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A REPRESENTATION THEORETIC DEFINITION OF BRANDT MATRIX ON A CERTAIN SPACE OF MODULAR FORMS

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Abstract

A representation theoretic definition of Brandt matrix is given and we prove that this definition is equivalent to the old one.

1. Introduction

Let N be a natural number and $\Gamma_0(N)$ be the congruence modular group of level N, that is, $\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) | c \equiv 0 \bmod N\right\}$. There is a close connection between the theory of modular forms of weight $k \geq 2$ on $\Gamma_0(N)$ and the arithmetical theory of a rational quaternion algebra. An order M of a quaternion algebra A over a local field k is called *primitive* if it satisfies one of the following conditions. If A is a division algebra, then M contains the full ring of integers of a quadratic extension field of k. If A is isomorphic to $\mathrm{Mat}_{2\times 2}(k)$, then M contains a subring which is isomorphic either to $\mathcal{O} \oplus \mathcal{O}$, where $\mathcal{O}$ is the ring of integers in k or to the full ring of integers in a quadratic extension field of k. In [4], the subspace

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of modular forms of weight 2 generated by theta series associated with certain orders was studied. However, he studied the subspace of cusp forms. In this paper, we investigate the properties of representation theoretic definition of Brandt matrices and Eisenstein series in the subspace of modular forms treated in [4].

2. Hecke Ring

Let A be a quaternion algebra over a number field k ramified precisely at a prime q and ∞. For simplicity, we restrict to the case which interest us at present, namely, fix an order $\mathcal{O}$ in A of level $N' = (q;\, L(p),\, \nu(p))$ [3]. Let J_A denote the idele group of A:

$$J_A = \left\{ \hat{a} = (a_l) \in \prod_l A_l^{\times} \mid a_l \in \mathcal{O}_l^{\times} \text{ for almost all } l \right\},$$

where the product is over all l finite and infinite.

Let $U = U(\mathcal{O}) = \{\hat{u} = (u_l) \in J_A \mid u_l \in \mathcal{O}_l^{\times} \text{ for all } l < \infty\}$. Since $\hat{a}U\hat{a}^{-1}$ is commensurable with U for all $\hat{a} \in J_A$, Hecke ring $H(U, J_A)$ can be defined as the free $\mathbb{Z}$-module generated by all double cosets $U\hat{a}U$, $\hat{a} \in J_A$ with multiplication defined as in [5].

Let $J_{\mathbb{Q}}$ denote the idele group of $\mathbb{Q}$ and put $U(\mathbb{Z}) = \{\hat{u} = (u_l) \in J_{\mathbb{Q}} \mid u_l \in \mathbb{Z}_l^{\times}$ for all $l < \infty\}$. The reduced norm $N : A \to \mathbb{Q}$ induces the reduced norm $N : J_A \to J_{\mathbb{Q}}$. For a positive integer n, we denote by $T(n)$ the element of $H(U, J_A)$ which is the sum of all double cosets $U\hat{a}U$ such that the left ideal $\mathcal{O}\hat{a}$ is integral and of norm n, i.e., such that $a_l \in \mathcal{O}_l$ for all $l < \infty$ and $N(\hat{a}) \in nU(\mathbb{Z})$, where $\hat{a} = (a_l)$.

Let $D_N = D_N(\mathcal{O}) = \{\hat{a} = (a_l) \in J_A \mid a_l \in \mathcal{O}_l, \forall l \mid N\}$ and let $H(U, D_N)$ denote the subring of $H(U, J_A)$ generated by all $U\hat{a}U$ with $\hat{a} \in D_N$. Note that $T(n) \in H(U, D_N)$.

It is well known that

$$J_A = \bigcup_{\lambda=1}^{h} U\hat{x}_{\lambda}A^{\times},$$

where h is the class number of $\mathcal{O}$.

