mathcal{P}(\Gamma^*)$ and $\sigma^+ : \mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$ by setting for every $i \in \{1, \ldots, m\}$

$$\sigma_\varepsilon(b_i) := \begin{cases} \{b_i\} & \text{if } \varepsilon \notin \sigma(b_i) \\ \{b_i, \varepsilon\} & \text{if } \varepsilon \in \sigma(b_i) \end{cases} \quad \text{and} \quad \sigma^+(b_i) := \sigma(b_i) \setminus \{\varepsilon\}.$$

We have $\sigma = \sigma^+ \circ \sigma_\varepsilon$ and $\sigma_\varepsilon = \sigma_\varepsilon \circ \sigma_\varepsilon$ since $\sigma(b_i) = \sigma^+(\sigma_\varepsilon(b_i))$ and $\sigma_\varepsilon(b_i) = \sigma_\varepsilon(\sigma_\varepsilon(b_i))$ for $i = \{1, \ldots, m\}$. Moreover, we have $\sigma \circ \sigma_\varepsilon = \sigma^+ \circ \sigma_\varepsilon \circ \sigma_\varepsilon = \sigma^+ \circ \sigma_\varepsilon = \sigma$.

Lemma 5.4. *The following assertions are equivalent*:

1. *The instance* (K_1, K_2, m, σ) *has a solution*.

2. *There exists a solution r of* (K_1, K_2, m, σ) *such that* $\sigma(L(r)) = \sigma^+(L(r))$.

Proof. (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2) Let t be solution of (K_1, K_2, m, σ). The key idea is to replace in t every letter b_i satisfying $\varepsilon \in (b_i)$ by $b_i \cup \varepsilon$. Hence, we apply σ_ε to t to construct some $r \in \mathsf{REX}(\Gamma)$ such that $\mathsf{sh}(r) = \mathsf{sh}(t)$ and $L(r) = \sigma_\varepsilon(L(t))$. We have

$$\sigma(L(r)) = \sigma(\sigma_\varepsilon(L(t))) = \sigma(L(t)) = \sigma^+(\sigma_\varepsilon(L(t))) = \sigma^+(L(r)).$$

From $\sigma(L(r)) = \sigma(L(t))$ follows that r is a solution of (K_1, K_2, m, σ). □

From the implication (1) $\Rightarrow$ (2) in the proof of Lemma 5.4, we get $\mathsf{sh}(K_1, K_2, m, \sigma) \geq \mathsf{sh}(K_1, K_2, m, \sigma^+)$. Indeed, if t is a solution of (K_1, K_2, m, σ), then r is a solution of (K_1, K_2, m, σ^+). However, there are instances satisfying $\mathsf{sh}(K_1, K_2, m, \sigma) > \mathsf{sh}(K_1, K_2, m, \sigma^+)$, as the following example shows.

Example 5.5. Let $\Sigma = \{a_1, a_2, a_3\}$ and $L \in \mathsf{REC}(\{a_1, a_2\}^*)$ be a language of large star height satisfying $\varepsilon \notin L$. It is easy to show that $\mathsf{sh}(L) = \mathsf{sh}(L \cup \{a_3\})$.

Let $m := 4$ and $\sigma(b_i) := \{a_i\}$ for $i \in \{1, 2, 3\}$, and further, $\sigma(b_4) := L \cup \{a_3, \varepsilon\}$.

Moreover, let $K_1 = K_2 := L \cup \{a_3\}$.

We have $\mathsf{sh}(K_1, K_2, m, \sigma^+) = 0$, since b_4 is a solution.

Now, let r be a solution of $(K_1, K_2, 4, \sigma)$. By contradiction, assume that b_4 occurs in $L(r)$. Let $u, v \in \Gamma^*$ such that $ub_4v \in L(r)$. If $u \neq \varepsilon$, then some word of the form $\Sigma^+ a_3 \Sigma^*$ occurs in $\sigma(L(r))$. If $v \neq \varepsilon$ then some word of the form $\Sigma^* a_3 \Sigma^+$ occurs in $\sigma(L(r))$. If $u = \varepsilon$ and $v = \varepsilon$ then $b_4 \in L(r)$, and hence, $\varepsilon \in \sigma(L(r))$. Anyway, $\sigma(L(r)) \neq L \cup \{a_3\}$. Consequently, b_4 does not occur in r.

Since σ is a bijection on $\{b_1, b_2, b_3\}^*$, we can transform r into a rational expression for L by preserving the star height. Hence, $\mathsf{sh}(K_1, K_2, 4, \sigma) = \mathsf{sh}(r) \geq \mathsf{sh}(L)$.

Conversely, we can transform every rational expression for L into some $r \in \mathsf{REX}(\Gamma)$ by preserving the star height such that $\sigma(L(r)) = L \cup \{a_3\}$, and hence, $\mathsf{sh}(K_1, K_2, 4, \sigma) \leq \mathsf{sh}(L)$.

To sum up, $\mathsf{sh}(K_1, K_2, 4, \sigma) = \mathsf{sh}(L)$. □

5.4. Upper Bounds on the Relative Inclusion Star Height

In this section, we construct automata which recognize $L = \{w \in \Gamma^* \mid \sigma(w) \subseteq K_2\}$ and the complement of L. We also construct an automaton for $\sigma(L)$ and decide the existence of a solution.

Proposition 5.6. *We can effectively construct a non-deterministic automaton* $\mathcal{A}_{\overline{L}}$ *which recognizes* $\Gamma^* \setminus L$. *In particular,* $\mathcal{A}_{\overline{L}}$ *has the same states, accepting and final states as some automaton* $\mathcal{A}_2$ *which recognizes* $\Sigma^* \setminus K_2$.

Proof. We let $\mathcal{A}_2 = [Q_2, E_2, I_2, F_2]$. We define a new set of transitions $E_{\overline{L}}$. For every $p, q \in Q_2$, $b \in \Gamma$, the triple (p, b, q) belongs to $E_{\overline{L}}$ there exists some word $w \in \sigma(b)$ such that $\mathcal{A}_2$ admits a path from p to q which is labeled with w. This condition is decidable in polynomial time since it means to decide whether the language of $[Q_2, E_2, p, q]$ and $\sigma(b)$ are disjoint. Let $\mathcal{A}_{\overline{L}} = [Q_2, E_{\overline{L}}, I_2, F_2]$.

Let $w \in \Gamma^* \setminus L$. We denote $w = c_1 \cdots c_{|w|}$. For $i \in \{1, ..., |w|\}$, there is some $u_i \in \sigma(c_i)$ such that $u_1 \cdots u_{|w|} \notin K_2$ Hence, $\mathcal{A}_2$ accepts $u_1 \cdots u_{|w|}$. For $i \in \{1, ..., |w|\}$, there are q_{i-1}, $q_i \in Q_2$ such that $\mathcal{A}_2$ admits a path from q_{i-1} to q_i which is labeled with u_i, and $q_0 \in I_2$, $q_{|w|} \in F_2$. The transitions $(q_{i-1}, c_i, q_i) \in E_{\overline{L}}$ form an accepting path for w in $\mathcal{A}_{\overline{L}}$.

Conversely, let $w = c_1 \cdots c_{|w|} \in L(\mathcal{A}_{\overline{L}})$. Let $(q_0, c_1, q_1) \cdots (q_{|w|-1}, c_{|w|}, q_{|w|})$ be an accepting path for w in $\mathcal{A}_{\overline{L}}$. By the definition of $E_{\overline{L}}$, $\mathcal{A}_2$ admits for every $i = \{1, ..., |w|\}$ a path from q_{i-1} to q_i which is labeled with some $u_i \in \sigma(c_i)$. Thus, $\mathcal{A}_2$ accepts the word $u_1 \cdots u_{|w|} \in \sigma(w)$, i.e., $w \notin L$. □

If $K_2 = L(\mathcal{A}_2)$ for some non-deterministic automaton $\mathcal{A}_2 = [Q_2, E_2, I_2, F_2]$, then we can complement $\mathcal{A}_2$ and apply Proposition 5.6. However, the number of states of $\mathcal{A}_{\overline{L}}$ is at most $2^{|Q_2|}$.

