**Far East Journal of Mathematical Sciences (FJMS)**
Volume 46, Number 2, 2010, Pages 143-156
Published Online: November 22, 2010
This paper is available online at http://pphmj.com/journals/fjms.htm


# GENERALIZED HYPERBOLIC AND TRIANGULAR FUNCTION SOLUTIONS OF NONLINEAR EVOLUTION EQUATIONS

**ZEID I. A. AL-MUHIAMEED[1] and EMAD A.-B. ABDEL-SALAM[1,2]**

[1]Department of Mathematics
Faculty of Science
Qassim University
Buraida, Saudi Arabia

[2]Department of Mathematics
New Valley Faculty of Education
Assiut University
El-Khargah, New Valley, Egypt

**Abstract**

The extended generalized hyperbolic function method is used to construct the exact traveling wave solutions of the nonlinear partial differential equations in a unified way. The main idea of this method is to take full advantage of the Riccati equation which has more new solutions by using the generalized hyperbolic functions and generalized triangular functions. More new solitons and periodic solutions for the Dodd-Bullough-Mikhailov, the Tzitzeica-Dodd-Bullough, the special type of the Dodd-Bullough-Mikhailov and the Liouville equations are formally derived.

## 1. Introduction

Nonlinear equations play a major role in scientific fields. Two classes of equations, namely,

2010 Mathematics Subject Classification: 35-XX.

Keywords and phrases: Riccati equation, generalized hyperbolic functions, generalized triangular functions, soliton solution, periodic solution.

Received July 16, 2010

$$u_{xt} + f(u) = 0 \tag{1}$$

and

$$u_{tt} - u_{xx} + f(u) = 0, \tag{2}$$

play a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory. The function $f(u)$ takes many forms such as

$$f(u) = \sin u, \quad f(u) = \sinh u, \quad f(u) = e^{u},$$
$$f(u) = e^{u} + e^{-2u}, \quad f(u) = e^{-u} + e^{-2u}, \tag{3}$$

that characterize the sine-Gordon equation, sinh-Gordon equation, Liouville equation, Dodd-Bullough-Mikhailov (DBM) equation, and the Tzitzeica-Dodd-Bullough (TDB) equation, respectively, [15, 16, 23]. The first two equations gained its importance when it gave kink and antikink solutions with the collisional behaviors of solitons. A kink is a solution with boundary values 0 and $2\pi$ at the left and the right infinity, respectively, [18]. However, antikink is a solution with boundary values 0 and $-2\pi$ at the left and the right infinity, respectively. In addition, these two equations are integrable, when boundary conditions are periodic, giving plenty of quasi periodic solutions. Moreover, these two equations appear in many fields such as the propagation of fluxons in Josephson junctions [15, 16, 23] between two superconductors, the motion of rigid pendula attached to a stretched wire, solid state physics, nonlinear optics, and dislocations in metals. The DBM equation and the TDB equation appear in problems varying from fluid flow to quantum field theory. Other equations for other forms of $f(u)$ appear such as the Klein-Gordon equation and the $\phi^4$ equation.

It has recently become more interesting to obtain exact solutions of nonlinear partial differential equations (NLPDEs) using symbolical computer programs such as Maple, Matlab, Mathematica that facilitate complex and tedious algebraical computations. Various effective methods have been developed to understand the mechanisms of these physical models such as Bäcklund transformation, Darboux transformation, Painlevé method, Exp-method, tanh method, sine-cosine method, Lucas Riccati method, projective Riccati method, homogeneous balance method, similarity reduction method and so on [1-14, 17-22].

The rest of this paper is organized as follows: in the following section, we introduce the extended generalized hyperbolic function (EGHF) method to NLPDEs. In Section 3, we solve the DBM, the TDB, the special type of the DBM and the Liouville equations by the EGHF method. Finally, we conclude the paper and give some features and comments.