2.1. The double coset representative $\hat{x}_\lambda = (x_{\lambda l})$ can be chosen so that $x_{\lambda l} \in \mathcal{O}_l^\times$ for all $l \mid N$. For $\lambda = 1$ to h, let $I_\lambda = \mathcal{O}\hat{x}_\lambda$, $\mathcal{O}_\lambda = \hat{x}_\lambda^{-1}\mathcal{O}\hat{x}_\lambda$, $U_\lambda = \hat{x}_\lambda^{-1}U\hat{x}_\lambda$ and $U_\lambda = U_\lambda \bigcap A^\times$. Then I_λ, $\lambda = 1, \ldots, h$ are representatives of all the distinct left $\mathcal{O}$ -ideal classes, $\mathcal{O}_\lambda$ is the right order of I_λ and $U_\lambda = \mathcal{O}_\lambda^\times$ is the unit group of $\mathcal{O}_\lambda$.

Fix an isomorphism $A_\infty \otimes \mathbb{C} = (A_\infty \otimes_{\mathbb{Q}} \mathbb{R}) \otimes \mathbb{C} \simeq \mathrm{Mat}_2(\mathbb{C})$ which gives a natural embedding of $A_\infty^\times$ into $GL_2(\mathbb{C})$. Let φ denote the projection $D_N \to A_\infty^\times$, the above embedding $A_\infty^\times \subset GL_2(\mathbb{C})$.

Denote by $M_2(\mathcal{O})$ the complex vector space of all continuous functions $f(x)$ on J_A, taking values in $F = \mathbb{C}$, which satisfy

$$f(ux\alpha) = f(x)$$

for all $u \in U$, $x \in J_A$, and $\alpha \in A^\times$.

We define a representation of the Hecke ring $H(U, D_N)$ on $M_2(\mathcal{O})$ as follows. For a double coset $UyU \in H(U, D_N)$, let $UyU = \bigcup_i Uy_i$ be its decomposition into disjoint right cosets.

We denote by $\rho(UyU)$ the operator defined by

$$\rho(UyU) f(x) = g(x), \quad \text{where } g(x) = \sum_i f(y_i x).$$

It is easy to see that $\rho(UyU)$ is an endomorphism on all of $H(U, D_N)$.

Lemma 2.1. *The structure of* $M_2(\mathcal{O})$ *is independent of the particular choice of* $\mathcal{O}$.

Proof. Let $\mathcal{O}'$ be another order in A having the same level as $\mathcal{O}$. Since $\mathcal{O}'_l$ is isomorphic to $\mathcal{O}_l$ for all $l < \infty$, let $\mathcal{O}' = \hat{\beta}\mathcal{O}\hat{\beta}^{-1}$, where $\hat{\beta} = (b_l) \in J_A$ with $b_\infty = 1$. Then $U(\mathcal{O}') = \hat{\beta}U(\mathcal{O})\hat{\beta}^{-1}$ and $D_N(\mathcal{O}') = \hat{\beta}D_N(\mathcal{O})\hat{\beta}^{-1}$. The map $\psi : f(x) \to f(\hat{\beta}x)$ induces a complex vector space isomorphism of $M_2(\mathcal{O}')$ onto $M_2(\mathcal{O}')$. Then

$M_2(\mathcal{O}')$ is isomorphic to $M_k(\mathcal{O})$ as Hecke modules. Clearly, the map $U(\mathcal{O})yU(\mathcal{O}) \to U(\mathcal{O}')\hat{\beta}y\hat{\beta}^{-1}U(\mathcal{O}')$ induces an isomorphism of $H(U(\mathcal{O}), D_N(\mathcal{O}))$ onto $H(U(\mathcal{O}'), D_N(\mathcal{O}'))$. It follows by an easy calculation that ψ is in fact an $H(U(\mathcal{O}), D_N(\mathcal{O}))$ module isomorphism. $\square$

3. Brandt Matrices

Elements of $M_2(\mathcal{O})$ are completely determined by their values at $\hat{x}_\lambda$, $\lambda = 1, \ldots, h$. In fact, for $f \in M_2(\mathcal{O})$, put $f_\lambda = f(\hat{x}_\lambda)$. Then the mapping

$$f \to (f_1, \ldots, f_h) \tag{3.1}$$

gives an isomorphism of M into $= F^h = F \times \cdots \times F$.

