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ON *M*-SYSTEMS IN ORDERED AG-GROUPOIDS

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Abstract

In this paper, we study ideals, *M*-systems, *N*-systems and *I*-systems of ordered AG-groupoids. We prove that if L is a left ideal of an ordered AG-groupoid with left identity, then L is quasi-prime if and only if $S\backslash L$ is an *M*-system; L is quasi-semiprime if and only if $S\backslash L$ is an *N*-system and L is quasi-irreducible if and only if $S\backslash L$ is an *I*-system. Moreover, we show that every quasi-semiprime left ideal of an ordered AG-groupoid with left identity is an intersection of some quasi-prime left ideals.

1. Introduction and Preliminaries

Abel-Grassmann's groupoid, abbreviated as AG-groupoid, is a groupoid whose

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elements satisfy the left invertive law: $(ab)c = (cb)a$. An AG-groupoid is the midway structure between a commutative semigroup and a groupoid. This structure is also known as left almost semigroup, abbreviated as LA-semigroup. A groupoid G is called *medial* if $(xa)(by) = (xb)(ay)$ for all $a, b, x, y \in G$, and is called *paramedial* if $(ax)(yb) = (bx)(ya)$ for all $a, b, x, y \in G$. It is well known that every AG-groupoid is medial [3] but in general AG-groupoid needs not be paramedial. However, every AG-groupoid with left identity is paramedial [8].

A nonempty subset M of a semigroup S is said to be an *M-system* if for all $a, b \in M$, there exists $x \in S$ such that $a(xb) \in M$ ([10]). *M*-systems in semigroups were studied in many papers ([1], [7], etc). Later, *M*-systems were studied in various kind of some generalizations of semigroups, for example, *M*-systems in ordered semigroups were studied by Kehayopulu ([4]), *M*-systems in LA-semigroups were studied by Mushtaq and Khan ([6]), *M*-systems in ordered Γ-semigroups were studied by Hila ([2]) and *M*-systems in Γ-AG-groupoids were studied by Shah and Rehman ([9]), etc.

Let S be a nonempty set, $\cdot$ be a binary operation on S and $\leq$ be relation on S. Then $(S, \cdot, \leq)$ is called an *ordered AG-groupoid* if $(S, \cdot)$ is an AG-groupoid, $(S, \leq)$ is a partially ordered set and for all $a, b, c \in S$, $a \leq b$ implies that $ac \leq bc$ and $ca \leq cb$. This structure is a generalization of AG-groupoids and commutative ordered semigroups. The following theorem follows by Theorem 1 in [5] and definitions of ordered AG-groupoids and ordered semigroups.

Theorem 1.1. *An ordered AG-groupoid S is an ordered semigroup if and only if* $a(bc) = (cb)a$ *for all* $a, b, c \in S$.

For $H \subseteq S$, let $(H] = \{t \in S \mid t \leq h \text{ for some } h \in H\}$. This lemma is similar to the case of ordered semigroups.

Lemma 1.2. *Let S be an ordered AG-groupoid and A, B be subsets of S. The following statements hold*:

(i) *If* $A \subseteq B$, *then* $(A] \subseteq (B]$.

(ii) $(A](B] \subseteq (AB]$.

(iii) $((A](B]] = (AB]$.

The aim of this paper is to study ideals, *M*-systems, *N*-systems and *I*-systems of ordered AG-groupoids.

2. Main Results

A nonempty subset A of an ordered AG-groupoid S is called a *left ideal* of S if $(A] \subseteq A$ and $SA \subseteq A$ and called a *right ideal* of S if $(A] \subseteq A$ and $AS \subseteq A$. A nonempty subset A of S is called an *ideal* of S if A is both left and right ideals of S.

Proposition 2.1. *Let S be an ordered AG-groupoid with left identity. Then every right ideal of S is a left ideal of S.*

Proof. Let R be a right ideal of S. Then $(R] \subseteq R$ and $RS \subseteq R$. We claim that $SR \subseteq R$, indeed, $SR = (eS)R = (RS)e \subseteq Re \subseteq R$. □

Let S be an ordered AG-groupoid. For $A \subseteq S$, let $\langle A \rangle_l$ denote the left ideal of S generated by A and for $a \in S$, $\langle \{a\} \rangle_l$ be denoted by $\langle a \rangle_l$.

