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FUNCTIONS WITH πg-CLOSED GRAPHS

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Abstract

In this paper, the concepts of πg-closed graphs for functions between topological spaces are introduced with the help of πg-open sets. Some properties of functions with a πg-closed graph have been obtained.

1. Introduction

Closed graph notion is now an active area of research and a large number of topologists have established its far-reaching effect on different concepts of point set topology. In 1969, Long [7] studied the properties of functions with closed graph in great detail. In 1983, Dube et. al [2] introduced the notion of semi-closed graph utilizing semi-open sets introduced by Levine [6]. Bandyopadhyay and Bhattacharyya [8] studied the notion of pre-closed graph utilizing pre-open sets. Dontchev and Noiri [1] have developed the concept Quasi Normal spaces and

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πg-closed sets in topological spaces. In this paper, we introduce πg-closed graphs with the aid of πg-open sets. Some properties of functions with a πg-closed graph have been discussed.

2. Preliminaries

Throughout the present paper (X, τ), (Y, σ) (or simply X, Y) will always denote topological spaces on which no separations axioms are assumed unless explicitly stated. If A is a subset of a space (X, τ), then the closure of A (resp. interior of A) is denoted by $cl(A)$ (resp. $\mathrm{int}(A)$). A subset A is said to be *regular open* (resp. *regular closed*) if $A = \mathrm{intcl}(A)$ (resp. $A = \mathrm{clint}(A)$). The finite union of regular open sets is said to be π-*open*.

Definition 2.1. A subset A of (X, τ) is called

(1) πg-*closed* [1] if $cl(A) \subset U$ whenever $A \subset U$ and U is π-open.

(2) πg-*open* [1] if X-A is πg-closed.

(3) πg-*clopen* if A is both πg-open and πg-closed.

The family of all πg-open sets containing x (resp. πg-closed sets) is denoted by $\pi GO(X, x)$ (resp. $\pi GC(X, x)$).

Definition 2.2. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(1) πg-*continuous* [3] (resp. πg-*irresolute*) if the inverse image of every open (resp. πg-open) subset of Y is πg-open in X.

(2) *Contra* πg-*continuous* [3] if the inverse image of every open subset of Y is πg-closed in X.

(3) $M - \pi g$ -open if $f(U)$ is πg-open for all $U \in \pi GO(X)$.

Definition 2.3 [4]**.** Let $f : (X, \tau) \to (Y, \sigma)$ be any function. Then the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the *graph* of *f*.

Definition 2.4 [4]**.** Let X, Y be topological spaces. A mapping $f(X, \tau) \to$

(Y, σ) is said to have a *closed graph* if its graph $G(f)$ is closed in the product space $X \times Y$.

Lemma 2.5 [4]. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be given. Then* $G(f)$ *is closed iff for each* $(x, y) \in X \times Y - G(f)$ *there exists* $U \in \sum(x)$ *in* X *and* $V \in \sum(y)$ *in* Y *such that* $f(U) \bigcap V = \phi$.

Definition 2.6. A space X is called

(1) $\pi g - T_1$ if for $x, y \in X$ such that $x \neq y$ there exist a πg-open set containing x but not y and a πg-open set containing y but not x.

(2) $\pi g - T_2$ if for $x, y \in X$ such that $x \neq y$ there exist $U \in \pi GO(X, x)$, $V \in \pi GO(Y, y)$ such that $U \bigcap V = \phi$.

(3) π*GO-compact* if every πg-open cover of X admits a finite subcover.

(4) π*g-connected* if X cannot be expressed as the disjoint union of two πg-open sets.

3. πg-closed Graphs

Definition 3.1. For a function $f : (X, \tau) \to (Y, \sigma)$, the graph $G(f)$ is said to be π*g-closed graph* if for each $(x, y) \in X \times Y - G(f)$ there exist $U \in \pi GO(X, x)$, $V \in \pi GO(Y, y)$ such that $(U \times V) \bigcap G(f) = \phi$.