Using this isomorphism, we consider ρ as giving a representation of $H(U, D_N)$ on F^h. Specifically, for $\xi \in H(U, D_N)$, we can represent $\rho(\xi)$ as the matrix

$$\rho(\xi) = \begin{pmatrix} \rho_{11}(\xi) & \cdots & \rho_{1h}(\xi) \\ \vdots & \vdots & \vdots \\ \rho_{h1}(\xi) & \cdots & \rho_{hh}(\xi) \end{pmatrix},$$

where $\rho_{ij}(\xi)$ is the linear map of ith coordinate, $F_i = (, \ldots, F, \ldots,)$, to jth coordinate, $F_j = (, \ldots, F, \ldots,)$, which is the composition of the canonical injection of F_i into F^h, the inverse of the isomorphism in (3.1), $\rho(\xi)$, the isomorphism in (3.1), and finally the canonical projection of $F_1 \times \cdots \times F_h$. Let

$$p_\lambda = e_\lambda^{-1} \sum_{\gamma \in U_\lambda} \varphi(\gamma), \tag{3.2}$$

where $e_\lambda = |U_\lambda|$.

p_i define a projection of F into F_i. Let i_i denote the canonical injection of F_i into F. For $\xi \in H(U, D_N)$, put

$$\beta_{ij}(\xi) = i_i \circ \rho_{ij}(\xi) \circ p_j.$$

Then $\beta_{ij}(\xi)$ is an endomorphism of F and

$$B(\xi) = \begin{pmatrix} \beta_{11}(\xi) & \cdots & \beta_{1h}(\xi) \\ \vdots & & \vdots \\ \beta_{h1}(\xi) & \cdots & \beta_{hh}(\xi) \end{pmatrix} \tag{3.3}$$

gives an endomorphism of $F^h = F \times \cdots \times F$ which is a matrix representation of $\rho(\xi)$.

If $\xi = T(n) \in H(U, D_N)$, then we next prove that $B(T(n))$ is just the Brandt matrix defined in [4].

Proposition 3.1. *For* $n > 0$, *the matrix* $B(T(n))$ *defined in* (3.3) *is identical to Brandt matrix. We assume that the primitive order* $\mathcal{O}$, *the same set of ideal class representations, the same embedding of* $A^{\times}$ *into* $GL_2(\mathbb{C})$, *and the same basis of F used to define* $B(T(n))$ *and* $B(n)$.

Proof. Let $\xi = UyU \in H(U, D_N)$. Then $\rho_{ij}(\xi)$ is the sum, is over all cosets $U_\lambda \hat{x}_\lambda^{-1} \hat{x}_\mu \alpha$ $(\alpha \in A^{\times})$ contained in $U_\lambda \hat{x}_\lambda^{-1} y \hat{x}_\lambda U_\lambda$. Thus

$$\begin{aligned} \beta_{ij}(\xi) &= i_i \circ \rho_{ij}(\xi) \circ p_j \\ &= \frac{1}{e_\mu} \sum_{\alpha} \sum_{\gamma} 1, \end{aligned}$$

where $\sum_\alpha$ is over all cosets $U_\lambda \hat{x}_\lambda^{-1} \hat{x}_\mu \alpha$ $(\alpha \in A^{\times})$ contained in $U_\lambda \hat{x}_\lambda^{-1} y \hat{x}_\lambda U_\lambda$ and $\sum_\gamma$ is over all $\gamma \in U_\mu$,

$$\beta_{ij}(\xi) = \frac{1}{e_i} \sum_{\eta} 1, \tag{3.4}$$

where the sum is over all $\eta \in A^{\times} \cap \hat{x}_\mu^{-1} U yU \hat{x}_\lambda$. Thus

$$\beta_{ij}(T(n)) = \frac{1}{e_i} \sum_{\xi} \sum_{\eta} 1, \tag{3.5}$$

where the sum $\sum_\xi$ is over all $\xi = UyU$ such that $y_l \in \mathcal{O}_l$ for all $l < \infty$ and $N(y) = n \bmod U(\mathbb{Z})$ and for fixed ξ, the sum $\sum_\eta$ is as in (3.5).

We claim

$$\beta_{ij}(T(n)) = \frac{1}{e_i}\sum_{\eta} 1, \tag{3.6}$$

where the sum is over all $\eta \in I_\mu^{-1} I_\lambda$ with $N(\eta) = nN(I_\lambda)/N(I_\mu)$. We recall that $I_\lambda = \mathcal{O}\hat{x}_\lambda$ and $N(I_\lambda)$ is the unique positive rational number in the coset $n(\hat{x}_\lambda)U(\mathbb{Z})$.

