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INTERTWINED BASINS OF ATTRACTION OF DYNAMICAL SYSTEMS ON A SMOOTH MANIFOLD

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Abstract

In this article, we investigate a kind of intertwining phenomenon of many attractors, and give some conditions to guarantee the existence intertwined attractors of the dynamical systems in a smooth two-dimensional manifold.

1. Introduction

The purpose of dynamical system theory is to study rules of change in state which depends on time. In the investigation of dynamical systems, one of very interesting topics is to determine the topological structure of the basin of attraction for an attractor (see [1, 7-9, 13]). As an example with topological complexity, the Lakes of Wada continuum can be a common basin boundary for three attracting

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fixed points [1]. For the case of planar flows, the situation is simpler. Recently, several authors have discussed the property of intertwined basins of attraction, to some extent it also leads to the obstruction to predictability (see [10-12]). In [9], intertwined basins of attraction are elucidated by examples without much theoretical analysis. In [10], the author gave a definition of intertwined basins of attractors.

Definition 1.1 ([10, Definition 2.1])**.** Let p be a regular point of system (2.1), and L is a transversal at p. We call that the system has intertwined basins of attraction beside p, if there exists an arc $L_1 \subset L$ such that p is an endpoint of L_1, and for any $\varepsilon > 0$ both

$$L_1 \cap B(A_1) \cap D(p, \varepsilon) \neq \emptyset \quad \text{and} \quad L_1 \cap B(A_2) \cap D(p, \varepsilon) \neq \emptyset$$

hold for two different attractors A_1 and A_2.

This definition characterizes a kind of intertwined attractors which is very interesting. But when p is not a regular point of the dynamical system (2.1) in [10], we find that nearby p, there still may exist the intertwining phenomenon of attractor basins. In [14], the authors gave a definition of intertwined basins of attractors when p is a singular point of the dynamical system in the plane. We now shall investigate the kind of intertwining phenomenon and give some conditions to guarantee the existence intertwined attractors of the dynamical systems on smooth manifold and get some related results. In Section 2, we fix some notations and definitions. In Section 3, we give the main results about intertwining phenomenon. In Section 4, we give an example to characterize the intertwining property of basins of attraction of the dynamical systems on a smooth manifold.

2. Preliminaries

First of all, we recall some basic notions. Let M be a smooth two-dimensional manifold with a metric d, on which there is a flow $f : M \times R \to M$ defined by the vector field:

$$\dot{x} = V(x), \quad x \in M, \tag{2.1}$$

where $V(x)$ is continuous, and assume that solutions of arbitrary initial value problems are unique. For $A \subset M$ and $I \subset R$, we denote $A \cdot I = \{ f(p, t) | \, p \in A, t \in I \}$ for brevity, in particular, $p \cdot t = f(p, t)$. A set A is invariant under the flow f if $A \cdot R = A$ holds. In particular, for a point $p \in M$, the orbit $p \cdot R$ is an invariant set. Throughout the paper for $A \subset M$, $\overline{A}$ and ∂A denote, respectively, the closure and boundary of A. The ω-limit set $\omega(p)$ of p (or of the orbit $p \cdot R$) is defined to be the set $\bigcap_{t \geq 0} \overline{p \cdot [t, +\infty)}$, equivalently, $q \in \omega(p)$ means that there is a sequence $\{t_n\}$ with $t_n \to +\infty$ such that $p \cdot t_n \to q$ as $n \to +\infty$. Similarly, we define the α-limit set $\alpha(p)$ of p by reversing the direction of time. $B(p, r) = \{x \mid d(x, p) < r\}$ denotes a ball with p the center and r the radius. In the literature, several different concepts of attractor are used by different authors. To avoid confusion, here we admit the following definitions.

Definition 2.1. An attractor, for the flow f, is a compact and invariant set $A \subset M$, satisfying the following property: A has a shrinking neighborhood, i.e., there is an open neighborhood U of A such that $\overline{U \cdot t} \subset U$ for $t > 0$ and $A = \bigcap_{t>0} U \cdot t$

Definition 2.2. If $A(\subset M)$ is an attractor, then its basin of attraction $B(A)$ is defined to be the set of initial points p such that $\omega(p) \subset A$, i.e., $d(p \cdot t, A) \to 0$ $(t \to +\infty)$, where $d(p \cdot t, A) = \inf\{d(p \cdot t, a) \mid a \in A\}$, d is the ordinary metric on M and with no confusion we also use it for the distance between a point and a set.

Observe that the basin of attraction $B(A)$ can be expressed as $\bigcup_{t<0} U \cdot t$ for a shrinking neighborhood U of A. Thus $B(A)$ is an open set.

Here, we give a new definition of intertwined basins of attractors.

Definition 2.3. Let M be a smooth two-dimensional manifold and $p \in M$. Then we call that the dynamical system (2.1) has intertwined basins of attraction beside p, providing there exists a small sector S, where p is the vertex, such that for any $\varepsilon > 0$ and any radius L in the sector S, such that both

$$L \cap B(A_1) \cap D(p, \varepsilon) \neq \emptyset \quad \text{and} \quad L \cap B(A_2) \cap D(p, \varepsilon) \neq \emptyset \tag{2.2}$$

hold for two different attractors A_1 and A_2.

From the above definition, beside p the basins $B(A_1)$ and $B(A_2)$ approach to p alternately, meanwhile they become narrower and narrower. By the continuity of dependence on initial conditions, we can say that the basins $B(A_1)$ and $B(A_2)$ intertwine together beside p.