For the rest of Section 5, we denote by $\mathcal{A}_{\overline{L}} = [Q_{\overline{L}}, E_{\overline{L}}, I_{\overline{L}}, F_{\overline{L}}]$ the non-deterministic automaton which is either constructed by Proposition 5.6, or by a complementation of $\mathcal{A}_2$ and an application of Proposition 5.6 depending how K_2 is given.

Proposition 5.7. *From* (K_1, K_2, m, σ), *we can effectively construct a non-deterministic automaton* $\mathcal{A}_L$ *which recognizes* L. *In this construction, we can bound the number of states of* $\mathcal{A}_L$ *as follows*:

σ	K_2	$\Sigma^* \setminus K_2$
singular	n_2	2^{n_2}
arbitrary	$2^{2^{n_2}}$	2^{n_2}

The columns of the table correspond to the representation of K_2: *In the column* "K_2", *we assume that* K_2 *is given by a non-deterministic automaton with* n_2 *states*. *In the column* "$\Sigma^* \setminus K_2$", *we assume that the complement of* K_2 *is given by a non-deterministic automaton with* n_2 *states. The rows of the table correspond to the case that* σ *is singular or not necessarily singular*.

Proof. *If* $\Sigma^* \setminus K_2$ is given by a non-deterministic automaton with n_2 states, then we can utilize the construction of $\mathcal{A}_{\overline{L}}$ from Proposition 5.6 and apply a complementation. Hence, the entries in the column "$\Sigma^* \setminus K_2$" are shown.

Assume that K_2 is given by a non-deterministic automaton $\mathcal{A}_2 = [Q_2, E_2, I_2, F_2]$ and σ is not necessarily singular. We can complement $\mathcal{A}_2$, construct the automaton $\mathcal{A}_{\overline{L}}$ from Proposition 5.6, and complement $\mathcal{A}_{\overline{L}}$. One can also construct $\mathcal{A}_L$ directly, but this construction utilizes sets of sets of states from $\mathcal{A}_2$, i.e., it gives the same bound on the number of states of $\mathcal{A}_{\overline{L}}$.

Now, assume that K_2 is given by a non-deterministic automaton $\mathcal{A}_2 = [Q_2, E_2, I_2, F_2]$ and σ is singular. We can construct $\mathcal{A}_L$ directly by defining a new set of edges E_L to $\mathcal{A}_2$. For every $p, q \in Q_2$, $b \in \Gamma$, the triple (p, b, q) belongs to E_L iff there exists some word $w \in \sigma(b)$ such that $\mathcal{A}_2$ admits a path from p to q which is labeled with w. It is easy to verify that $\mathcal{A}_L = [Q_2, E_L, I_2, F_2]$ recognizes L. □

From now on, we denote by $\mathcal{A}_L = [Q_L, E_L, I_L, F_L]$ the automaton constructed in Proposition 5.7.

We have $\sigma(\sigma_\varepsilon(L)) = \sigma(L) \subseteq K_2$, and hence, $\sigma_\varepsilon(L) \subseteq L$, i.e., $\sigma_\varepsilon(L) = L$.

Consequently, $\sigma^+(L) = \sigma^+(\sigma_\varepsilon(L)) = \sigma(L)$.

Lemma 5.8 ([12])**.** *The instance* (K_1, K_2, m, σ) *has a solution iff* $K_1 \subseteq \sigma(L)$ *iff* $K_1 \subseteq \sigma^+(L)$. *In this case, we have* $\mathsf{sh}(K_1, K_2, m, \sigma) \leq \mathsf{sh}(L)$.

Proof. The latter claim follows from $\sigma(L) = \sigma^+(L)$.

If (K_1, K_2, m, σ) has a solution r, then we have $K_1 \subseteq \sigma(L(r)) \subseteq K_2$, and hence, $K_1 \subseteq \sigma(L(r)) \subseteq \sigma(L) \subseteq K_2$. Consequently, $K_1 \subseteq \sigma(L)$.

Conversely, if $K_1 \subseteq \sigma(L)$, then the inclusion $K_1 \subseteq \sigma(L) \subseteq K_2$ implies the existence of a solution of (K_1, K_2, m, σ) and moreover, $\mathsf{sh}(K_1, K_2, m, \sigma) \leq \mathsf{sh}(L)$. □

Since $\mathsf{sh}(L) \leq |Q_L|$, the table in Proposition 5.7 gives an upper bound on $\mathsf{sh}(K_1, K_2, m, \sigma)$.

Proposition 5.9. *From* (K_1, K_2, m, σ), *we can effectively construct a non-deterministic automaton which recognizes* $\sigma(L)$ *and has at most* $|Q_L| \cdot (n_\sigma - 2m + 1)$ *states.*

Thus, we can effectively decide in space polynomial in $\mathcal{O}(n_1 \cdot |Q_L| \cdot (n_\sigma - 2m + 1))$ *whether* (K_1, K_2, m, σ) *has a solution.*

Proof. By Lemma 5.8, it suffices to decide whether $K_1 \subseteq \sigma^+(L)$. By Proposition 4.3, we construct a non-deterministic automaton which recognizes $\sigma^+(L)$. We can decide in $\mathcal{O}(n_1 \cdot |Q_L| \cdot (n_\sigma - 2m + 1))$ space whether K_1 is a subset of $\sigma^+(L)$. □

Due to the factor $|Q_L|$, the complexity in Proposition 5.9 crucially depends on the representation of K_2 and on whether σ is singular.

5.5. The $T_{d,h}(P, R)$-hierarchy

Let $\mathcal{A}_{\overline{L}} = [Q_{\overline{L}}, E_{\overline{L}}, I_{\overline{L}}, F_{\overline{L}}]$ be the automaton recognizing $\Gamma^* \setminus L$ by Proposition 5.6.

Let $\delta_{\overline{L}} : \mathcal{P}(Q_{\overline{L}}) \times \Gamma^* \to \mathcal{P}(Q_{\overline{L}})$ be defined by $\delta_{\overline{L}}(P, w) := \{r \in Q_{\overline{L}} \mid P \overset{w}{\rightsquigarrow} r \neq \emptyset\}$ for every $P \subseteq Q_{\overline{L}}$, $w \in \Gamma^*$. For every P, $R \subseteq Q_{\overline{L}}$ let $\mathcal{T}(P, R) := \{w \in \Gamma^+ \mid \delta_{\overline{L}}(P, w) \subseteq R\}$. Consequently, $\mathcal{T}(I_{\overline{L}}, Q_{\overline{L}} \setminus E_{\overline{L}}) = \Gamma^+ L(\mathcal{A}_{\overline{L}}) = L \setminus \{\varepsilon\}$.

Let $d \geq 1$ and P, $R \subseteq Q_{\overline{L}}$. We define $T_{d,0}(P, R) := \{w \in \Gamma^+ \mid \delta_{\overline{L}}(P, w) \subseteq R, \mid w \mid \leq d\}$. We have

$$T_{d,0}(P, R) = \bigcup_{\substack{1 \leq c \leq d, \\ P_0, \ldots, P_c \subseteq Q_{\overline{L}} \\ P = P_0, P_c \subseteq R}} T_{1,0}(P_0, P_1) T_{1,0}(P_1, P_2) \cdots T_{1,0}(P_{c-1}, P_c).$$

It is easy to see that $\mathcal{T}(P, R) = \bigcup_{d \geq 1} T_{d,0}(P, R)$.