## 2. The Extended Generalized Hyperbolic Function Method

Consider a given NLPDE

$$H(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0. \tag{4}$$

The EGHF method proceeds in the following four steps:

**Step 1.** We seek its traveling wave solution of Eq. (4) in the form

$$u(x, t) = u(\xi), \quad \xi = \alpha(x - \omega t), \tag{5}$$

where $\alpha$ and $\omega$ are constants to be determined later. Substituting Eq. (5) into Eq. (4) yields an ordinary differential equation (ODE)

$$\tilde{H} = (u, u', u'', \ldots) = 0, \quad u' = \frac{du}{d\xi}, \ldots, \text{ etc.}, \tag{6}$$

where $\tilde{H}$ is a polynomial of $u$ and its various derivatives. If $\tilde{H}$ is not a polynomial of $u$ and its various derivatives, then we may use new variables $v = v(\xi)$ which makes $\tilde{H}$ become polynomial of $v$ and its various derivatives.

**Step 2.** Suppose that $u(\xi)$ can be expressed by a finite power series of $F(\xi)$

$$u(\xi) = \sum_{j=0}^{n} a_j F^j(\xi), \;\; a_n \neq 0, \tag{7}$$

where $n$ is a positive integer which can be determined by balancing the highest derivative term with the nonlinear term(s) in Eq. (6) and $a_j$ are some parameters to be determined. The function $F(\xi)$ satisfies the Riccati equation

$$F'(\xi) = A + BF^2(\xi), \;\; ' = \frac{d}{d\xi}, \tag{8}$$

where $A$, $B$ are constants.

**Step 3.** Substituting Eq. (7) with Eq. (8) into the ODE Eq. (6), then the left-hand side of Eq. (6) can be converted into a polynomial in $F(\xi)$. Setting all coefficients of the polynomial to zero yields system of algebraic equations for $a_0, a_1, \ldots, a_n$, $\alpha$ and $\omega$.

**Step 4.** Solving this system obtained in Step 3, then $a_0, a_1, \ldots, a_n$, $\alpha$ and $\omega$ can be expressed by $A$, $B$. Substituting these results into Eq. (7), then a general formulae of traveling wave solutions of Eq. (4) can be obtained. Choose properly $A$ and $B$ in ODE Eq. (8) such that the corresponding solution $F(\xi)$ is one of the generalized hyperbolic function (GHF) and generalized triangle function (GTF) [9, 17] given below, some definitions and properties are given in the Appendix.

**Case 1.** If $A = k$ and $B = k$, then Eq. (8) possesses a solution $\tan_{pqk}(\xi)$.

**Case 2.** If $A = k$ and $B = -k$, then Eq. (8) possesses a solution $\cot_{pqk}(\xi)$.

**Case 3.** If $A = k$ and $B = -k$, then Eq. (8) possesses solutions $\tanh_{pqk}(\xi)$, $\coth_{pqk}(\xi)$.

**Case 4.** If $A = \frac{k}{2}$, $B = -\frac{k}{2}$, $p = l \left(\text{or } p = \frac{1}{l}\right)$ and $q = \frac{1}{l}$ (or $q = l$), then Eq. (8) possesses a solution $\frac{\tanh_{pqk}(\xi)}{1 \pm \operatorname{sech}_{pqk}(\xi)}$.

**Case 5.** If $A = \frac{k}{2}$, $B = \frac{k}{2}$, $p = l \left(\text{or } p = \frac{1}{l}\right)$ and $q = \frac{1}{l}$ (or $q = l$), then Eq. (8) possesses solutions $\tan_{pqk}(\xi) \pm \sec_{pqk}(\xi)$, $\frac{\tan_{pqk}(\xi)}{1 \pm \sec_{pqk}(\xi)}$.

**Case 6.** If $A = k$, $B = 4k$, $p = l \left(\text{or } p = \frac{1}{l}\right)$ and $q = \frac{1}{l}$ (or $q = l$), then Eq. (8) possesses a solution $\frac{\tan_{pqk}(\xi)}{1 + \tanh^2_{pqk}(\xi)}$.