Assume that $\eta \in I_\mu^{-1} I_\lambda$ with $N(\eta) = nN(I_\lambda)/N(I_\mu)$. Then letting $\hat{x}_\lambda = (x_{\lambda_l})$, etc., we have $\eta \in I_\mu^{-1} I_\lambda = \hat{x}_\mu^{-1}\mathcal{O}\hat{x}_\lambda$ so for all $l < \infty$, $\eta = \hat{x}_\mu^{-1} y_l \hat{x}_\lambda$ for some $y_l \in \mathcal{O}_l$, where $N(y_l) = n \bmod \mathbb{Z}_l^\times$. Thus letting $y = (y_l)$, we have $N(y) \in n \bmod U(\mathbb{Z})$ and $\eta \in A^\times \cap \hat{x}_\mu^{-1} U yU \hat{x}_\lambda$.

Conversely, if

$$\eta \in A^\times \cap \hat{x}_\mu^{-1} U yU \hat{x}_\lambda, \tag{3.7}$$

where $y_l \in \mathcal{O}_l$ for all $l < \infty$ and $n(y) = n \bmod U(\mathbb{Z})$, then we have $\eta \in \hat{x}_\mu^{-1}\mathcal{O}\hat{x}_\lambda = I_\mu^{-1} I_\lambda$ with $N(\eta) \equiv N(\hat{x}_\mu^{-1})N(y)N(\hat{x}_\lambda) \equiv nN(I_\lambda)/N(I_\mu) \bmod U(\mathbb{Z})$. Since the double coset $U\eta U$ is uniquely determined by η, we see that the two sums are identical.

Thus, $B_{\lambda\mu}(T(n))$ equals the $\lambda\mu$th the entry of Brandt matrix which shows that the matrices $B(T(n))$ and $B(n)$ are identical. □

4. Eisenstein Series

In this section, we determine the Eisenstein series part $M_2^e(\mathcal{O})$ of $M_2(\mathcal{O})$ and explicitly determine the action of $T(n)$ on this subspace.

Denote by $M_2^e(\mathcal{O})$ the subspace of $M_2(\mathcal{O})$ consisting of those f which factor through the norm, i.e., such that there exists a function $g : J_{\mathbb{Q}}^+ \to F$ with $f(x) = g(N(x))$ for all $x \in J_A$, here $J_{\mathbb{Q}}^+ = N(J_A) = \{y = (y_l) \in J_{\mathbb{Q}} \mid y_\infty > 0\}$.

Any modular form f is determined by its values on a set of representatives for the double cosets $U \backslash J_A / A^{\times}$. Thus, if $f(x) = g(N(x))$, then f is determined by its values on a set of representatives for

$$N(U) \backslash N(J_A)/N(A^{\times}) = J_{\mathbb{Q}}^{+}/N(U)\mathbb{Q}_{+}^{\times} = U(\mathbb{Z})\mathbb{Q}_{+}^{\times}/N(U)\mathbb{Q}_{+}^{\times}. \tag{4.1}$$

Now, $U(\mathbb{Z})\mathbb{Q}_{+}^{\times}/N(U)\mathbb{Q}_{+}^{\times} \simeq \mathbb{Z}_p^{\times}/N(\mathcal{O}_p^{\times})$. In particular, it was shown that the cardinality of $\mathbb{Z}_p^{\times}/N(\mathcal{O}_p^{\times})$ is always 1 or 2 so that $M_2^e(\mathcal{O})$ has dimension at most 2.

Theorem 4.1. *Let $\mathcal{O}$ be a primitive order of level $N' = (q;\, L(p),\, \nu(p))$.*

(1) *$L(p)$ is unramified over $\mathbb{Q}_p$. Then $M_2^e(\mathcal{O})$ is one dimensional and the action of $T(n)$, $(n, qp) = 1$, on $f \in M_2^e(\mathcal{O})$ is given by $T(n) f = \deg T(n) f$.*

(2) *$L(p)$ is ramified over $\mathbb{Q}_p$. Then $M_2^e(\mathcal{O})$ is two dimensional and there exist bases $f_1, f_2 \in M_2^e(\mathcal{O})$ such that for $(n, qp) = 1$, $T(n) f_1 = \deg T(n) f_1$ and $T(n) f_2 = \left(\frac{n}{p}\right) \deg T(n) f_2$.*

