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A UNIFIED PRESENTATION OF CERTAIN SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS

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Abstract

In this paper, authors introduce and study a unified class $HS(m, n; q; \alpha; \lambda)$ of harmonic univalent functions in the open unit disk. A number of results are obtained which include the coefficient estimates, sharp distortion theorems and extreme points of functions belonging to the class $HS(m, n; q; \alpha; \lambda)$. Results concerning the convolutions of functions of this class with univalent, harmonic and convex functions in the unit disc and harmonic functions having positive real part are obtained. Relevant connections of the results presented here with various known results are briefly indicated.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be *harmonic* in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are

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analytic in D. We call h the *analytic part* and g the *co-analytic part* of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$, see Clunie and Sheil-Small [2].

Denote by S_H the class of functions $f = h + \overline{g}$ that is harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

Denote by $HS(m, n; q; \alpha; \lambda)$ the class of all functions of the form (1.1) that satisfies the condition

$$\sum_{k=2}^{\infty} (1 - \lambda + \lambda k)(k^m - \alpha k^n) |a_k| + \sum_{k=1}^{\infty} (1 - \lambda + \lambda k)\{k^m - (-1)^q \alpha k^n\} |b_k| \le 1 - \alpha, \tag{1.2}$$

where

$$m \in N, \ n \in N_0, \ m > n, \ q \in \{0, 1\}, \ 0 \le \lambda \le 1 \text{ and } 0 \le \alpha < 1.$$

The class $HS(m, n; q; \alpha; \lambda)$ with $b_1 = 0$ will be denoted by $HS^0(m, n; q; \alpha; \lambda)$.

We note that by specializing the parameters we obtain the following known subclasses which have been studied by various authors.

1. The class $HS(m, n; 0; \alpha; 0) \equiv HS(m, n; \alpha)$ was studied by Dixit and Porwal [3].

2. The class $HS(1, 0; 1; \alpha; \lambda) \equiv T_H(\alpha, \lambda, k)$ was studied by Joshi and Darus [5].

3. The classes $HS(1, 0; 0; \alpha; 0) \equiv HS(\alpha)$ and $HS(2, 1; 0; \alpha; \lambda) \equiv HC(\alpha)$ were studied by Öztürk and Yalcin [6].

4. The classes $HS(1, 0; 0; 0; 0) \equiv HS$ and $HS(2, 1; 0; 0; \lambda) \equiv HC$ were studied by Avci and Zlotkiewicz [1].

If h, g, H, G are of the form (1.1) and if $f(z) = h(z) + \overline{g(z)}$ and $F(z) = H(z) + \overline{G(z)}$, then the convolution of f and F is defined to be the function

$$(f * F)(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k},$$

while the integral convolution is defined by

$$(f \lozenge F)(z) = z + \sum_{k=2}^{\infty} \frac{a_k A_k}{k} z^k + \overline{\sum_{k=1}^{\infty} \frac{b_k B_k}{k} z^k}.$$

2. Main Results

First, we show that the class $HS(m, n; q; \alpha; \lambda)$ consists of sense-preserving and harmonic univalent mappings in *U*.

Theorem 2.1. *The class* $HS(m, n; q; \alpha; \lambda)$ *consists of univalent sense-preserving harmonic mappings*.

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}
\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| \\
&= 1 - \left| \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{z_1 - z_2 + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right| \\
&> 1 - \frac{\sum_{k=1}^{\infty} k |b_k|}{1 - \sum_{k=2}^{\infty} k |a_k|} \\
&\geq 1 - \frac{\sum_{k=1}^{\infty} \frac{(1 - \lambda + \lambda k)\{k^m - (-1)^q \alpha k^n\}}{1 - \alpha} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{(1 - \lambda + \lambda k)(k^m - \alpha k^n)}{1 - \alpha} |a_k|} \\
&\geq 0,
\end{aligned}$$

which proves univalence.

Note that *f* is sense-preserving in *U*, this is because

$$\begin{aligned}
|h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \\
&> 1 - \sum_{k=2}^{\infty} k|a_k| \\
&> 1 - \sum_{k=2}^{\infty} \frac{(1-\lambda+\lambda k)(k^m - \alpha k^n)}{1-\alpha}|a_k| \\
&\geq \sum_{k=1}^{\infty} \frac{(1-\lambda+\lambda k)\{k^m - (-1)^q \alpha k^n\}}{1-\alpha}|b_k| \\
&> \sum_{k=1}^{\infty} k|b_k| \\
&\geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \\
&\geq |g'(z)|. \qquad \square
\end{aligned}$$

In the following theorem, we determine the distortion bounds for the functions of $HS(m, n; q; \alpha; \lambda)$.