Now, let $h \in \mathbb{N}$, and assume by induction that for every P, $R \subseteq Q_{\overline{L}}$, $T_{d,h}(P, R)$ is already defined. We define $T_{d,h+1}(P, R) :=$

$$\bigcup_{\substack{1 \leq c \leq d, \\ P_0, \ldots, P_c \subseteq Q_{\overline{L}}, \\ P = P_0, P_c \subseteq R}} T_{1,0}(P_0, P_1)(T_{d,h}(P_1, P_1))^* T_{1,0}(P_1, P_2) \cdots T_{1,0}(P_{c-1}, P_c).$$

Let $d \geq 1$, $h \in \mathbb{N}$, and P, $R \subseteq Q_{\overline{L}}$ be arbitrary. We have $\varepsilon \notin T_{d,h}(P, R)$.

Lemma 5.10. *Let $d \geq 1$, $h \in \mathbb{N}$, and P, $R \subseteq Q_{\overline{L}}$. We have*

$$(T_{d,h}(P, P))^* T_{1,0}(P, P)(T_{d,h}(P, P))^* \subseteq (T_{d,h}(P, P))^*.$$

Proof. The assertion follows, because $T_{1,0}(P, P) \subseteq T_{d,h}(P, P)$ and $(T_{d,h}(P, P))^*$ is closed under concatenation. □

From the definition, it follows immediately for every $R \subseteq R' \subseteq Q_{\overline{L}}$, $T_{d,h}(P, R) \subseteq T_{d,h}(P, R')$.

It is easy to show by an induction on h that for every $d' \geq d$, $T_{d,h}(P, R) \subseteq T_{d',h}(P, R)$. Moreover, for every $h' \geq h$, we have $T_{d,h}(P, R) \subseteq T_{d,h'}(P, R)$. To sum up, for every $d' \geq d$ and $h' \geq h$, $T_{d,h}(P, R) \subseteq T_{d',h'}(P, R)$. For fixed P, $R \subseteq Q_{\overline{L}}$, the sets $T_{d,h}(P, R)$ form a two-dimensional hierarchy. Whenever we use the notion $T_{d,h}(P, R)$ -hierarchy, we regard P, $R \subseteq Q_{\overline{L}}$ and $h \in \mathbb{N}$ as fixed, i.e., it is a one-dimensional hierarchy with respect to the parameter $d \geq 1$.

By induction, we can easily construct a string expression r with $L(r) = T_{d,h}(P, R)$ such that $\mathsf{sh}(r) \leq h$ and $\mathsf{dg}(r) \leq d$, and hence, $\mathsf{sh}(T_{d,h}(P, R)) \leq h$. However, we cannot assume that there is a string expression r with $L(r) = T_{d,h}(P, R)$ such that $\mathsf{sh}(r) = h$ and $\mathrm{dg}(r) = d$. In the inductive construction of r, several sets $T_{1,0}(P_{i-1}, P_i)$ may be empty, and then, the star-height (resp. degree) of r is possibly smaller than h (resp. d). Just consider the case $T_{d,h}(P, R) = \{a\}$ but $h > 1$, $d > 1$.

Lemma 5.11. *Let* $d \geq 1$, $h \in \mathbb{N}$, and P, $R \subseteq Q_{\overline{L}}$. *We have* $T_{d,h}(P, R) \subseteq \mathcal{T}(P, R)$.

Proof. We fix some arbitrary $d \geq 1$ for the entire proof.

For $h = 0$, the claim follows directly from the definitions of $T_{d,0}(P, R)$ and $\mathcal{T}(P, R)$.

Let $h \in \mathbb{N}$ and assume by induction that the claim is true for h. Consequently, for every $P' \subseteq Q_{\overline{L}}$ and $u \in (T_{d,h}(P', P'))^*$, the inclusion $\delta(P', u) \subseteq P'$. holds.

Let P, $R \subseteq Q_{\overline{L}}$ and $w \in T_{d,h+1}(P, R)$ be arbitrary. We show $\delta(P, w) \subseteq R$. According to the definition of $T_{d,h+1}(P, R)$, there are some $1 \leq c \leq d$ and $P = P_0, ..., P_c \subseteq R$ with the following property: there are $a_1, ..., a_c \in \Gamma$ and $w_1, ..., w_{c-1} \in \Gamma^*$ such that $w = a_1 w_1 a_2 w_2 \cdots w_{c-1} a_c$ and

1. for every $1 \leq i \leq c$, we have $a_i \in T_{1,0}(P_{i-1}, P_i)$, and

2. for every $1 \leq i \leq c$, we have $w_i \in (T_{d,h}(P_i, P_i))^*$.

By the definition of $T_{1,0}$, we have for every $1 \le i \le c$, $\delta(P_{i-1}, a_i) \subseteq P_i$. As seen above, we have for every $1 \le i \le c$, $\delta(P_i, w_i) \subseteq P_i$. Consequently, $\delta(P_0, w) \subseteq P_c$, i.e., $\delta(P, w) \subseteq R$. □

We have for every $h \in \mathbb{N}$ and $P, R \subseteq Q_{\overline{L}}$:

$$\mathcal{T}(P, R) = \bigcup_{d \ge 1} T_{d,0}(P, R) \subseteq \bigcup_{d \ge 1} T_{d,h}(P, R) \subseteq \mathcal{T}(P, R).$$

5.6. The Collapse of the $T_{d,h}(P, R)$-hierarchy

We say that the $T_{d,h}(P, R)$-hierarchy collapses for some $h \in \mathbb{N}$ if there is some $d \ge 1$ such that $T_{d,h}(P, R) = \mathcal{T}(P, R)$. Below, we will observe that the $T_{d,h}(P, R)$ -hierarchy collapses for some $h \ge \mathsf{sh}(\mathcal{T}(P, R))$.

For the relative inclusion star height problem, we are rather interested in $\sigma^+(T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}}))$ than in $T_{d,h}(P, R)$. In particular, it is interesting whether for some given $h \in \mathbb{N}$, there exists some d such that $K_1 \subseteq \sigma^+(T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}}))$. For this, the following lemma will be very useful.

Lemma 5.12. *Let r be a string expression, $d \ge \mathsf{dg}(r)$, and $h \ge \mathsf{sh}(r)$. Let P, $R \subseteq Q_{\overline{L}}$ such that $L(r) \subseteq \mathcal{T}(P, R)$. We have $L(r) \subseteq T_{d,h}(P, R)$.*

Proof. We assume $L(r) \ne \emptyset$. By $L(r) \subseteq \mathcal{T}(P, R)$, we have $\varepsilon \notin L(r)$.

Assume $\mathsf{sh}(r) = 0$. There are some $k \ge 1$ and $w_1, ..., w_k \in \Gamma^+$ such that $r = w_1 \cup \cdots \cup w_k$ and for every $1 \le i \le k$, we have $|w_i| \le d$, and moreover, $\delta(P, w_i) \subseteq R$. By the definition of $T_{d,0}(P, R)$, we have $w_i \in T_{d,0}(P, R)$, i.e., $L(r) \subseteq T_{d,0}(P, R) \subseteq T_{d,h}(P, R)$.

Now, let $\mathsf{sh}(r) \ge 1$, and assume that the claim is true for every string expression r' with $\mathsf{sh}(r') < \mathsf{sh}(r)$.