Lemma 2.2. *Let S be an ordered AG-groupoid with left identity and $A \subseteq S$. Then $S(SA) = SA$ and $S(SA] \subseteq (SA]$.*

Proof. Since S has a left identity, $S = SS$. Then by definition of AG-groupoids and paramedial law, we have $S(SA) = (SS)(SA) = (AS)(SS) = (AS)S = (SS)A = SA$. Thus $S(SA) = SA$. By Lemma 1.2, we have $S(SA] = (S](SA] \subseteq (S(SA)] = (SA]$. □

Lemma 2.3. *Let S be an ordered AG-groupoid with left identity and $a \in S$. Then $\langle a \rangle_l = (Sa]$.*

Proof. Since S has a left identity, $a \in (Sa]$. By Lemma 2.2, we have $S(Sa] \subseteq (Sa]$. So $(Sa]$ is a left ideal of S containing a. Next, let L be another left ideal containing a. Thus $Sa \subseteq L$, so $(Sa] \subseteq L$. □

An ordered AG-groupoid is called *fully idempotent* if all ideals of S are idempotent. For $A \subseteq S$, let $\langle A \rangle_i$ denote the ideal of S generated by A and for $a \in S$, $\langle \{a\} \rangle_i$ denoted by $\langle a \rangle_i$.

Proposition 2.4. *Let S be an ordered AG-groupoid with left identity e and A, B be ideals of S. If S is fully idempotent, then $A \cap B = \langle AB \rangle_i$ and the ideals of S form a semilattice $(L_S, \wedge)$, where $A \wedge B = \langle AB \rangle_i$.*

Proof. Since $AB \subseteq A \cap B, \langle AB \rangle_i \subseteq A \cap B.$ Conversely, let $a \in A \cap B.$ Thus $a \in \langle a \rangle_i = \langle a \rangle_i \langle a \rangle_i \subseteq AB \subseteq \langle AB \rangle_i.$ Thus $A \cap B \subseteq \langle AB \rangle_i.$ Hence $A \cap B = \langle AB \rangle_i.$

Since $A \wedge B = \langle AB \rangle = A \cap B$ and $\cap$ is associative, commutative and idempotent binary operation, $(L_S, \wedge)$ is a semilattice. □

Let S be an ordered AG-groupoid. A nonempty subset M of S is called an *M-system* of S if for each $a, b \in M$, there exist $x \in S$ and $c \in M$ such that $c \leq a(xb).$ Equivalent definition: for each $a, b \in M$, there exists $c \in M$ such that $c \in (a(Sb)].$

Remark. (i) If $(S, \cdot)$ is an AG-groupoid, we endow S with the order relation $\leq := id_S$, then $(S, \cdot, \leq)$ is an ordered AG-groupoid. Moreover, the set M is an M-system of an AG-groupoid $(S, \cdot)$ if and only if M is an M-system of an ordered AG-groupoid $(S, \cdot, \leq).$

(ii) If an ordered AG-groupoid S is an ordered semigroup, then the set M is an M-system of an ordered AG-groupoid S if and only if M is an M-system of an ordered semigroup S.

A nonempty subset P of an ordered AG-groupoid S is called *quasi-prime* if and only if for any left ideals A, B of S, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P.$

Lemma 2.5. *Let L be a left ideal of an ordered AG-groupoid S with identity e. Then L is quasi-prime if and only if for all $a, b \in S$, $a(Sb) \subseteq L$ implies $a \in L$ or $b \in L.$*