**Case 7.** If $A = k$, $B = -4k$, $p = l \left(\text{or } p = \frac{1}{l}\right)$ and $q = \frac{1}{l}$ (or $q = l$), then Eq. (8) possesses a solution $\frac{\tanh_{pqk}(\xi)}{1 + \tanh^2_{pqk}(\xi)}$, where $l$ is an arbitrary constant.

## 3. Applications

### 3.1. The Dodd-Bullough-Mikhailov equation

The DBM equation is given by

$$u_{xt} + e^{u} + e^{-2u} = 0. \tag{9}$$

In order to apply the EGHF method, we use the Painlevé transformation

$$v = e^{u} \quad \text{or} \quad u = \ln v. \tag{10}$$

This transformation will change Eq. (9) into

$$vv_{xt} - v_x v_t + v^3 + 1 = 0. \tag{11}$$

Using the traveling wave $\xi = \alpha(x - \omega t)$ carries Eq. (11) into the ODE

$$v^3 + 1 - \alpha^2 \omega v v'' + \alpha^2 \omega (v')^2 = 0. \tag{12}$$

Balancing the linear terms of highest order in the last equation with the highest order nonlinear terms, we have $n = 2.$ This gives the solution in the form

$$v = a_0 + a_1 F(\xi) + a_2 F^2(\xi). \tag{13}$$

Substituting Eq. (13) with Eq. (8) into Eq. (12) yields the system of algebraic equations for $a_0$, $a_1$, $a_2$, $\alpha$ and $\omega$:

$$\begin{aligned}
&a_2^2(a_2 - 2\alpha^2\omega B^2) = 0,\\
&a_1 a_2(3a_2 - 4\alpha^2\omega B^2) = 0,\\
&-\alpha^2\omega a_1^2 B^2 + 3a_1^2 a_2 + 3a_0 a_2^2 - 6\alpha^2\omega a_2 B^2 a_0 = 0,\\
&a_1(a_1^2 + 6a_0 a_2 - 2\alpha^2\omega B^2 a_0 - 2\alpha^2\omega A a_2 B) = 0,\\
&-8\alpha^2\omega a_2 AB a_0 + 3a_2 a_0^2 + 2\alpha^2\omega a_2^2 A^2 + 3a_1^2 a_0 = 0,\\
&a_1(3a_0^2 - 2\alpha^2\omega BA a_0 + 2\alpha^2\omega A^2 a_2) = 0,\\
&1 - 2\alpha^2\omega a_2 A^2 a_0 + \alpha^2\omega a_1^2 A^2 + a_0^3 = 0.
\end{aligned} \tag{14}$$

Solving this system, with the aid of Maple, we have

$$a_0 = \frac{1}{2}, \quad a_1 = 0, \quad a_2 = \frac{3B}{2A}, \quad \omega = \frac{3}{4\alpha^2 BA}. \tag{15}$$

Substituting Eqs. (15) and Eq. (13) into Eq. (10), we have the following general formulae of traveling wave solutions of the DBM equation:

$$u = \ln\left[\frac{1}{2} + \frac{3B}{2A} F^2\left(\alpha\left(x - \frac{3t}{4\alpha^2 BA}\right)\right)\right]. \tag{16}$$

By selecting the special values of the $A$, $B$ and the corresponding function $F(\xi)$, we have the following solutions of the DBM equation:

$$u_1 = \ln\left[\frac{1}{2} + \frac{3}{2} \tan^2_{pqk}\left(\alpha\left(x - \frac{3t}{4\alpha^2 k^2}\right)\right)\right], \tag{17}$$

$$u_2 = \ln\left[\frac{1}{2} - \frac{3}{2} \cot^2_{pqk}\left(\alpha\left(x + \frac{3t}{4\alpha^2 k^2}\right)\right)\right], \tag{18}$$

$$u_3 = \ln\left[\frac{1}{2} - \frac{3}{2} \tanh^2_{pqk}\left(\alpha\left(x + \frac{3t}{4\alpha^2 k^2}\right)\right)\right], \tag{19}$$

$$u_4 = \ln\left[\frac{1}{2} - \frac{3}{2} \coth^2_{pqk}\left(\alpha\left(x + \frac{3t}{4\alpha^2 k^2}\right)\right)\right] \tag{20}$$