Theorem 2.2. *If* $f \in HS(m, n; q; \alpha; \lambda)$, *then*

$$|f(z)| \leq |z|(1+|b_1|) + |z|^2 \left\{ \frac{1-\alpha}{(1+\lambda)(2^m - \alpha 2^n)} - \frac{1-(-1)^q \alpha}{(1+\lambda)(2^m - \alpha 2^n)} |b_1| \right\},$$

and

$$|f(z)| \geq |z|(1-|b_1|) - |z|^2 \left\{ \frac{1-\alpha}{(1+\lambda)(2^m - \alpha 2^n)} - \frac{1-(-1)^q \alpha}{(1+\lambda)(2^m - \alpha 2^n)} |b_1| \right\}.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f \in HS(m, n; q; \alpha; \lambda)$.

Taking the absolute value of f we obtain

$$|f(z)| \leq |z|(1+|b_1|) + \sum_{k=2}^{\infty}(|a_k| + |b_k|)|z|^k$$

$$\leq |z|(1+|b_1|) + |z|^2 \sum_{k=2}^{\infty}(|a_k| + |b_k|)$$

$$= |z|(1+|b_1|) + |z|^2 \frac{1-\alpha}{(1+\lambda)(2^m - \alpha 2^n)} \sum_{k=2}^{\infty} \frac{(1+\lambda)(2^m - \alpha 2^n)}{1-\alpha}(|a_k| + |b_k|)$$

$$\leq |z|(1+|b_1|) + |z|^2 \frac{1-\alpha}{(1+\lambda)(2^m - \alpha 2^n)} \sum_{k=2}^{\infty} \frac{(1+\lambda)(k^m - \alpha k^n)}{1-\alpha}(|a_k| + |b_k|)$$

$$\leq |z|(1+|b_1|) + |z|^2 \frac{1-\alpha}{(1+\lambda)(2^m - \alpha 2^n)} \sum_{k=2}^{\infty} \left\{ \frac{(1+\lambda)(k^m - \alpha k^n)}{1-\alpha} |a_k| \right.$$

$$\left. + \frac{(1+\lambda)\{k^m - (-1)^q \alpha k^n\}}{1-\alpha} |b_k| \right\}$$

$$\leq |z|(1+|b_1|) + |z|^2 \frac{1-\alpha}{(1+\lambda)(2^m - \alpha 2^n)} \left\{ 1 - \frac{1-(-1)^q \alpha}{1-\alpha} |b_1| \right\}$$

$$\leq |z|(1+|b_1|) + |z|^2 \left\{ \frac{1-\alpha}{(1+\lambda)(2^m - \alpha 2^n)} - \frac{1-(-1)^q \alpha}{(1+\lambda)(2^m - \alpha 2^n)} |b_1| \right\}.$$

The following covering result follows from the left hand inequality in Theorem 2.2.

Corollary 2.3. *Let f of the form* (1.1) *be so that* $f \in HS(m, n; q; \alpha; \lambda)$. *Then*

$$\left\{ \omega : |\omega| < \left(\frac{(1+\lambda)(2^m - \alpha 2^n) - (1-\alpha)}{(1+\lambda)(2^m - \alpha 2^n)} \right. \right.$$

$$\left. \left. - \frac{(1+\lambda)(2^m - \alpha 2^n) - \{1-(-1)^q \alpha\}}{(1+\lambda)(2^m - \alpha 2^n)} |b_1| \right) \right\} \subset f(U).$$

Theorem 2.4. *The extreme points of* $HS^0(m, n; q; \alpha; \lambda)$ *are only the functions of the form* $z + a_k z^k$ *or* $z + \overline{b_l z^l}$ *with*

$$|a_k| = \frac{1-\alpha}{(1-\lambda+\lambda k)(k^m - \alpha k^n)}, \quad |b_l| = \frac{1-\alpha}{(1-\lambda+\lambda k)\{l^m - (-1)^q \alpha l^n\}}, \quad 0 \le \alpha < 1.$$