Clearly, it suffices to consider the case that r is a single string expression. Let $c \ge 2$ and $a_1, ..., a_c \in \Gamma$ and $r_1, ..., r_{c-1}$ be string expressions of a star height less than $\mathsf{sh}(r)$ such that $r = a_1 r_1^* a_2 r_2^* \cdots r_{c-1}^* a_c$. Let $d \ge \mathsf{dg}(r)$ and $h \ge \mathsf{sh}(r)$. Let P, $R \subseteq Q_{\overline{L}}$ such that $L(r) \subseteq \mathcal{T}(P, R)$.

Let $P_0 := P$, and for $1 \le i < c$, let $P_i := \delta(P_{i-1}, a_i L(r_i^*))$. Finally, let $P_c := \delta(P_{c-1}, a_c)$. To show $L(r) \subseteq T_{d,h}(P, R)$, we apply the definition of $T_{d,h}(P, R)$ with $P_0, ..., P_c$. We defined $P_0 = P$, and we can easily show $P_c = \delta(P_0, L(r)) \subseteq R$. Clearly, $c \le d$. To complete the proof, we show the following two assertions:

1. for every $1 \le i \le d'$, we have $a_i \in T_{1,0}(P_{i-1}, P_i)$, and

2. for every $1 \le i < d'$, we have $L(r_i) \subseteq T_{d,h-1}(P_i, P_i)$.

(1) Clearly, $\delta(P_{i-1}, a_i) \subseteq \delta(P_{i-1}, a_i L(r_i)^*) = P_i$. Hence, $a_i \in T_{1,0}(P_{i-1}, P_i)$ follows from the definition of $T_{1,0}(P_{i-1}, P_i)$.

(2) We have $\mathsf{sh}(r_i) < h$ and $\mathsf{dg}(r_i) \le d$. In order to apply the inductive hypothesis, we still have to show $\delta(P_i, L(r_i)) \subseteq P_i$. We have $a_i L(r_i)^* L(r_i) \subseteq a_i L(r_i)^*$. Thus, we obtain

$$\begin{aligned}\delta(P_i, L(r_i)) &= \delta(\delta(P_{i-1}, a_i L(r_i)^*), L(r_i)) \\ &= \delta(P_{i-1}, a_i L(r_i)^* L(r_i)) \subseteq \delta(P_{i-1}, a_i L(r_i)^*) = P_i. \qquad \square\end{aligned}$$

Let $P, R \subseteq Q_{\overline{L}}$ and $h \ge \mathsf{sh}(\mathcal{T}(P, R))$. By Lemma 5.1, there is a string expression r such that $L(r) = \mathcal{T}(P, R)$ and $h \ge \mathsf{sh}(r)$. Let $d := \mathsf{dg}(r)$. We have

$$\mathcal{T}(P, R) = L(r) \overset{\text{Lemma 5.12}}{\subseteq} T_{d,h}(P, R) \overset{\text{Lemma 5.11}}{\subseteq} \mathcal{T}(P, R),$$

i.e., the $T_{d,h}(P, R)$-hierarchy collapses for h.

Conversely, let $h \in \mathbb{N}$, $P, R \subseteq Q_{\overline{L}}$ and assume that the $T_{d,h}(P, R)$-hierarchy collapses for h. Let $d \ge 1$ such that $T_{d,h}(P, R) = \mathcal{T}(P, R)$. As already seen, we can construct a string expression r such that $L(r) = T_{d,h}(P, R)$, $\mathsf{sh}(r) \le h$, and $\mathsf{dg}(r) \le d$. Thus, $h \ge \mathsf{sh}(\mathcal{T}(P, R))$.

To sum up, the $T_{d,h}(P, R)$-hierarchy collapses for h iff $h \ge \mathsf{sh}(\mathcal{T}(P, R))$.

Proposition 5.13. *Let* $h \in \mathbb{N}$. *There exists some* $d \geq 1$ *such that* $K_1 \subseteq \sigma^+(T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}}))$ *iff* $\mathsf{sh}(K_1, K_2, m, \sigma) \leq h$.

Proof. $\cdots \Rightarrow \cdots$ Let r be a string expression such that $L(r) = T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$, $\mathsf{sh}(r) \leq h$, and $\mathsf{dg}(r) \leq d$. From $L(r) \subseteq \mathcal{T}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}}) = L \setminus \{\varepsilon\}$, it follows $\sigma(L(r)) \subseteq \sigma(L) \subseteq K_2$.

Moreover, we have $K_1 \subseteq \sigma^+(L(r)) \subseteq \sigma(L(r))$. Consequently, $h \geq \mathsf{sh}(r) \geq \mathsf{sh}(K_1, K_2, m, \sigma)$.

$\cdots \Leftarrow \cdots$ Let s be a solution of (K_1, K_2, m, σ). By Lemma 5.4, we can assume $\sigma(L(s)) = \sigma^+(L(s))$. Thus,

$$K_1 \subseteq \sigma^+(L(s)) \subseteq K_2.$$

Our aim is to apply Lemma 5.12 to show that $L(s)$ is subsumed by the set $T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$ for some $d \geq 1$. However, the empty word causes some trouble. Since $\varepsilon \notin K_1$, we obtain

$$K_1 \subseteq \sigma^+(L(s) \setminus \{\varepsilon\}) \subseteq K_2.$$

By Lemmas 5.1 and 5.2, we can transform s into a string expression r by preserving the star height such that $L(r) = L(s) \setminus \{\varepsilon\}$. Thus,

$$K_1 \subseteq \sigma^+(L(r)) \subseteq K_2.$$

From $L(r) \subseteq L(s) \subseteq L$ and $\varepsilon \notin L(r)$, it follows $L(r) \subseteq L \setminus \{\varepsilon\} = \mathcal{T}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$.

Let $d := \mathsf{dg}(r)$. Since $h \geq \mathsf{sh}(r)$, we can apply Lemma 5.12 and get $L(r) \subseteq T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$, i.e.,

$$K_1 \subseteq \sigma^+(L(r)) \subseteq \sigma^+(T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})). \qquad \square$$

5.7. A Reduction to Limitedness

In this section, we construct for given $h \in \mathbb{N}$ and $P, R \subseteq Q_{\overline{L}}$ a $(h+1)$-nested distance desert automaton $\mathcal{A}_h(P, R)$ over the alphabet Γ. This automaton associates to each word $w \in \Gamma^+$ the least integer d such that $w \in T_{d+1,h}(P, R)$. It computes ∞ if such an integer d does not exist, i.e., if $w \notin \mathcal{T}(P, R)$.

The automaton $\mathcal{A}_h(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$ will be of particular interest. By applying the construction from Section 4.2, we transform $\mathcal{A}_h(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$ to a distance desert automaton which associates to each word $w \in \Sigma^*$ the least integer d such that $w \in \sigma^+(T_{d+1,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}}))$.

In combination with Proposition 5.13 and the decidability of limitedness (Theorem 3.3), this construction allows us to decide whether $\mathrm{sh}(K_1, K_2, m, \sigma) \le h$.

Proposition 5.14. *Let* $h \in \mathbb{N}$ *and* $P, R \subseteq Q_{\overline{L}}$. *We can construct an* $(h+1)$-*nested distance desert automaton* $\mathcal{A}_h(P, R) = [Q, E, q_I, q_F, \theta]$ *with the following properties*:

1. $E \subseteq (Q \setminus \{q_F\}) \times \Gamma \times (Q \setminus \{q_I\})$,

2. $|Q| \le k^{h+1} + \dfrac{k^h - 1}{k - 1} + 1$ *where* $k = 2^{|Q_{\overline{L}}|}$,

3. *for every* $(p, a, q) \in E$, *we have* $\theta((p, a, q)) = \curlyvee_h$ *if* $p = q_I$, *and* $\theta((p, a, q)) \in \{\curlyvee_0, ..., \curlyvee_{h-1}, \angle_0, ..., \angle_h\}$ *if* $p \ne q_I$,

4. *for every* $w \in \Gamma^*$, $\Delta_{\mathcal{A}}(w) + 1 = \min\{d \le 1 \mid w \in T_{d,h}(P, R)\}$.