and

$$u_5 = \ln\left[\frac{1}{2} - \frac{3}{2}\left[\frac{\tanh_{pqk}\left(\alpha\left(x + \frac{3t}{\alpha^2 k^2}\right)\right)}{1 \pm \operatorname{sech}_{pqk}\left(\alpha\left(x + \frac{3t}{\alpha^2 k^2}\right)\right)}\right]^2\right], \tag{21}$$

$$u_6 = \ln\left[\frac{1}{2} + \frac{3}{2}\left[\frac{\tan_{pqk}\left(\alpha\left(x - \frac{3t}{\alpha^2 k^2}\right)\right)}{1 \pm \sec_{pqk}\left(\alpha\left(x - \frac{3t}{\alpha^2 k^2}\right)\right)}\right]^2\right], \tag{22}$$

$$u_7 = \ln\left[\frac{1}{2} + \frac{3}{2}\left[\tan_{pqk}\left(\alpha\left(x - \frac{3t}{\alpha^2 k^2}\right)\right) \pm \sec_{pqk}\left(\alpha\left(x - \frac{3t}{\alpha^2 k^2}\right)\right)\right]^2\right], \tag{23}$$

$$u_8 = \ln\left[\frac{1}{2} + 6\left[\frac{\tanh_{pqk}\left(\alpha\left(x - \frac{3t}{16\alpha^2 k^2}\right)\right)}{1 + \tanh^2_{pqk}\left(\alpha\left(x - \frac{3t}{16\alpha^2 k^2}\right)\right)}\right]^2\right], \tag{24}$$

$$u_9 = \ln\left[\frac{1}{2} - 6\left[\frac{\tanh_{pqk}\left(\alpha\left(x + \frac{3t}{16\alpha^2 k^2}\right)\right)}{1 + \tanh^2_{pqk}\left(\alpha\left(x + \frac{3t}{16\alpha^2 k^2}\right)\right)}\right]^2\right], \tag{25}$$

with $p = l\left(\text{or } p = \frac{1}{l}\right)$ and $q = \frac{1}{l}$ (or $q = l$).

### 3.2. The Tzitzeica-Dodd-Bullough equation

We consider here the TDB equation

$$u_{xt} + e^{-u} + e^{-2u} = 0. \tag{26}$$

The Painlevé transformation

$$v = e^{-u} \quad \text{or} \quad u = \operatorname{arcsinh}\frac{v^{-1} - v}{2}, \tag{27}$$

this transformation changes Eq. (26) into

$$-vv_{xt} + v_x v_t - v^3 - v^4 = 0, \tag{28}$$

the traveling wave carries Eq. (28) into the ODE

$$v^3 + v^4 + \alpha^2\omega v v'' - \alpha^2\omega(v')^2 = 0. \tag{29}$$

By using the same manner, we have $n = 1$. This gives the solution in the form

$$v = a_0 + a_1 F(\xi). \tag{30}$$

Substituting Eq. (30) with Eq. (8) into Eq. (29) yields the system of algebraic equations for $a_0$, $a_1$, $\alpha$ and $\omega$:

$$a_1^2(\alpha^2\omega B^2 + a_1^2) = 0,$$

$$a_1(4a_0a_1^2 + a_1^2 + 2\alpha^2\omega B^2 a_0) = 0,$$

$$a_1^2 a_0(6a_0 + 3) = 0,$$

$$a_1 a_0(4a_0^2 + 3a_0 + 2\alpha^2\omega BA) = 0,$$

$$a_0^2(a_0^2 + a_0 - \alpha^2\omega A^2) = 0. \tag{31}$$

Solving this system of equations, we obtain

$$a_1 = \sqrt{\frac{-B}{4A}}, \quad a_0 = -\frac{1}{2}, \quad \omega = \frac{1}{4BA\alpha^2}. \tag{32}$$