Proof. Suppose that

$$f(z) = z + \sum_{k=2}^{\infty} (a_k z^k + \overline{b_k z^k})$$

is such that

$$\sum_{k=2}^{\infty} \frac{(1-\lambda+\lambda k)(k^m - \alpha k^n)}{1-\alpha} |a_k|$$

$$+ \sum_{k=1}^{\infty} \frac{(1-\lambda+\lambda k)\{k^m - (-1)^q \alpha k^n\}}{1-\alpha} |b_k| < 1, \quad a_k > 0.$$

Then, if $\xi > 0$ is small enough we can replace a_k by $a_k - \xi$, $a_k + \xi$ and we obtain two functions that satisfy the same condition for which one obtains $f(z) = \frac{1}{2}[f_1(z) + f_2(z)]$. Hence, f is not a possible extreme point of $HS^0(m, n; q; \alpha; \lambda)$.

Now, let $f \in HS^0(m, n; q; \alpha; \lambda)$ be such that

$$\sum_{k=2}^{\infty} \frac{(1-\lambda+\lambda k)(k^m - \alpha k^n)}{1-\alpha} |a_k| + \sum_{k=1}^{\infty} \frac{(1-\lambda+\lambda k)\{k^m - (-1)^q \alpha k^n\}}{1-\alpha} |b_k| = 1, \quad (2.1)$$

$a_k \ne 0$, $b_l \ne 0$.

If $\xi > 0$ is small enough and if μ, τ with $|\mu| = |\tau| = 1$ are properly, chosen complex numbers, then leaving all but a_k, b_l coefficients of $f(z)$ unchanged and replacing a_k, b_l by

$$a_k + \xi \frac{1-\alpha}{(1-\lambda+\lambda k)(k^m - \alpha k^n)} \mu, \quad b_l - \xi \frac{1-\alpha}{(1-\lambda+\lambda l)\{(l^m - (-1)^q \alpha l^n)\}} \tau,$$

$$a_k - \xi \frac{1-\alpha}{(1-\lambda+\lambda k)(k^m - \alpha k^n)} \mu, \quad b_l + \xi \frac{1-\alpha}{(1-\lambda+\lambda l)\{(l^m - (-1)^q \alpha l^n)\}},$$

we obtain functions $f_1(z)$, $f_2(z)$ that satisfy (2.1) such that

$$f(z) = \frac{1}{2}[f_1(z) + f_2(z)].$$

In this case f cannot be an extreme point. Thus for

$$|a_k| = \frac{1-\alpha}{(1-\lambda+\lambda k)(k^m - \alpha k^n)},$$

$$|b_l| = \frac{1-\alpha}{(1-\lambda+\lambda l)\{l^m - (-1)^q \alpha l^n\}},$$

$f(z) = z + a_k z^k$ and $f(z) = z + \overline{b_l z^l}$ are extreme points of $HS^0(m, n; q; \alpha; \lambda)$. □

Remark 1.

1. If $q = 0, \lambda = 0,$ then the extreme points of the class $HS^0(m, n, \alpha)$ are obtained.

2. If $m = 1, n = 0, q = 1,$ then the extreme points of the class $T_H(\alpha, \lambda, k)$ are obtained.

3. If $m = 1, n = 0, q = 0, \lambda = 0,$ then the extreme points of the class $HS^0(\alpha)$ are obtained.

4. If $m = 2, n = 1, q = 0, \lambda = 0,$ then the extreme points of the class $HC^0(\alpha)$ are obtained.

Let K_H^0 denote the class of harmonic univalent functions of the form (1.1) with $b_1 = 0$ that map U onto convex domains. It is known [2, Theorem 5.10] that the sharp inequalities $|A_k| \le \frac{k+1}{2}, |B_k| \le \frac{k-1}{2}$ are true. These results will be used in next theorem.

Theorem 2.5. *Suppose that*

$$F(z) = z + \sum_{k=2}^{\infty}(A_k z^k + \overline{B_k z^k})$$

belongs to K_H^0. *Then if* $f \in HS^0(m, n; q; \alpha; \lambda)$, *then* $f * F \in HS^0(m-1, n-1; q; \alpha; \lambda)$, *provided* $n \ge 1$ *and* $f \lozenge F \in HS^0(m, n; q; \alpha; \lambda)$.