Proof. Suppose that $a(Sb) \subseteq L.$ We get $S(a(Sb)) \subseteq SL \subseteq L$ and by medial law, paramedial law and the definition of AG-groupoid, we have

$$S(a(Sb)) = (SS)(a(Sb)) = (Sa)(S(Sb)) = (Sa)((SS)(Sb))$$

$$= (Sa)((bS)(SS)) = (Sa)((bS)S) = (Sa)((SS)b) = (Sa)(Sb).$$

So $S(a(Sb)) = (Sa)(Sb).$ Since L is a left ideal of S, $(Sa](Sb] \subseteq ((Sa)(Sb)] = (S(a(Sb))] \subseteq L.$ Since $(Sa]$ and $(Sb]$ are left ideals of S and L is quasi-prime, $(Sa] \subseteq L$ or $(Sb] \subseteq L.$ By Lemma 2.3, $a \in L$ or $b \in L.$ Conversely, let A and B be left ideals of S such that $AB \subseteq L$ and $A \not\subseteq L.$ Then there exists an element x in S

such that $x \in A$ but $x \notin L$. Now for all $y \in B$, we have $x(Sy) \subseteq A(SB) \subseteq AB \subseteq L$. Hence by assumption, $y \in L$ for all $y \in B$. Hence $B \subseteq L$, this implies that L is quasi-prime. □

Theorem 2.6. *Let S be an ordered AG-groupoid with left identity and L be a proper left ideal of S. Then L is quasi-prime if and only if* $S \backslash L$ *is an M-system.*

Proof. Assume L is quasi-prime and let $a, b \in S \backslash L$. Suppose $c \notin (a(Sb)]$ for all $c \in S \backslash L$. Then $(a(Sb)] \subseteq L$, this implies $a(Sb) \subseteq L$. By Lemma 2.5, $a \in L$ or $b \in L$, which is impossible. Then there exists $c \in S \backslash L$ such that $c \in (a(Sb)]$. Hence, $S \backslash L$ is an *M*-system.

Conversely, assume that $S \backslash L$ is an *M*-system. Let $a, b \in S$ such that $a(Sb) \subseteq L$. Suppose that $a, b \in S \backslash L$. Since $S \backslash L$ is an *M*-system, there exist $c \in S \backslash L$ and $x \in S$ such that $c \leq a(xb) \in a(Sb) \subseteq L$. Since L is a left ideal of S, we have $c \in L$, which is impossible. Hence $a \in L$ or $b \in L$. By Lemma 2.5, L is quasi-prime. □

Let S be an ordered AG-groupoid. A nonempty subset N of S is called an *N-system* of S if for each $a \in N$, there exist $x \in S$ and $c \in N$ such that $c \leq a(xa)$. Equivalent definition: for each $a \in N$, there exists $c \in N$ such that $c \in (a(Sa)]$.

Remark. (i) In [6], definition of *N*-systems in AG-groupoids is called a *P-system*. If $(S, \cdot)$ is an AG-groupoid, we endow S with the order relation $\leq := id_S$, then $(S, \cdot, \leq)$ is an ordered AG-groupoid. Moreover, the set N is a *P*-system of an AG-groupoid $(S, \cdot)$ if and only if N is an *N*-system of an ordered AG-groupoid $(S, \cdot, \leq)$.

(ii) If an ordered AG-groupoid S is an ordered semigroup, then the set N is an *N*-system of an ordered AG-groupoid S if and only if N is an *N*-system of an ordered semigroup S.

(iii) Let S be an ordered AG-groupoid. Each *M*-system of S is an *N*-system of S.

A nonempty subset P of an ordered AG-groupoid S is called *quasi-semiprime* if for any left ideal A of S, $A^2 \subseteq P$ implies that $A \subseteq P$. It is obvious that a quasi-prime subset of S is a quasi-semiprime subset of S.

Lemma 2.7. *Let L be a left ideal of an ordered AG-groupoid S with identity e. Then L is quasi-semiprime if and only if for all* $a \in S$, $a(Sa) \subseteq L$ *implies* $a \in L$.

Proof. Suppose that $a(Sa) \subseteq L$. We get $S(a(Sa)) \subseteq SL \subseteq L$ and by similar in the proof of Lemma 2.5, we have $S(a(Sa)) = (Sa)(Sa)$. Since L is a left ideal of S, $(Sa](Sa] \subseteq ((Sa)(Sa)] = (S(a(Sa))] \subseteq L$. Since $(Sa]$ is a left ideal of S and L is quasi-semiprime, $(Sa] \subseteq L$. By Lemma 2.3, $a \in L$. Conversely, let A be a left ideal of S such that $A^2 \subseteq L$. Now for all $x \in A$, we have $x(Sx) \subseteq A(SA) \subseteq A^2 \subseteq L$. Hence by assumption, $x \in L$ for all $x \in A$. Hence $A \subseteq L$, this implies that L is quasi-semiprime. □