The general formulae of the traveling wave solution TDB equation

$$u = \operatorname{arcsinh}\left[\frac{\left[1 + \sqrt{\frac{-B}{A}}F\left(\alpha\left(x - \frac{t}{4BA\alpha^2}\right)\right)\right]^2 - 4}{4\left[1 + \sqrt{\frac{-B}{A}}F\left(\alpha\left(x - \frac{t}{4BA\alpha^2}\right)\right)\right]}\right]. \tag{33}$$

By selecting the special values of the $A$, $B$ and the corresponding function $F(\xi)$, we have the following solutions of the TDB equation:

$$u_1 = \operatorname{arcsinh}\left[\frac{\left[1 + \tanh_{pqk}\left(\alpha\left(x + \frac{t}{4k^2\alpha^2}\right)\right)\right]^2 - 4}{4\left[1 + \tanh_{pqk}\left(\alpha\left(x + \frac{t}{4k^2\alpha^2}\right)\right)\right]}\right], \tag{34}$$

$$u_2 = \operatorname{arcsinh}\left[\frac{\left[1 + \coth_{pqk}\left(\alpha\left(x + \frac{t}{4k^2\alpha^2}\right)\right)\right]^2 - 4}{4\left[1 + \coth_{pqk}\left(\alpha\left(x + \frac{t}{4k^2\alpha^2}\right)\right)\right]}\right]. \tag{35}$$

We omitted the reminder solutions for simplicity.

### 3.3. Special type of the Dodd-Bullough-Mikhailov equation

The special type of the DBM equation takes the form

$$u_{xx} - u_{tt} + e^u + e^{-2u} = 0, \tag{36}$$

we use the Painlevé transformation

$$v = e^u \quad \text{or} \quad u = \ln v. \tag{37}$$

This transformation will change Eq. (37) to

$$vv_{tt} - vv_{xx} - v_t^2 + v_x^2 + v^3 + 1 = 0. \tag{38}$$

The traveling wave transforms Eq. (38) into the ODE

$$v^3 + 1 + \alpha^2(\omega^2 - 1)vv'' - \alpha^2(\omega^2 - 1)(v')^2 = 0. \tag{39}$$

We have $n = 2$. This gives the solution

$$v = a_0 + a_1 F(\xi) + a_2 F^2(\xi). \tag{40}$$

Substituting Eq. (40) with Eq. (8) into Eq. (39) yields a system of algebraic equations for $a_0$, $a_1$, $a_2$ $\alpha$ and $\omega$. Solving this system of equations, we obtain

$$a_1 = 0, \quad a_0 = \frac{1}{2}, \quad a_2 = \frac{3B}{2A}, \quad \alpha = \frac{1}{2}\sqrt{\frac{3}{BA(\omega^2 - 1)}}. \tag{41}$$

The general formulae of traveling wave solution of the special type of DBM equation

$$u = \ln\left[\frac{1}{2} + \frac{3B}{2A}F^2\left(\frac{1}{2}\sqrt{\frac{3}{BA(\omega^2 - 1)}}(x - \omega t)\right)\right]. \tag{42}$$

By selecting the special values of the $A$, $B$ and the corresponding function $F(\xi)$, we have the following solutions of the special type of DBM equation:

$$u_1 = \ln\left[\frac{1}{2} + \frac{3}{2}\tan^2_{pqk}\left(\frac{1}{2k}\sqrt{\frac{3}{\omega^2 - 1}}(x - \omega t)\right)\right], \tag{43}$$

$$u_2 = \ln\left[-\frac{1}{2} - \frac{3}{2}\cot^2_{pqk}\left(\frac{1}{2k}\sqrt{\frac{3}{1 - \omega^2}}(x - \omega t)\right)\right], \tag{44}$$

$$u_3 = \ln\left[-\frac{1}{2} - \frac{3}{2}\tanh^2_{pqk}\left(\frac{1}{2k}\sqrt{\frac{3}{1 - \omega^2}}(x - \omega t)\right)\right], \tag{45}$$

$$u_4 = \ln\left[-\frac{1}{2} - \frac{3}{2}\coth^2_{pqk}\left(\frac{1}{2k}\sqrt{\frac{3}{1 - \omega^2}}(x - \omega t)\right)\right]. \tag{46}$$

We omitted the reminder solutions for simplicity.