Proof. If $L(p)$ is unramified, then $|\mathbb{Z}_p^{\times}/N(\mathcal{O}_p^{\times})| = 1$ and $M_2^e(\mathcal{O})$ has dimension at most 1. Consider the mapping $f : J_A \to \mathbb{C}$ defined as follows: for $\tilde{\alpha} \in J_A$, $N(\tilde{\alpha}) = \tilde{u}a$ for a unique $\tilde{u} = (u_l) \in U(\mathbb{Z})$ and $a \in \mathbb{Q}_{+}^{\times}$. Let $f(\tilde{\alpha}) = (u_p)$. Then it is clear that $M_2^e(\mathcal{O})$ is one dimensional.

Define $g(z)$ by $g(z) = g(N(z)) = f(x)$. Let $(n, N) = 1$ and recall that $(T(n) f)(x) = \sum_i f(y_i x)$, where the sum is over representatives $y_i \in J_A$ of all cosets $U y_i$ such that $\mathcal{O} y_i$ is an integral ideal of norm n. In particular, $N(y_i)/n \in U(\mathbb{Z})$. Also, the number of such y_i is $\deg(T(n))$. Now, let $x \in J_A$ with $N(x) = \tilde{u}a$ for a unique $\tilde{u} \in U(\mathbb{Z})$ and $a \in \mathbb{Q}_{+}^{\times}$. Then

$$\begin{aligned} T(n) f &= \sum_i f(y_i x) = \sum_i g(N(y_i) \cdot \tilde{u} \cdot a) \\ &= \sum_i \left(\frac{N(y_i)}{n}\right) f(x) = \sum_i f(x) \text{ by (4.1)} \\ &= \deg T(n) f(x). \end{aligned}$$

Next, if $L(p)$ is ramified, then $|\mathbb{Z}_p^\times/N(\mathcal{O}_p^\times)| = 2$. Define $\left(\frac{*}{p}\right) : J_A \to \{\pm 1\}$ as follows: for $\tilde{a} \in J_A$, $N(\tilde{a}) = \tilde{u} \cdot a$ for a unique $\tilde{u} = (u_l) \in U(\mathbb{Z})$ and $a \in \mathbb{Q}_+^\times$. Then let $f_1(\tilde{a}) = (u_p)$ and $f_2(\tilde{a}) = \left(\left(\frac{\tilde{a}}{p}\right)\right) f_1(\tilde{a}) = \left(\frac{u_p}{p}(u_p)\right)$. It is easy to check that f_1 and f_2 form a basis for $M_2^e(\mathcal{O})$ so that $M_2^e(\mathcal{O})$ has dimension 2. For the action of $T(n)$, as in the unramified case, let $g_j(N(z)) = f_j(z)$. Assume $(n, N) = 1$,

$$\begin{aligned} T(n) f_j &= \sum_i f_j(y_i x) \\ &= \sum_i g_j\left(\frac{N(y_i)}{n} \cdot \tilde{u} \cdot a \cdot n\right) \\ &= \sum_i g_j\left(\frac{N(y_i)}{n} \cdot \tilde{u}\right) \text{ by (4.1).} \end{aligned}$$

If $j = 1$, then this equals

$$\sum_i \left(\frac{N(y_i)}{n}\right) f_1(x) = \deg(T(n))\, f_1$$

while if $j = 2$, then

$$\sum_i \left(\frac{N(y_i)}{n}\right)\left(\frac{N(y_i)_p/n}{p}\right) f_2(x),$$

where $N(y_i)_p$ denotes the p component of the idele $N(y_i) \in J_{\mathbb{Q}}$. Now, $\mathcal{O}y_i$ is an integral ideal of norm n, where $(n, N) = 1$ so that $y_i = (y_{il}) \in J_A$ with $y_{ip} \in \mathcal{O}_p^\times$. Hence, $N(y_i)_p = N(y_{ip})$ is a residue mod p and

$$\left(\frac{N(y_i)_p/n}{p}\right) = \left(\frac{1/n}{p}\right) = \left(\frac{n}{p}\right).$$

Thus, $T(n) f_2 = \deg T(n)\, f_2(x)$. $\square$

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