Lemma 3.2. *The function* $f : (X, \tau) \to (Y, \sigma)$ *has a* π*g-closed graph iff for each* $(x, y) \in X \times Y - G(f)$ *there exist* $U \in \pi GO(X, x)$, $V \in \pi GO(Y, y)$ *such that* $f(U) \bigcap V = \phi$.

Proof. It follows from definition and the fact that for any subsets $U \subset X$ and $V \subset Y$, $(U \times V) \bigcap G(f) = \phi$ iff $f(U) \bigcap V = \phi$.

Theorem 3.3. *Every closed graph is* π*g-closed graph*.

Proof. Straightforward.

Converse of the above is not true as seen in the following example.

Example 3.4. Let $X = \{a, b\}$, $Y = \{a, b, c, d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}\}$ and $\sigma = \{\phi, Y, \{c, d\}\}$, respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be the mapping defined by $f(a) = a$, $f(b) = b$. Then $G(f)$ is πg-closed but not closed.

Remark 3.5. Functions having πg-closed graph need not be πg-continuous.

Example 3.6. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\phi, X, \{c, d\}\}$ and $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then $G(f)$ is πg-closed but f is not πg-continuous.

Remark 3.7. A πg-continuous function need not have a πg-closed graph as shown by the following example.

Example 3.8. Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\phi, X, \{a, b\}\}$ and $f : (X, \tau) \to (X, \sigma)$ be the mapping defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is πg-continuous but $G(f)$ is not πg-closed.

Remark 3.9. Examples 3.6 and 3.8 show that πg-closed graph and πg-continuous function are independent concepts.

Theorem 3.10. *Let $f : (X, \tau) \to (Y, \sigma)$ be πg-irresolute surjection, where X is an arbitrary topological space and Y is $\pi g - T_2$. Then $G(f)$ is πg-closed.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Since Y is $\pi g - T_2$ there exist πg-open sets U, V in Y such that $f(x) \in U$, $y \in V$ and $U \cap V = \phi$. Since f is πg-irresolute, $W = f^{-1}(U) \in \pi GO(X, x)$. Hence $f(W) = f(f^{-1}(U)) \subset U$ implies $f(W) \cap V = \phi$. Hence by Lemma 3.2, $G(f)$ is πg-closed.

Theorem 3.11. *Let $f : (X, \tau) \to (Y, \sigma)$ be πg-continuous surjection, where X is an arbitrary topological space and Y is T_2. Then $G(f)$ is πg-closed.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Since Y is T_2, there exist open sets U and V containing $f(x)$ and y, respectively such that $U \cap V = \phi$. Since f is πg-continuous, $f^{-1}(U) = W \in \pi GO(X, x)$. Since f is surjection, $f(W) = f(f^{-1}(U)) \subset U$. Hence $f(W) \cap V = \phi$. By Lemma 3.2, $G(f)$ is πg-closed.

Remark 3.12. From Example 3.8, we find that the condition πg-irresolute in Theorem 3.10 cannot be replaced by πg-continuous.

Theorem 3.13. *Let $f : (X, \tau) \to (Y, \sigma)$ be any surjection with $G(f)$ πg-closed. Then Y is $\pi g - T_1$.*

Proof. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is surjective, there exist $x_1 \in X$ such that $f(x_1) = y_2$. Now $(x_1, y_1) \in X \times Y - G(f)$. Since $G(f)$ is πg-closed, there exist a πg-open set U_1 containing x_1 and a πg-open set V_1 containing y_1 such that $f(U_1) \cap V_1 = \phi$. Now $x_1 \in U_1 \Rightarrow f(x_1) = y_2 \in f(U_1) \cdot y_2 \in f(U_1)$ and $f(U_1) \cap V_1 = \phi \Rightarrow y_2 \notin V_1$. Again, since f is surjective, there exist a point $x_2 \in X$ such that $f(x_2) = y_1$. Now $(x_2, y_2) \in X \times Y - G(f)$. Since $G(f)$ is πg-closed, there exist $U_2 \in \pi GO(X, x_2)$ and $V_2 \in \pi GO(Y, y_2)$ such that $f(U_2) \cap V_2 = \phi$. Now $x_2 \in U_2 \Rightarrow f(x_2) = y_1 \in f(U_2)$. Now $y_1 \in f(U_2)$ and $f(U_2) \cap V_2 = \phi \Rightarrow y_1 \notin V_2$. Thus we obtain sets $V_1, V_2 \in \pi GO(Y)$ such that $y_1 \in V_1$ but $y_2 \notin V_1$ while $y_2 \in V_2$, $y_1 \notin V_2$. Hence Y is $\pi g - T_1$.