Proof. We employ the mapping $\delta_{\overline{L}}$ from the beginning of Section 5.5. We proceed by induction on h. Let $P, R \subseteq Q_{\overline{L}}$ be arbitrary.

Let $h = 0$. At first, we construct an automaton which accepts every word w with $\delta(P, w) \subseteq R$. We use $\mathcal{P}(Q_{\overline{L}})$ as set of states. For every $S, T \subseteq Q_{\overline{L}}$, $b \in \Gamma$, we set a transition (S, b, T) iff $\delta_{\overline{L}}(S, b) \subseteq T$. The initial state is P, every non-empty subset of R is an accepting state. We apply to this automaton a standard construction to get an automaton $[Q, E, q_I, q_F]$ which satisfies (1) where $Q = \mathcal{P}(Q_{\overline{L}}) \,\dot\cup\, \{q'_I, q'_F\}$. Hence, $|Q| = |\mathcal{P}(Q_{\overline{L}})| + 2 = k + 2$, i.e., (2) is satisfied. For every transition $(q_I, b, q) \in E$, we set $\theta((q_I, b, q)) = \curlyvee_0$. For every transition $(p, b, q) \in E$ with $p \ne q_I$, we set $\theta((q_I, b, q)) = \angle_0$. This completes the construction of $\mathcal{A}_0(P, R) = [Q, E, q_I, q_F, \theta]$, and (3) is satisfied.

We show (4). For every $w \in \Gamma^*$ with $w \notin \mathcal{T}(P, R)$, the equation in (4) reduces to $\infty = \infty$ by the construction of $\mathcal{A}_0(P, R)$ and Lemma 5.11. For $w \in \mathcal{T}(P, R)$, the equation in (4) reduces to $|w| = |w|$ by the construction of $\mathcal{A}_0(P, R)$ and the definition of $T_{d,0}(P, R)$.

Now, let $h \in \mathbb{N}$. We assume that the claim is true for h and show the claim for $h + 1$. At first, we construct an automaton $\mathcal{A}' := [Q', E', q_I, q_F]$. Let $Q' := \mathcal{P}(Q_{\overline{L}}) \dot{\cup} \{q_I, q_F\}$.

Let $b \in \Gamma$ and $S, T \subseteq Q_{\overline{L}}$ be arbitrary. If $S \neq T$ and $\delta(S, b) \subseteq T$, then we put the transition (S, b, T) into E'. If $\delta(P, b) \subseteq T$, then we put the transition (q_I, b, T) into E'. If $\delta(S, b) \subseteq R$, then we put the transition (S, b, q_F) into E'. Finally, if $\delta_{\overline{L}}(P, b) \subseteq R$, then we put the transition (q_I, b, q_F) into E'. For every word w which $\mathcal{A}'$ accepts, we have $w \in \mathcal{T}(P, R)$.

We define $\theta' = E' \to \{\curlyvee_{h+1}, \angle_{h+1}\}$. For every transition $(q_I, b, q) \in E'$, let $\theta'((q'_I, b, q)) = \curlyvee_{h+1}$. For every transition $(p, b, q) \in E'$ with $p \neq q_I$, we set $\theta'((q'_I, b, q)) = \angle_{h+1}$.

We construct $\mathcal{A}_{h+1}(P, R)$. For every $S \subseteq Q_{\overline{L}}$, we assume by induction an automaton $\mathcal{A}_h(S, S)$ which satisfies $(1, ..., 4)$. We assume that the sets of states of the automata $\mathcal{A}_h(S, S)$ are mutually disjoint. We construct $\mathcal{A}_{h+1}(P, R) = [Q, E, q_I, q_F, \theta]$ as a disjoint union of $\mathcal{A}'$ and the automata $\mathcal{A}_h(S, S)$ for every $S \subseteq Q_{\overline{L}}$ and unifying both the initial and accepting state of $\mathcal{A}_h(S, S)$ with the state S in $\mathcal{A}'$. Because we did not allow self loops in $\mathcal{A}'$, the union of the transitions is disjoint, and hence, θ arises in a natural way as union of θ' and the corresponding mappings of the automata $\mathcal{A}_h(S, S)$. If $\theta(t) \in \{\curlyvee_{h+1}, \angle_{h+1}\}$ for some $t \in E$, then t stems from $\mathcal{A}'$. Conversely, if $\theta(t) \in \{\curlyvee_0, ..., \curlyvee_h, \angle_0, ..., \angle_h\}$ for some $t \in E$, then t stems from some automaton $\mathcal{A}_h(S, S)$.

Let π be some path in $\mathcal{A}_{h+1}(P, R)$ and assume that for every transition t in π, we have $\theta(t) \in \{\curlyvee_0, ..., \curlyvee_h, \angle_0, ..., \angle_h\}$. Then, the entire path π stems from some automaton $\mathcal{A}_h(S, S)$, i.e., π cannot visit states in $\mathcal{P}(Q_{\overline{L}}) \setminus \{S\}$. Conversely, if π is a path in $\mathcal{A}_{h+1}(P, R)$, and two states $S, T \subseteq Q_{\overline{L}}$ with $S \neq T$ occur in π, then π contains some transition t with $\theta(t) = \angle_{h+1}$.

Clearly, $\mathcal{A}_{h+1}(P, R)$ satisfies (1) and (3). We show (2). The states of $\mathcal{A}_{h+1}(P, R)$ are q_I, q_F, k states from $\mathcal{P}(Q_{\overline{L}})$, and the states of the k inserted automata $\mathcal{A}_h(S, S)$. We obtain

$$|Q| \leq 2 + k + k\underbrace{\left(k^{h+1} + \frac{k^h - 1}{k-1} - 1\right)}_{(*)} = \cdots$$

$(*)$ is the bound on the number of states of one $\mathcal{A}_h(S, S)$ by the inductive hypothesis (2) reduced by two states which are lost by identification.

$$\cdots = 2 + k + k^{h+2} + \frac{k^h - k}{k-1} - k = k^{h+2} + \frac{k^{h+1} - k}{k-1} + 1 + 1 = \cdots$$

$$\cdots = k^{h+2} + \frac{k^{h+1} - k}{k-1} + k\,\frac{k-1}{k-1} + 1 = k^{h+2} + \frac{k^{h+1} - 1}{k-1} + 1$$

Thus, we have shown (2).

To prove (4), we show the following two claims:

4a. Let $d \geq 1$. For every $w \in T_{d,h+1}(P, R)$, there is a successful path π in $\mathcal{A}_{h+1}(P, R)$ with the label w and $\Delta(\theta(\pi)) + 1 \leq d$.

4b. Let π be a successful path in $\mathcal{A}_{h+1}(P, R)$ with the label w. We have $w \in T_{\Delta(\theta(\pi))+1,\,h+1}(P, R)$.

Claim (4a) (resp. 4b) proves "$\cdots \leq \cdots$" (resp. "$\cdots \geq \cdots$") in (4). Thus, (4) is a conclusion from (4a) and (4b).

We show (4a). We decompose w according to the definition of $T_{d,h+1}(P, R)$. There are some $1 \leq c \leq d$ and $P_0, ..., P_c \subseteq Q_{\overline{L}}$ with $P_0 = P$ and $P_c \subseteq R$. For every $1 \leq i \leq c$, there is some $a_i \in T_{1,0}(P_{i-1}, P_i)$, and for every $1 \leq i < c$ there is some $w_i \in (T_{d,h}(P_i, P_i))^*$ such that $w = a_1 w_1 a_2 w_2 \cdots a_c$. By Lemma 5.10, we can assume $P_{i-1} \neq P_i$ for every $2 \leq i < c$.