**3.4. The Liouville equation**

Finally, we consider the Liouville equation

$$u_{xt} + e^{u} = 0, \tag{47}$$

the Painlevé transformation

$$v = e^{u} \quad \text{or} \quad u = \ln v. \tag{48}$$

This transformation will change Eq. (47) into the form

$$vv_{xt} - v_x v_t + v^3 = 0. \tag{49}$$

The traveling wave carries Eq. (49) into the ODE

$$v^3 - \alpha^2 \omega v v'' + \alpha^2 \omega (v')^2 = 0. \tag{50}$$

We have $n = 2$. This gives the solution

$$v = a_0 + a_1 F(\xi) + a_2 F^2(\xi). \tag{51}$$

Substituting Eq. (51) with Eq. (8) into Eq. (50) yields the system of algebraic equations for $a_0$, $a_1$, $a_2$ $\alpha$ and $\omega$. Solving this system of equations, we obtain

$$a_1 = 0, \quad a_2 = \frac{a_0 B}{A}, \quad \alpha = \sqrt{\frac{a_0}{2\omega AB}}. \tag{52}$$

The general formulae of traveling wave solution of the Liouville equation:

$$u = \ln\left[a_0 + \frac{a_0 B}{A} F^2\left(\sqrt{\frac{a_0}{2\omega AB}}(x - \omega t)\right)\right]. \tag{53}$$

By selecting the special values of the $A$, $B$ and the corresponding function $F(\xi)$, we have the following solutions of the special type of the Liouville equation:

$$u_1 = \ln\left[a_0 + a_0 \tan^2_{pqk}\left(\frac{1}{k}\sqrt{\frac{a_0}{2\omega}}(x - \omega t)\right)\right], \tag{54}$$

$$u_2 = \ln\left[a_0 - a_0 \cot^2_{pqk}\left(\frac{1}{k}\sqrt{-\frac{a_0}{2\omega}}(x - \omega t)\right)\right], \tag{55}$$

$$u_3 = \ln\left[a_0 - a_0 \tanh^2_{pqk}\left(\frac{1}{k}\sqrt{-\frac{a_0}{2\omega}}(x - \omega t)\right)\right], \tag{56}$$

$$u_4 = \ln\left[a_0 - a_0 \coth^2_{pqk}\left(\frac{1}{k}\sqrt{-\frac{a_0}{2\omega}}(x - \omega t)\right)\right]. \tag{57}$$

**Remark 1.** To our knowledge, the solutions by the GHFs and GTFs have been not found before.

**Remark 2.** If $p = q = k = 1$, then the obtained results reduced to the well-known solution obtained by tanh function method, extended tanh function method and other methods.

## 4. Summary and Discussion

In this paper, the EGHF method is used to obtain analytical solutions of the DBM, the TDB and the Liouville equations. These solutions include solitary wave solution, soliton like solutions and periodic solutions. The obtained solutions may be of important significance for the explanation of some practical physical problems. We can successfully recover the known solitary wave solutions that had been found by the tanh-function method and other methods. The EGHF method can be applied to other NLPDEs.

## 5. Appendix

### 5.1. The generalized hyperbolic functions

The generalized hyperbolic sine, cosine and tangent functions are

$$\sinh_{pqk}(\xi) = \frac{pe^{k\xi} - qe^{-k\xi}}{2},$$

$$\cosh_{pqk}(\xi) = \frac{pe^{k\xi} + qe^{-k\xi}}{2},$$

$$\tanh_{pqk}(\xi) = \frac{pe^{k\xi} - qe^{-k\xi}}{pe^{k\xi} + qe^{-k\xi}}, \tag{58}$$

where $\xi$ is an independent variable, $p$, $q$ and $k$ are arbitrary constants greater than zero [9, 17]. The generalized hyperbolic cotangent, secant and cosecant functions are

$$\coth_{pqk}(\xi) = \frac{1}{\tanh_{pqk}(\xi)}, \quad \operatorname{sech}_{pqk}(\xi) = \frac{1}{\cosh_{pqk}(\xi)}, \quad \operatorname{csch}_{pqk}(\xi) = \frac{1}{\sinh_{pqk}(\xi)}.$$