Theorem 3.14. *Let $f : (X, \tau) \to (Y, \sigma)$ be any $M - \pi g$ -open surjection with $G(f)$ πg-closed. Then Y is $\pi g - T_2$.*

Proof. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is surjective, there exist $x_1 \in X$ such that $f(x_1) = y_2$. Then $(x_1, y_1 \in X \times Y) - G(f)$. Since $G(f)$ is πg-closed by Lemma 3.2, there exist $U \in \pi GO(X, x_1)$ and $V \in \pi GO(Y, y_1)$ such that $f(U) \cap V = \phi$. Since f is $M - \pi g$ -open, $f(U)$ is πg-open in Y. Now $x_1 \in U$ $\Rightarrow f(x_1) = y_2 \in f(U)$. Therefore, there exist $V \in \pi GO(Y, y_1)$ and $f(U) \in \pi GO(Y, y_2)$ such that $f(U) \cap V = \phi$. Hence Y is $\pi g - T_2$ space.

Theorem 3.15. *Let $f : (X, \tau) \to (Y, \sigma)$ be injective with $G(f)$ πg-closed. Then X is $\pi g - T_1$.*

Proof. Let $x_1, x_2 (\neq x_1) \in X$. Since f is injective, $f(x_1) \neq f(x_2)$. Hence $(x_1, f(x_2)) \in X \times Y - G(f)$. Since $G(f)$ is πg-closed by Lemma 3.2 there exist $U \in \pi GO(X, x_1)$ and $V \in \pi GO(Y, f(x_2))$ such that $f(U) \cap V = \phi$. $f(x_2) \in V$ and $f(U) \cap V = \phi \Rightarrow f(x_2) \notin f(U)$ and so $x_2 \notin U$. Similarly for $(x_2, f(x_1)) \in$

$X \times Y - G(f)$ there exist $U_1 \in \pi GO(X, x_2)$ and $V_1 \in \pi GO(Y, f(x_1))$ such that $f(U_1) \cap V_1 = \phi$. Therefore $f(x_1) \notin f(U_1)$ and so $x_1 \notin U_1$. Hence we obtain πg-open sets U and U_1 in X respectively such that $x_1 \in U$ but $x_2 \notin U$ and $x_2 \in U_1$ but $x_1 \notin U_1$. Thus X is $\pi g - T_1$.

Corollary 3.16. *Let $f : (X, \tau) \to (Y, \sigma)$ be bijective and $G(f)$ be πg-closed. Then both X and Y are $\pi g - T_1$.*

Proof. It follows from Theorems 3.13 and 3.15.

Theorem 3.17. *If $f : (X, \tau) \to (Y, \sigma)$ is injective, πg-irresolute with a πg-closed graph, then X is $\pi g - T_2$.*

Proof. Let $x_1, x_2 (\neq x_1) \in X$. Since f is injective, $f(x_1) \neq f(x_2)$. Hence $(x_1, f(x_2)) \in X \times Y - G(f)$. Since $G(f)$ is πg-closed by Lemma 3.2, there exist $U \in \pi GO(X, x_1)$ and $V \in \pi GO(Y, f(x_2))$ such that $f(U) \cap V = \phi$. Hence $U \cap f^{-1}(V) = \phi$. Since f is πg-irresolute, $f^{-1}(V) \in \pi GO(X, x_2)$. Hence there exist πg-open sets U and $f^{-1}(V)$ in X containing x_1 and x_2 respectively such that $U \cap f^{-1}(V) = \phi$. Therefore, X is $\pi g - T_2$.