If $c = 1$, then w is a letter. We set $\pi := (q_I, w, q_F)$. Then, $\theta(\pi) = \angle_{h+1}$ and $\Delta(\theta(\pi)) = 0$ which proves (4a). We assume $c \geq 2$ in the rest of the proof of (4a).

Let $t_1 := (q_I, a_1, P_1)$ and $t_c := (P_{c-1}, a_c, q_F)$. For every $2 \leq i < c$, let $t_i := (P_{i-1}, a_i, P_i)$. Clearly, $t_1, ..., t_c$ are transitions in $\mathcal{A}_{h+1}(P, R)$, $\theta(t_1) = \angle_{h+1}$, and for $2 \leq i \leq c$, $\theta(t_i) = \angle_{h+1}$.

Let $1 \leq i < c$. We decompose w_i. There is some $n_i \in \mathbb{N}$ and $w_{i,1}, ..., w_{i,n_i} \in T_{d,h}(P_i, P_i)$ such that $w_i = w_{i,1}, ..., w_{i,n_i}$.

Let $1 \leq i < c$ and $1 \leq j \leq n_i$. Then, $w_{i,j} \in T_{d,h}(P_i, P_i)$. By the inductive hypothesis, there is a path $\tilde{\pi}_{i,j}$ in $\mathcal{A}_h(P_i, P_i)$ with the label $w_{i,j}$ and $\Delta(\theta(\tilde{\pi}_{i,j}))$ $+1 \leq d$. The first transition of this path is marked $\angle_h$, any other transition is marked by some member in $\{\angle_0, ..., \angle_{h-1}, \angle_0, ..., \angle_h\}$. We rename the first and the last state in $\tilde{\pi}_{i,j}$ to P_i and call the resulting path $\pi_{i,j}$. Since $\mathcal{A}_{h+1}(P, R)$ contains $\mathcal{A}_h(P_i, P_i)$, $\pi_{i,j}$ is a path in $\mathcal{A}_{h+1}(P, R)$. Let $\pi_i := \pi_{i,1}, ..., \pi_{i,n_i}$. Clearly, π_i is a path in $\mathcal{A}_{h+1}(P, R)$ from P_i to P_i with the label w_i. The transitions of π_i are marked by members in $\{\angle_0, ..., \angle_h, \angle_0, ..., \angle_h\}$. In the particular case $w_i = \varepsilon$, π_i is simply the empty path from P_i to P_i.

Clearly, $\pi := t_1\pi_1 t_2\pi_2 \cdots t_c$ is a successful path in $\mathcal{A}_{h+1}(P, R)$ with the label w. It remains to show $\Delta(\theta(\pi)) + 1 \leq d$. We apply the definition of Δ from Section 2.2. Let π' be an arbitrary factor of $\theta(\pi)$. We have $|\pi'|_{h+1} + 1 \leq |\theta(\pi)|_{h+1} + 1 = c \leq d$. Let $0 \leq g \leq h$, and assume $\pi' \in \{\angle_0, ..., \angle_{g-1}, \angle_0, ..., \angle_g\}^*$. Then, π' is a factor of $\theta(\pi_{i,j})$ for some $1 \leq i < c$, $1 \leq j \leq n_i$. Since $\Delta(\theta(\tilde{\pi}_{i,j})) + 1 \leq d$, we have $|\pi'|_g + 1$ $\leq d$. Consequently, $\Delta(\theta(\pi)) + 1 \leq d$.

We show (4b). Let π be a successful path in $\mathcal{A}_{h+1}(P, R)$ with the label w. The first transition of π is marked $\angle_{h+1}$, any other transitions are marked by some member of $\{\angle_0, ..., \angle_h, \angle_0, ..., \angle_{h+1}\}$. Let $c \geq 1$ and factorize π into $\pi = t_1\pi_1 t_2\pi_2 \cdots t_c$ such that $t_2, ..., t_c$ are the transitions in π which are marked by $\angle_{h+1}$. We have $\Delta(\theta(\pi)) \geq c - 1$, i.e., $c \leq \Delta(\theta(\pi)) + 1$.

We denote the labels of $t_1, ..., t_c$ and $\pi_1, ..., \pi_{c-1}$ by $a_1, ..., a_c$ and $w_1, ..., w_{c-1}$, resp., i.e., $w = a_1 w_1 q_2 w_2 \cdots a_c$. Every transition $t_1, ..., t_c$ starts and ends at some state in $\mathcal{P}(Q_{\overline{L}})$ except t_1 which starts in q_I and t_c which ends in q_F.

Let $1 \le i < c$. Let P_i be the state in which π_i starts. Since the transitions of π_i are marked by members in $\{\angle_0, ..., \angle_h, \angle_0, ..., \angle_h\}$, π_i is a path inside $\mathcal{A}_h(P_i, P_i)$. Clearly, π_i ends in the same state in which t_{i+1} starts, i.e., π_i ends in some state in $\mathcal{P}(Q_{\overline{L}})$. To sum up, π_i ends in P_i.

Let $P_0 := P$ and $P_c := R$. By the construction of $\mathcal{A}_{h+1}(P, R)$, (in particular by the definition of E'), we have for every $1 \le i \le c$, $\delta(P_{i-1}, a_i) \subseteq P_i$, and thus, $a_i \in T_{1,0}(P_{i-1}, P_i)$.

To show $w \in T_{\Delta(\theta(\pi))+1, h+1}(P, R)$, we show for every $1 \le i < c$, $w_i \in (T_{\Delta(\theta(\pi))+1, h}(P_i, P_i))^*$.

Let $1 \le i < c$. We decompose π_i into cycles. There are some $n_i \in \mathbb{N}$, and non-empty paths $\pi_{i,1}, ..., \pi_{i,n_i}$ such that $\pi_i := \pi_{i,1}, ..., \pi_{i,n_i}$ and every path among $\pi_{i,1}, ..., \pi_{i,n_i}$ starts and ends at P_i, but none of the paths $\pi_{i,1}, ..., \pi_{i,n_i}$ contains the state P_i inside.

Let $1 \le j \le n_i$. We denote the label of $\pi_{i,j}$ by $w_{i,j}$. In order to show $w_i \in (T_{\Delta(\theta(\pi))+1, h}(P_i, P_i))^*$, we show $w_{i,j} \in T_{\Delta(\theta(\pi))+1, h}(P_i, P_i)$. We rename the first (resp. last) state of $\pi_{i,j}$ to q_F (resp. q_F) and obtain a path which we call $\tilde{\pi}_{i,j}$. Clearly, $\tilde{\pi}_{i,j}$ is an accepting path in $\mathcal{A}_h(P_i, P_i)$ with the label $w_{i,j}$.

Let d be the weight which $\mathcal{A}_h(P_i, P_i)$ computes on $w_{i,j}$. We have $d \le (\theta(\tilde{\pi}_{i,j})) = \Delta(\theta(\pi_{i,j})) \le \Delta(\theta(\pi))$. By induction, or more precisely, by (4) for $\mathcal{A}_h(P_i, P_i)$, we have $w_{i,j} \in T_{d+1, h}(P_i, P_i)$, and thus, $w_{i,j} \in T_{\Delta(\theta(\pi))+1, h}(P_i, P_i)$. □

Proposition 5.15. *Let $h \in \mathbb{N}$. We can construct an $(h+2)$-nested distance desert automaton $\mathcal{A}$ over Σ such that for every $w \in \Sigma^*$*

$$\Delta_{\mathcal{A}}(w) + 1 = \min\{d \ge 1 \mid w \in \sigma^+(T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}}))\}.$$

In particular, $\mathcal{A}$ *has at most*

$$\left(k^{h+1} + \frac{k^h - 1}{k - 1} + 1\right)(n_\sigma - 2n + 1)$$

states where $k = 2^{|Q_{\overline{L}}|}$.