Theorem 2.8. *Let S be an ordered AG-groupoid with left identity and L be a proper left ideal of S. Then L is quasi-semiprime if and only if* $S \backslash L$ *is an N-system.*

Proof. Assume L is quasi-semiprime and let $a \in S \backslash L$. Suppose $c \notin (a(Sa)]$ for all $c \in S \backslash L$. Thus $(a(Sa)] \subseteq L$, this implies $a(Sa) \subseteq L$. By Lemma 2.7, $a \in L$, which is impossible. So there exists $c \in S \backslash L$ such that $c \in (a(Sa)]$. Hence, $S \backslash L$ is an N-system.

Conversely, assume that $S \backslash L$ is an N-system. Let $a \in S$ such that $a(Sa) \subseteq L$. Suppose that $a \in S \backslash L$. Since $S \backslash L$ is an N-system, there exist $c \in S \backslash L$ and $x \in S$ such that $c \leq a(xa) \in a(Sa) \subseteq L$. Then $c \in L$, which is impossible. Therefore $a \in L$. By Lemma 2.7, L is quasi-semiprime. □

The intersection of quasi-prime left ideals of an ordered AG-groupoids S (if it is non empty) needs not to be quasi-prime left ideals of S. The following proposition shows that it becomes quasi-semiprime.

Proposition 2.9. *Let* J_i *be any set of quasi-prime left ideals of an ordered AG-groupoid for all* $i \in I$. *If* $P = \bigcap_{i \in I} J_i \neq \emptyset$, *then P is a quasi-semiprime left ideal of S.*

Proof. Let L be a left ideal of S such that $L^2 \subseteq P$. Then $L^2 \subseteq J_i$ for all $i \in J$. This implies $L \subseteq J_i$ for all $i \in J$. So $L \subseteq P$. Hence, P is a quasi-semiprime left ideal of S. □

Theorem 2.10. *Every quasi-semiprime left ideal of an ordered AG-groupoid with left identity is an intersection of some quasi-prime left ideals.*

Proof. Let L be a quasi-semiprime left ideal of S and $\{J_i \mid i \in I\}$ be the set of all quasi-prime left ideals of S containing L. This set is not empty because S itself is a quasi-prime left ideal of S. Let $a \in S \backslash L$. Then $a(Sa) \not\subseteq L$, take $a_1 \in a(Sa) \subseteq (a(Sa)]$ but $a_1 \notin L$. From $a_1(Sa_1) \not\subseteq L$, we have $a_2 \in S$ such that $a_2 \in a_1(Sa_1) \subseteq (a_1(Sa_1)]$ but $a_2 \notin L$. We continue this way, take $a_i \in (a_{i-1}(Sa_{i-1})]$ but $a_i \notin L$. We put $a = a_0$ and let $A = \{a_0, a_1, a_2, ...\}$. So $A \cap L = \varnothing$. Next, we claim that M is an M-system. Let $a_i, a_j \in M$. Let us assume that $i \leq j$. If $i = j$, then $a_{i+1} \in (a_i(Sa_i)] = (a_j(Sa_i)]$. If $i < j$, then $a_{j+1} \in (a_j(Sa_j)] \subseteq (a_j(S(a_{j-1}(Sa_{j-1}))))] \subseteq (a_j(Sa_{j-1})] \subseteq \cdots \subseteq (a_j(Sa_i)]$ by Lemma 2.2 and Lemma 1.2. A similar argument takes care of the case in which $i > j$. Now we have that A is an M-system and $A \cap L = \varnothing$. Let $T = \{M \mid M$ is an M-system of S such that $a \in M$ and $M \cap L = \varnothing\}$. Then $T \neq \varnothing$ because $A \in T$. By Zorn's Lemma, there exists a maximal element, say M' in T. Again let $X = \{J \mid J$ is a left ideal of S such that $J \cap M' = \varnothing$ and $L \subseteq J\}$. Then $X \neq \varnothing$ because $L \in X$. By Zorn's Lemma, there exists a maximal element, say J' in X. Let $x, y \in S \backslash J'$. Then $(Sx \cup J'] \cap M' \neq \varnothing$ and $(Sy \cup J'] \cap M' \neq \varnothing$. So there exist $s, t \in M'$ such that $s \leq ux$ and $t \leq vy$ for some $u, v \in S$. Since M' is an M-system, there exists $m \in M$ such that $m \leq S(wt)$ for some $w \in S$. Thus $m \leq (ux)(w(vy)) = ((vy)x)(wu) = (uw)(x(vy)) = (e(uw))(x(vy)) = (ex)((uw)(vy)) = x((yw)(vu)) = x(((vu)w)y)$. Then $S \backslash J'$ is an M-system. From maximality of M', $S \backslash J' = M'$ and so J' is a quasi-prime left ideal of S containing L. Since $a \notin J'$, $L = \bigcap\{J_i \mid i \in I\}$. □