These functions satisfy the following relations:

$$\cosh^2_{pqk}(\xi) - \sinh^2_{pqk}(\xi) = pq,$$

$$\sinh_{pqk}(\xi) = \sqrt{pq}\,\sinh\left(k\xi - \frac{1}{2}\ln\frac{q}{p}\right). \tag{59}$$

Note that, $\sinh_{pqk}(\xi)$ is not odd and $\cosh_{pqk}(\xi)$ is not even:

$$\sinh_{pqk}(-\xi) = -pq\sinh_{\frac{1}{p}\frac{1}{q}k}(\xi), \quad \cosh_{pqk}(-\xi) = pq\cosh_{\frac{1}{p}\frac{1}{q}k}(\xi), \quad p, q \neq 1. \tag{60}$$

Also, from the above definition, we give the derivative formulas of GHFs as follows:

$$(\sinh_{pqk}(\xi))' = k\cosh_{pqk}(\xi), \quad (\cosh_{pqk}(\xi))' = k\sinh_{pqk}(\xi),$$

$$(\tanh_{pqk}(\xi))' = kpq\,\mathrm{sech}^2_{pqk}(\xi), \quad (\coth_{pqk}(\xi))' = -kpq\,\mathrm{csch}^2_{pqk}(\xi). \tag{61}$$

**5.2. The generalized triangular functions**

The generalized triangular sine, cosine and tangent functions are

$$\sin_{pqk}(\xi) = \frac{pe^{ik\xi} - qe^{-ik\xi}}{2i},$$

$$\cos_{pqk}(\xi) = \frac{pe^{ik\xi} + qe^{-ik\xi}}{2},$$

$$\tan_{pqk}(\xi) = \frac{pe^{ik\xi} - qe^{-ik\xi}}{pe^{ik\xi} + qe^{-k\xi}}, \tag{62}$$

where $\xi$ is an independent variable, $p$, $q$ and $k$ are arbitrary constants greater than zero [9, 17]. The generalized triangular cotangent, secant and cosecant functions are $\cot_{pqk}(\xi) = \frac{1}{\tan_{pqk}(\xi)}$, $\sec_{pqk}(\xi) = \frac{1}{\cos_{pqk}(\xi)}$, $\csc_{pqk}(\xi) = \frac{1}{\sin_{pqk}(\xi)}$. These functions satisfy the following relations:

$$\cos^2_{pqk}(\xi) + \sin^2_{pqk}(\xi) = pq, \quad \sin_{pqk}\xi = \sqrt{pq}\,\sin\left(k\xi + \frac{i}{2}\ln\frac{q}{p}\right). \tag{63}$$

Note that, $\sin_{pqk}(\xi)$ is not odd and $\cos_{pqk}(\xi)$ is not even:

$$\sin_{pqk}(-\xi) = -pq\sinh_{\frac{1}{p}\frac{1}{q}k}(\xi), \quad \cos_{pqk}(-\xi) = pq\cos_{\frac{1}{p}\frac{1}{q}k}(\xi), \quad p, q \neq 1. \tag{64}$$

Also, from the above definition, we give the derivative formulas of GTFs as follows:

$$(\sin_{pqk}(\xi))' = k\cos_{pqk}(\xi), \ (\cos_{pqk}(\xi))' = -k\sin_{pqk}(\xi),$$

$$(\tanh_{pqk}(\xi))' = kpq\sec^2_{pqk}(\xi), \ (\cot_{pqk}(\xi))' = -kpq\csc^2_{pqk}(\xi). \tag{65}$$

**References**


[1] E. A.-B. Abdel-Salam, Periodic structures based on the symmetrical Lucas function of the $(2+1)$-dimensional dispersive long-wave system, Z. Naturforsch., A: Phys. Sci. 63 (2008), 671-678.