Corollary 3.18. *If $f : (X, \tau) \to (Y, \sigma)$ is bijective, $M - \pi g$ -open, πg-irresolute and $G(f)$ is πg-closed, then both X and Y are $\pi g - T_2$.*

Proof. Follows from Theorems 3.14 and 3.17.

Definition 3.19. A function $f : (X, \tau) \to (Y, \sigma)$ is sub contra-πg-continuous provided there exist an open base B for the topology on Y such that $f^{-1}(V)$ is πg-closed in X for every $V \in B$.

Theorem 3.20. *If $f : (X, \tau) \to (Y, \sigma)$ is sub contra-πg-continuous function and Y is T_1. Then $G(f)$ is πg-closed.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Let B be an open base for the topology on Y. Since f is sub contra-πg-continuous, $f^{-1}(V)$ is πg-closed in X for every $V \in B$. Since Y is T_1, there exist a $V \in B$ such that $y \in V$ and $f(x) \notin V$. Then $(x, y) \in (X - f^{-1}(V)) \times V \subset X \times Y - G(f)$. Hence $G(f)$ is πg-closed.

Corollary 3.21. *If* $f:(X, \tau) \to (Y, \sigma)$ *is contra-πg-continuous function and* Y *is* T_1, *then* $G(f)$ *is πg-closed.*

Proof. It follows from the fact that every contra-πg-continuous function is sub contra-πg-continuous.

4. πG-connectedness

Definition 4.1. A function $f:(X, \tau) \to (Y, \sigma)$ is said to be *πg-connected* if for every πg-connected set U, $f(U)$ is πg-connected.

Definition 4.2. A topological space X is locally πG-connected if for each $x \in X$ and each $U \in \pi GO(X, x)$ there exist a $V \in \pi GO(X, \tau)$ such that $x \in V \subset U$, where V is πg-connected.

Definition 4.3. Two subsets A and B of a space X are called *πg-separated* iff $A \cap \pi g - cl(B) = \phi$ and $\pi g - cl(A) \cap B = \phi$.

Lemma 4.4. *If* E *is a πg-connected subset of a topological space* X *such that* $E \subset A \cup B$, *where* A *and* B *are πg-separated sets, then either* $E \subset A$ *or* $E \subset B$.

Lemma 4.5. *In a topological space, if* E *is a πg-connected and* F *is any other set such that* $E \subset F \subset \pi g - cl(E)$, *then* F *is πg-connected.*

Proof. Suppose F is not πg-connected. Then F can be written as the disjoint union of non-empty πg-closed sets G and H such that $F = G \cup H$. Since $E \subset F$ and E is πg-connected, $E \subset G$ or $E \subset H$. Let $E \subset G$. Then $\pi g - cl(E) \subset \pi g - cl(G) \Rightarrow \pi g - cl(E) \cap H \subset \pi g - cl(G) \cap H \Rightarrow \pi g - cl(E) \cap H = \phi$. Also, $F \subset \pi g - cl(E) \Rightarrow G \cup H \subset \pi g - cl(E) \Rightarrow H \subset \pi g - cl(E) \Rightarrow H \subset \phi \Rightarrow H = \phi$ which is a contradiction. Hence F is πg-connected.