Proof. Since $f \in HS^0(m, n; q; \alpha; \lambda)$, we have

$$\sum_{k=2}^{\infty}(1-\lambda+\lambda k)(k^m - \alpha k^n)|a_k| + (1-\lambda+\lambda k)\{k^m - (-1)^q \alpha k^n\}|b_k| \le (1-\alpha). \quad (2.2)$$

Using (2.2), we have

$$\sum_{k=2}^{\infty}(1-\lambda+\lambda k)(k^{m-1} - \alpha k^{n-1})|a_k A_k| + (1-\lambda+\lambda k)(k^{m-1} - (-1)^q \alpha k^{n-1})|b_k B_k|$$

$$= \sum_{k=2}^{\infty}(1-\lambda+\lambda k)(k^m - \alpha k^n)|a_k|\left|\frac{A_k}{k}\right| + (1-\lambda+\lambda k)(k^m - (-1)^q \alpha k^n)|b_k|\left|\frac{B_k}{k}\right|$$

$$\le \sum_{k=2}^{\infty}(1-\lambda+\lambda k)(k^m - \alpha k^n)|a_k| + (1-\lambda+\lambda k)(k^m - (-1)^q \alpha k^n)|b_k|$$

$$\le 1-\alpha.$$

It follows that

$$f * F \in HS^0(m-1, n-1; q; \alpha; \lambda).$$

Next, again using (2.2), we have

$$\sum_{k=2}^{\infty}(1-\lambda+\lambda k)(k^m - \alpha k^n)\left|\frac{a_k A_k}{k}\right| + (1-\lambda+\lambda k)\{k^m - (-1)^q \alpha k^n\}\left|\frac{b_k B_k}{k}\right|$$

$$= \sum_{k=2}^{\infty}(1-\lambda+\lambda k)(k^m - \alpha k^n)|a_k|\left|\frac{A_k}{k}\right| + (1-\lambda+\lambda k)\{k^m - (-1)^q \alpha k^n\}|b_k|\left|\frac{B_k}{k}\right|$$

$$\le \sum_{k=2}^{\infty}(1-\lambda+\lambda k)(k^m - \alpha k^n)|a_k| + (1-\lambda+\lambda k)(k^m - (-1)^q \alpha k^n)|b_k|$$

$$\le 1-\alpha.$$

Thus, we have

$$f \lozenge F \in HS^0(m, n; q; \alpha; \lambda).$$

This completes the proof of Theorem 2.5. $\square$

Let S denote the class of analytic univalent functions of the form $F(z) = z + \sum_{k=2}^{\infty} A_k z^k$. It is well known that the sharp inequality $|A_k| \leq k$ is true. It is required in next theorem.

Theorem 2.6. *If* $f \in HS^0(m, n; q; \alpha; \lambda)$ *and* $F \in S$, *then for*

$$|\varepsilon| \leq 1, \; f * (F + \varepsilon\overline{F}) \in HS^0(m-1, n-1; q; \alpha; \lambda), \quad \text{if } n \geq 1.$$

Proof. The proof of this theorem is much akin to that of Theorem 2.5, therefore we omit the details involved.

Let P_H^0 denote the class of functions F complex and harmonic in U, $f = h + \overline{g}$ such that $Re\; f(z) > 0$, $z \in U$ and $H(z) = 1 + \sum_{k=1}^{\infty} A_k z^k$, $G(z) = \sum_{k=2}^{\infty} B_k z^k$.

It is known [4, Theorem 3] that the sharp inequalities $|A_k| \leq k+1$, $|B_k| \leq k-1$ are true.

Theorem 2.7. *Suppose that*

$$F(z) = 1 + \sum_{k=1}^{\infty} (A_k z^k + \overline{B_k z^k})$$

belongs to P_H^0. *Then* $f \in HS^0(m, n; q; \alpha; \lambda)$ *and for* $\frac{3}{2} \leq |A_1| \leq 2$, $\frac{1}{A_1} f * F \in HS^0(m, n; q; \alpha; \lambda)$ *if* $n \geq 1$ *and* $\frac{1}{A_1} f \lozenge F \in HS^0(m, n; q; \alpha; \lambda)$.

Proof. The proof of this theorem is similar to that of Theorem 2.5, therefore, we omit details involved. □

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