Proof. The initial point of our construction is the automaton $\mathcal{A}_h(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$ from Proposition 5.14. We denote its mapping by $\Delta_{\mathcal{A}_h}$.

We consider the following mapping $\Delta' : \Sigma^* \to \mathbb{N} \cup \{\infty\}$

$$\Delta'(w) := \min\{\Delta_{\mathcal{A}_h}(u) \mid u \in \Gamma^*,\ w \in \sigma^+(u)\}.$$

If $\Delta'(w) \in \mathbb{N}$ then there exists some $u \in \Gamma^*$ such that $w \in \sigma^+(u)$ and $\Delta_{\mathcal{A}_h}(u) = \Delta'(w)$. By Proposition 5.14(4), we have $u \in T_{\Delta_{\mathcal{A}}(u)+1,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}}) \subseteq T_{\Delta'(w)+1,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$. Thus, $w \in \sigma^+(T_{\Delta'(w)+1,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}}))$.

Conversely, let $d \geq 1$ and assume $w \in \sigma^+(T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}}))$. There is some $u \in T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$ such that $w \in \sigma^+(u)$. By Proposition 5.14 (4), we have $\Delta_{\mathcal{A}_h}(u) + 1 \leq d$, and hence, $\Delta'(w) + 1 \leq d$.

To prove the proposition, we just need an $(h+2)$-nested distance desert automaton $\mathcal{A}$ which computes Δ. We can construct such an automaton by Proposition 3.4. The bound on the number of states follows from Propositions 3.4 and 5.14(2). □

5.8 Decidability and Complexity

In this section, we show the decidability of the relative inclusion star height problem and we prove the complexity bounds stated in Section 3.2.

Given $h \in \mathbb{N}$, an algorithm can decide whether $\mathsf{sh}(K_1, K_2, m, \sigma) \leq h$ as follows.

At first, the algorithm decides by Proposition 5.9 whether $\mathsf{sh}(K_1, K_2, m, \sigma)$ has a solution. More precisely, it constructs the automaton $\mathcal{A}_L$ which recognizes $L = \{w \in \Gamma^* \mid \sigma(w) \subseteq K_2\}$. From $\mathcal{A}_L$, it constructs an automaton which recognizes

$\sigma(L)$ and decides whether $K_1 \subseteq \sigma(L)$. If $K_1 \nsubseteq \sigma(L)$, then the algorithm answers "no".

If $K_1 \subseteq \sigma(L)$, then the algorithm constructs $\mathcal{A}_{\overline{L}}$. From $\mathcal{A}_{\overline{L}}$, it constructs the automaton $\mathcal{A}$ in Proposition 5.15. Then, it decides by Theorem 3.3 whether $\mathcal{A}$ is limited on K_1. If so, the algorithm answers "yes", otherwise the algorithm answers "no".

Assume $\mathrm{sh}(K_1, K_2, m, \sigma) \leq h$. By Proposition 5.13, there is some $d \in \mathbb{N}$ such that $K_1 \subseteq \sigma^+(T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}}))$. By Proposition 5.15, the output of $\mathcal{A}$ on words in K_1 is less than d, i.e., $\mathcal{A}$ is limited on K_1.

Conversely, assume that $\mathcal{A}$ is limited on K_1 and let d be the largest output of $\mathcal{A}$ on K_1. We have $d \in \mathbb{N}$ since $K_1 \subseteq \sigma(L) = L(\mathcal{A})$. From Proposition 5.15, it follows $K_1 \subseteq \sigma^+(T_{d,h}(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}}))$, and by Proposition 5.13, $\mathrm{sh}(K_1, K_2, m, \sigma) \leq h$.

The reader should be aware that $\mathcal{A}$ might be limited even if (K_1, K_2, m, σ) has no solution. Just consider the extremal case that $L = \emptyset$ but $K_1 \neq \emptyset$. Then, (K_1, K_2, m, σ) has no solution. However, $\mathcal{A}$ is limited on K_1 since $\mathcal{A}$ does not accept any word.

5.8.1. On the Relative Inclusion Star Height Problem

To prove the bounds on the space complexity of the relative inclusion star height problem shown in Table 1 in Section 3.2.1, we summarize the results from Section 5 in the following table:

In the lines of the table we consider the same cases as in Table 1.

Table 5

	σ	$\lvert Q_L \rvert$	$\lvert Q_{\overline{L}} \rvert$	$A_h(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$
K_2	sing.	n_2	2^{n_2}	$2^{h2^{n_2}}$
	arb.	$2^{2^{n_2}}$	2^{n_2}	$2^{h2^{n_2}}$
$\Sigma^* \setminus K_2$	arb.	2^{n_2}	n_2	2^{hn_2}
both	sing.	n_2	n_2	2^{hn_2}

In the column $|Q_L|$ resp. $|Q_{\overline{L}}|$, we state the bounds on the number of states of $\mathcal{A}_L$ resp. $\mathcal{A}_{\overline{L}}$ as shown in Propositions 5.6 and 5.7. In the case "both sing.", we just choose the minimum from the more general cases. If (K_1, K_2, m, σ) has a solution r, then $\mathsf{sh}(r) \leq \mathsf{sh}(L)$. From any proof of Kleene's theorem, we get $\mathsf{sh}(L) \leq |Q_L|$. Hence, the entries in the column "bound" in Table 1 are the entries in column $|Q_L|$ in Table 5.

According to Proposition 5.9, we can decide in $\mathcal{O}(n_1 \cdot |Q_L| \cdot (n_\sigma - 2m + 1))$ space whether (K_1, K_2, m, σ) has a solution. We can estimate $(n_\sigma - 2m + 1)$ by n_σ. In this way, we achieve the entries in the column "existence" in Table 1.

The column "$A_h(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$" gives, up to a constant factor, an upper bound to the number of states of the automaton $A_h(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$ according to Proposition 5.14(2). We have to multiply this bound by n_σ to get an upper bound for the number of states of $\mathcal{A}$ in Proposition 5.15. Then, we multiply the bound by n_1 (the number of states of $\mathcal{A}_1$) to decide whether $\mathcal{A}$ is limited on K_1 (cf. Theorem 3.3). In this way, we achieve the entries in the column "$\mathsf{sh}(K_1, K_2, m, \sigma) \leq h$" in Table 1.

If h is larger than or equal to the entry in the column "bound", then $\mathsf{sh}(K_1, K_2, m, \sigma) \leq h$ iff (K_1, K_2, m, σ) has a solution. Thus, we can assume that h is less than the entry in the column "bound" in our analysis of the space complexity of the test whether "$\mathsf{sh}(K_1, K_2, m, \sigma) \leq h$".

Consequently, we can absorb the factor h into $2^{\mathcal{O}(n_2)}$ in the line "K_1 sing." as follows: $h2^{n_2} \leq n_2 2^{n_2} = 2^{\mathrm{ld}(n_2)+n_2} \in 2^{\mathcal{O}(n_2)}$. In the other three lines, such an absorption just worsens the bounds.

We already explained the entries in the column "$\mathsf{sh}(K_1, K_2, m, \sigma) = ?$" in Section 3.2.1.

5.8.2. On the Relative Star Height Problem

We show the complexity bounds for the relative star height problem given in Table 2.

The entries in Table 6 are essentially taken from Table 5. The entries in line “both arb.” Are taken from line “$\Sigma^* \setminus K_2$ arb.” in Table 5.

As for the relative inclusion star height problem, the complexity to decide the existence of a solution is the product of $|Q_L|$ and nn_σ. In lines 2 and 4 in the column “existence” in Table 2, the factor n is absorbed by $2^{\mathcal{O}(n)}$ resp. $2^{2^{\mathcal{O}(n)}}$.