Theorem 4.6. *If* $f:(X, \tau) \to (Y, \sigma)$ *is πg-connected, injective,* $M - \pi g$ *-open map and* $G(f)$ *is πg-closed, then* X *is* $\pi g - T_2$ *provided it is* T_1 *and locally πg-connected.*

Proof. Let $x_1, x_2 (\neq x_1) \in X$. Since f is injective, $f(x_1) \neq f(x_2)$. Hence $(x_1, f(x_2)) \in X \times Y - G(f)$. Since $G(f)$ is πg-closed by Lemma 3.2, there exist $U \in \pi GO(X, x_1)$ and $V \in \pi GO(Y, f(x_2))$ such that $f(U) \cap V = \phi$. Since X is locally πg-connected there exist a πg-connected set U_1 such that $x \in U_1 \subset U$. Then it follows that $f(U_1) \cap V = \phi$. Since f is $M - \pi g$ -open, $f(U_1)$ is πg-open. Claim: $x_2 \notin \pi g - cl(U_1)$ suppose $x_2 \in \pi g - cl(U_1)$. Since X is T_1, $\{x_2\}$ is a closed set and hence is πg-closed. Thus $U_1 \subset U_1 \cup \{x_2\} \subset \pi g - cl(U_1 \cup \{x_2\}) \subset \pi g - cl(U_1)$. Hence by Lemma 4.5 $U_1 \cup \{x_2\}$ is πg-connected. Since f is πg-connected, $f(U_1 \cup \{x_2\}) = f(U_1) \cap f(\{x_2\})$ is πg-connected in Y which is absurd as $f(U_1)$ and V are πg-open sets such that $f(U_1) \cap V = \phi$, which is a contradiction. Hence $x_2 \notin \pi g - cl(U_1)$. Setting $U_0 = X - \pi g - cl(U_1)$ we find $x_2 \in U_0$. Thus $U_1 \in \pi gO(X, x_1)$ and $U_0 \in \pi gO(X, x_2)$ with $U_1 \cap U_0 = \phi$. Hence X is $\pi g - T_2$.

Lemma 4.7. *Every πg-closed subset of a πGO-compact space is πGO-compact relative to X.*

Proof. Let A be a πg-closed subset of a πGO-compact space (X, τ). Let $\{U_i : i \in \wedge\}$ be a cover of A by πg-open subset of X. So $A \subset \bigcup\{U_i : i \in \wedge\}$ and then $(X - A) \cup \{U_i : i \in \wedge\} = X$. Since X is πGO-compact there exist a finite subset $\wedge_o$ of $\wedge$ such that $(X - A) \cup \{U_i : i \in \wedge_o\} = X$. Then $A \subset \bigcup\{U_i : i \in \wedge_o\}$ and hence A is πGO-compact relative to X.

Theorem 4.8. *If for the function $f : (X, \tau) \to (Y, \sigma)$, where Y is πGO-compact relative to Y, $G(f)$ is πg-closed in $X \times Y$, then f is πg-continuous.*

Proof. Let $x \in X$. Let V be open in Y and $y \in Y - V$. Then $(x, y) \in X \times Y - G(f)$. Since $G(f)$ is πg-closed, there exist $U_y \in \pi GO(X, x)$ and $V_y \in \pi GO(Y, y)$, such that $f(U_y) \cap V_y = \phi$. This holds for every $y \in Y - V$. Clearly $l = \{V_y : y \in Y - V\}$ is a cover of Y-V by πg-open sets. Now Y is πGO-compact Y-V is πGO-closed. Hence by Lemma 4.7 Y-V is πGO-compact relatively to Y. So l has a finite sub family $\{V_{yi} : i = 1 \cdots n\}$ such that $Y - V \subset \bigcup_{i=1}^{n} V_{yi}$. Let $\{U_{yi} : i = 1 \cdots n\}$ be the corresponding sets of $\pi GO(X, x)$ satisfying $f(U_{yi}) \cup V_{yi} = \phi$. Set $U = \bigcap_{i=1}^{n} U_{yi}$. Now, $U \in \pi GO(X)$. If $\alpha \in U$, then $f(\alpha) \in V_{yi}$, for all $i = 1 \cdots n$. This implies $f(\alpha) \notin \bigcup V_{yi}$. So that $f(\alpha) \notin Y - V$ and hence $f(\alpha) \in V$. Since α is arbitrary, it follows that $f(U) \subset V$ which implies f is πg-continuous.

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