Theorem 2.11. *Let S be an ordered AG-groupoid. If N is an N-system of S and $a \in N$, then there exists an M-system M of S such that $a \in M \subseteq N$.*

Proof. Since N is an N-system and $a \in N$, there exists $c_1 \in N$ such that $c_1 \in (a(Sa)]$. So $(a(Sa)] \cap N \neq \varnothing$, take $a_1 \in (a(Sa)] \cap N$. Since N is an N-system, there exists $c_2 \in N$ such that $c_2 \in (a_1(Sa_1)]$. So $(a_1(Sa_1)] \cap N \neq \varnothing$, take $a_2 \in (a_1(Sa_1)] \cap N$. We continue this way, take $a_i \in (a_{i-1}(Sa_{i-1})] \cap N$. We put $a = a_0$ and let $M = \{a_0, a_1, a_2, ...\}$. We have M is an M-system and $a \in M \subseteq N$. □

Let S be an ordered AG-groupoid with left identity and a nonempty subset of S be called *quasi-irreducible* if for any left ideals A, B of S, $A \cap B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

Let S be an ordered AG-groupoid with left identity. A nonempty subset I of S is called an ***I-system*** of S if for each $a, b \in I$, $(\langle a \rangle_l \cap \langle b \rangle_l) \cap I \neq \emptyset$.

Theorem 2.12. *Let S be an ordered AG-groupoid with left identity and L be a proper left ideal of S. Then the following statements are equivalent.*

(1) *L is quasi-irreducible.*

(2) *For all $a, b \in S$, $\langle a \rangle_l \cap \langle b \rangle_l \subseteq L$ implies $a \in L$ or $b \in L$.*

(3) *$S \backslash L$ is an I-system.*

Proof. (1) $\Rightarrow$ (2): Assume L is quasi-irreducible and let $a, b \in S$ such that $\langle a \rangle_l \cap \langle b \rangle_l \subseteq L$. Thus $\langle a \rangle_l \in L$ or $\langle b \rangle_l \in L$. Then $a \in L$ or $b \in L$.

(2) $\Rightarrow$ (3): Let $a, b \in S \backslash L$. Suppose $(\langle a \rangle_l \cap \langle b \rangle_l) \cap (S \backslash L) = \emptyset$. This implies $\langle a \rangle_l \cap \langle b \rangle_l \subseteq L$. So $a \in L$ or $b \in L$, it is impossible. Hence $(\langle a \rangle_l \cap \langle b \rangle_l) \cap (S \backslash L) \neq \emptyset$. Therefore, $S \backslash L$ is an I-system.

(3) $\Rightarrow$ (1): Let A, B be left ideals of S such that $A \cap B \subseteq L$. Suppose $A \not\subseteq L$ and $B \not\subseteq L$. Let $a \in A \backslash L$ and $b \in B \backslash L$. This implies that $a, b \in S \backslash L$. By hypothesis, $(\langle a \rangle_l \cap \langle b \rangle_l) \cap (S \backslash L) \neq \emptyset$. Then there exists an element $c \in S$ such that $c \in \langle a \rangle_l \cap \langle b \rangle_l$ and $c \in S \backslash L$. It shows that $c \in \langle a \rangle_l \cap \langle b \rangle_l \subseteq A \cap B \subseteq L$, it is impossible. Thus $A \subseteq L$ or $B \subseteq L$. Hence, L is quasi-irreducible. □

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