Table 6

	σ	$\lvert Q_L \rvert$	$\lvert Q_{\overline{L}} \rvert$	$A_h(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$
K	sing.	N	2^n	2^{h2^n}
	arb.	2^{2^n}	2^n	$2^{h2^{n2}}$
both	arb.	n	n	2^{hn}
	sing.	2^n	n	2^{hn}

Since $K_1 = K_2$, the automaton $\mathcal{A}$ in Proposition 5.15 recognizes $K \setminus \{\varepsilon\}$. Hence, the algorithm has just to decide whether $\mathcal{A}$ is limited rather than whether $\mathcal{A}$ is limited on K. Consequently, we can omit the factor n_1 in the complexity in the two right columns. Hence, the space complexity of the problem to decide “$\mathsf{sh}(K, m, \sigma) \leq h$” is determined by the number of states of $\mathcal{A}$ in Proposition 5.15, i.e., the product of the number of states of $A_h(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$ and n_σ.

5.8.3. On the Inclusion Star Height Problem

Let (K_1, K_2) be an instance of the relative inclusion star height problem. To consider (K_1, K_2) as an instance of the relative inclusion star height problem, we set $m := |\Sigma|$. We can freely assume $\Gamma = \Sigma$ and set $\sigma(b) := \{b\}$ for every $b \in \Gamma$.

Table 7

	$\lvert Q_L \rvert$	$\lvert Q_{\overline{L}} \rvert$	$A_h(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$
K_2	n_2	2^{n_2}	$2^{h2^{n_2}}$
$\Sigma^* \setminus K_2$	2^{n_2}	n_2	2^{hn_2}
both	n_2	n_2	2^{hn_2}

Since $L = K_2$ in this approach, we can use the automaton $\mathcal{A}_2$ resp. its complementation to construct $\mathcal{A}_L$ and $\mathcal{A}_{\overline{L}}$.

In our approach to the relative inclusion star height problem, we replaced transitions by automata which recognize $\sigma^+(b)$ for some $b \in \Gamma$. The factor $(n_\sigma - 2m + 1)$ in Proposition 5.15 arose due to this replacement. For the inclusion star height problem, we do not need this replacement. Indeed, the factor $(n_\sigma - 2m + 1)$ reduces to 1 since $n_\sigma = 2|\Sigma|$. Consequently, the space complexity to decide $\mathsf{sh}(K_1, K_2) \leq h$ is the product of the number of states of $\mathcal{A}_h(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$ and n_1.

5.8.4. On the Star Height Problem

For a summary, we can essentially use Table 7 by setting $n := n_2$.

As for the relative star height problem, we have to decide whether $\mathcal{A}$ in Proposition 5.15 is limited rather than whether $\mathcal{A}$ is limited on K_1. Hence, the space complexity to decide whether $\mathsf{sh}(K) \leq h$ is polynomial in the number of states of $\mathcal{A}_h(I_{\overline{L}}, Q_{\overline{L}} \setminus F_{\overline{L}})$.

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References

[1] S. Bala, Regular language matching and other decidable cases of the satisfiability problem for constraints between regular open terms, V. Diekert and M. Habib, eds., STACS'04 Proceedings, volume 2996 of LNCS, Springer-Verlag, Berlin (2004), 596-607.

[2] S. Bala, Complexity of regular language matching and other decidable cases of the satisfiability problem for constraints between regular open terms, Theory of Computing Systems, special issue of selected best papers from STACS 2004, 39(1) (2006), 137-163.

[3] J. Berstel, Transductions and Context-Free Languages, B. G. Teubner, Stuttgart 1979.

[4] R. S. Cohen, Star height of certain families of regular events, Journal of Computer and System Sciences 4(2) (1970), 81-297.

[5] L. C. Eggan, Transition graphs and the star height of regular events, Michigan Math. Journal 10 (1963), 385-397.

[6] S. Eilenberg, Automata, Languages, and Machines, Vol. A, Academic Press, New York, 1974.

[7] K. Hashiguchi, A decision procedure for the order of regular events, Theoretical Computer Science 8 (1979), 69-72.

[8] K. Hashiguchi, Limitedness theorem on finite automata with distance functions, Journal of Computer and System Sciences 24 (1982), 233-244.

[9] K. Hashiguchi, Regular languages of star height one, Information and Control 53 (1982), 199-210.

[10] K. Hashiguchi, Representation theorems of regular languages, Journal of Computer and System Sciences 27(1) (1983), 101-115.

[11] K. Hashiguchi, Algorithms for determining relative star height and star height, Information and Computation 78 (1988), 124-169.

[12] K. Hashiguchi, Algorithms for determining relative inclusion star height and inclusion star height, Theoretical Computer Science 91 (1991), 85-100.

[13] D. Kirsten, Desert automata and the finite substitution problem, V. Diekert and M. Habib, eds., STACS'04 Proceedings, volume 2996 of LNCS, Springer-Verlag, Berlin (2004), 305-316.

[14] D. Kirsten, Distance desert automata and the star height problem. R.A.I.R.O. – Informatique Théorique et Applications, special issue of selected best papers from FoSSaCS 2004, 39(3) (2005), 455-509.

[15] D. Kirsten, A Burnside approach to the finite substitution problem, Theory of Computing Systems, special issue of selected best papers from STACS 2004, 39(1) (2006), 15-50.

[16] D. Kirsten. Distance desert automata and star height substitutions, Habilitationsschrift, Universität Leipzig, Fakultät für Mathematik und Informatik, 2006.

[17] S. Lombardy, Approche structurelle de quelques problèmes de la théorie des automates, Ph.D Thesis, École nationale supérieure des télécommunications, Paris, 2001.

[18] R. McNaughton, The loop complexity of pure-group events, Information and Control 11 (1967), 167-176.

[19] J. Sakarovitch, Eléments de théorie des automates, Vuibert, 2003.

[20] J. Sakarovitch, Elements of Automata Theory, Encyclopedia of Mathematics and Applications, Cambridge University Press, 2009.

[21] I. Simon, Limited subsets of a free monoid, Proceedings of the 19th IEEE Annual Symposium on Foundations of Computer Science, IEEE Computer Society Press, Long Beach, CA, (1978) 143-150.

[22] S. Yu. Regular Languages, G. Rozenberg and A. Salomaa, eds., Handbook of Formal Languages, Word, Language, Grammar, Springer-Verlag, Berlin 1 (1997), 41-110.

[23] P. A. Abdulla, P. Krcal and W. Yi, R-automata, F. van Breugel and M. Chechik, eds., CONCUR'08 Proceedings, Vol. 5201 of LNCS, Springer-Verlag, Berlin, 2008, pp. 67-81.

[24] P. A. Abdulla, P. Krcal and W. Yi, Universality of R-automata with value copying, INFINITY'08 Proceedings, 2008.

[25] T. Colcombet and C. Löding, The nesting-depth of disjunctive mu-calculus for tree languages and the limitedness problem, M. Kaminski and S. Martini, eds., CSL'08 Proceedings, Vol. 5213 of LNCS, Springer-Verlag, Berlin, 2008, pp. 416-430.

[26] T. Colcombet and C. Löding, The non-determinstic Mostowski hierarchy and distance-parity automata, I. Damgard et al., eds., ICALP'08 Proceedings, Vol. 5126 of LNCS, Springer-Verlag, Berlin, 2008, pp. 398-409.

[27] T. Colcombet, The theory of stabilization monoids and regular cost functions, S. Albers et al., eds., ICALP'09 Proceedings, Vol. 5556 of LNCS, Springer-Verlag, Berlin, 2009, pp. 139-150.