[2] E. A.-B. Abdel-Salam, Quasi-periodic structures based on symmetrical Lucas function of $(2+1)$-dimensional modified dispersive water-wave system, Commun. Theor. Phys. 52 (2009), 1004-1012.

[3] E. A.-B. Abdel-Salam, Quasi-periodic, periodic waves, and soliton solutions for the combined KdV-mKdV equation, Z. Naturforsch., A: Phys. Sci. 64 (2009), 639-645.

[4] E. A.-B. Abdel-Salam and D. Kaya, Application of new triangular functions to nonlinear partial differential equations, Z. Naturforsch., A: Phys. Sci. 64 (2009), 1-7.

[5] E. A.-B. Abdel-Salam and Z. I. A. Al-Muhiameed, Generalized Jacobi elliptic function method and non-travelling wave solutions, Nonlinear Science Letters A 1(4) (2010), 363-372.

[6] M. J. Ablowitz, B. M. Herbst and C. Schober, On the numerical solution of the sine-Gordon equation, J. Comput. Phys. 126 (1996), 299-314.

[7] C. Q. Dai and J. F. Zhang, Jacobian elliptic function method for nonlinear differential-difference equations, Chaos Solitons Fractals 27(4) (2006), 1042-1047.

[8] C. Q. Dai and J. F. Zhang, Stochastic exact solutions and two-soliton solution of the Wick-type stochastic KdV equation, Europhys. Lett. 86(4) (2009), 40006.

[9] M. F. El-Sabbagh, M. M. Hassan and E. A.-B. Abdel-Salam, Quasi-periodic waves and their interactions in the $(2+1)$-dimensional modified dispersive water-wave system, Phys. Scr. 80 (2009), 15006-15014.

[10] E. G. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277(4-5) (2000), 212-218.

[11] I. A. Hassanien, R. A. Zait and E. A.-B. Abdel-Salam, Multicnoidal and multitravelling wave solutions for some nonlinear equations of mathematical physics, Phys. Scr. 67(6) (2003), 457-463.

[12] J. H. He, A new approach to nonlinear partial differential equations, Commun. Nonlinear Sci. Numer. Simul. 2 (1997), 230-235.

[13] B. Li, Exact soliton solutions for the higher-order nonlinear Schrödinger equation, Internat. J. Modern Phys. C 16(8) (2005), 1225-1237.

[14] W. Malfliet, Solitary wave solutions of nonlinear wave equations, Amer. J. Phys. 60(7) (1992), 650-654.

[15] A. C. Newell, Solitons in Mathematics and Physics, Philadelphia, SIAM, 1985.

[16] J. K. Perring and T. H. Skyrme, A model unified field equation, Nucl. Phys. 31 (1962), 550-555.

[17] Y. Ren and H. Zhang, New generalized hyperbolic functions and auto-Bäcklund transformation to find new exact solutions of the (2+1)-dimensional NNV equation, Phys. Lett. A 357(6) (2006), 438-448.

[18] M. Wadati, Introduction to solitons, Pramana: J. Phys. 57(5-6) (2001), 841-847.

[19] A.-M. Wazwaz, Partial Differential Equations: Methods and Applications, Netherlands, Balkema Publishers, 2002.

[20] A.-M. Wazwaz, New solitary-wave special solutions with compact support for the nonlinear dispersive $K(m, n)$ equations, Chaos Solitons Fractals 13(2) (2002), 321-330.

[21] A.-M. Wazwaz, Exact specific solutions with solitary patterns for the nonlinear dispersive $K(m, n)$ equations, Chaos Solitons Fractals 13(1) (2001), 161-170.

[22] A.-M. Wazwaz, General compactons solutions for the focusing branch of the nonlinear dispersive $K(n, n)$ equations in higher dimensional spaces, Appl. Math. Comput. 133(2/3) (2002), 213-227.

[23] N. J. Zabusky and M. D. Kruskal, Interaction of solitons in a collision less plasma and the recurrence of initial states, Phys. Rev. Lett. 15 (1965), 240-243.