Denote O to be a saddle point of system (2.1), then the stable manifold $W^s(O)$ and unstable manifold $W^u(O)$ are defined to be the following sets:

$$W^s(O) = \{p \in M \mid p \cdot t \to O \text{ as } t \to +\infty\},$$

$$W^u(O) = \{p \in M \mid p \cdot t \to O \text{ as } t \to -\infty\}.$$

We note that the existence of a saddle point with its two branches of unstable manifold approaching different attractors plays an essential role in occurrence of intertwined basins of attraction.

3. Main Results

We now assume that the flow f defined by the system (2.1) has a saddle point O and two attractors A_1 and A_2, where A_1 and A_2 need not be equilibria as in the system (2.1). Let $B(A_1)$ and $B(A_2)$ be, respectively, the basins of A_1 and A_2. Denote by $W_1^s(O)$ and $W_2^s(O)$ the two branches of the stable manifold $W^s(O)$, similarly, $W_1^u(O)$ and $W_2^u(O)$, respectively, denote the two branches of the unstable manifold $W^u(O)$.

Theorem 3.1. *Suppose that the system* (2.1) *has a saddle point O and its two branches of unstable manifold connect two distinct attractors A_1 and A_2. If the* α*-limit set $\alpha(q)$ of $q \in W^s(O) \setminus \{O\}$ has at least two points, then the system* (2.1) *has the intertwining property.*

Proof. Now let $M^* = M \cup \{\infty\}$ be the one-point compactification of M. Extend the dynamical system f on M to a dynamical system f^* on M^*, where f^* is given by $f^*(x, t) = f(x, t)$ for $x \in M$, $t \in R$ and $f^*(\infty, t) = \infty$ for all $t \in R$. The α-limit set $\alpha(q)$ of $q \in W^s(O) \setminus \{O\}$ is a compact and connected set in M by Theorem 3.6 [2, p. 23]. Since $\alpha(q)$ has at least two points, $\alpha(q)$ contains uncountable points. So there is a connected component containing uncountable points, furthermore, there at least exists an arc S connecting two points in this connected component. Otherwise, arbitrarily choose two distinct points p_1 and p_2 in the connected component, there exists a $\delta(> 0)$ such that $B(p_1, \delta)$ and $B(p_2, \delta)$

are relatively compact sets and there is no point of $\alpha(q)$ on $\partial B(p_1, \delta)$ and $\partial B(p_2, \delta)$, which, respectively, are the boundaries of $B(p_1, \delta)$ and $B(p_2, \delta)$. Since p_1 and p_2 in $\alpha(q)$, there exist sequences $\{t'_n\}(t'_n \to -\infty)$ and $\{t''_n\}$ $(t''_n \to -\infty)$ such that $qt'_n \in B(p_1, \delta)$ and $qt''_n \in B(p_2, \delta)$. Now choose a sequence $\{\tau_n\}(t'_n < \tau_n < t''_n)$ such that $q\tau_n$ in $\partial B(p_1, \delta)$. By the compactness of $\overline{B(p_1, \delta)}$, $\partial B(p_1, \delta)$ is compact, so $\{q\tau_n\}$ has a convergence subsequence $\{q\tau'_n\}$ whose convergence point in $\partial B(p_1, \delta)$. That is, there exists a point in $\alpha(q) \cap \partial B(p_1, \delta)$, which is a contradiction. Hence there exists an arc *C* containing at least two points in some connected component of $\alpha(q)$. Now we arbitrarily choose one arc *C* of $\alpha(q)$. Also, choose a point $p \in C$ which is not the end point of *C*. Since $p \in \alpha(q)$, choose a sector *S* beside the arc *C*, where *p* is the vertex of *S*, such that the intersection of *S* and *C* has only a point *p*, and $S \cap \alpha(q) = \{p\}$. For any radius *L* in the sector *S*, there exists a $t_1 = t_1(L) < 0$ such that $qt_1 \in L$, otherwise, by the continuous axiom, we can show that *L* is contained in $\alpha(q)$, which contradicts with $S \cap \alpha(q) = \{p\}$. Without loss of generality, qt_1 is the other end point of *L*. Hence we can assert that all the orbits cross *L* in the same direction beside the point *p* by the continuous dependence on initial conditions when *L* is sufficiently small. Then the negative semi-orbit $q \cdot R^-$ crosses *L* successively at t_i with $0 > t_1 > t_2 > \cdots (t_i \to -\infty)$ and $q \cdot t_i$ tends monotonously to *p* along *L* (see [3, Chapter 7]). On the other hand, obviously, one branch of unstable manifold $W^u(O) \backslash \{O\}$ lies in $B(A_1)$, and the other in $B(A_2)$. Thus $B(A_1)$ and $B(A_2)$ are, respectively, open neighborhoods of two branches of unstable manifold $W^u(O) \backslash \{O\}$. Now, for any $\varepsilon > 0$, we have $q \cdot t_k \in D(p, \varepsilon/2)$ for a sufficiently large $|t_k|$. By the continuity of dependence on initial conditions, there exists a $\delta > 0$ such that $D(q \cdot t_k, \delta) \subset D(p, \varepsilon)$ holds. Consider the diffeomorphism $F = f(*, 1) : M \to M$. By the Inclination Lemma [4, p. 82], it is easy to see that both $F^n(D(qt_k, \delta)) \cap B(A_1) \neq \varnothing$ and $F^n(D(qt_k, \delta))$ $\cap B(A_2) \neq \varnothing$ hold for a sufficiently large *n*. Hence we obtain that $f(D(qt_k, \delta), n)$ $\cap B(A_1) \neq \varnothing$ and $f(D(qt_k, \delta), n) \cap B(A_2) \neq \varnothing$. It implies that two components of $D(q \cdot t_k, \delta) \backslash q \cdot R$ also lie, respectively, in $B(A_1)$ and $B(A_2)$. Thus it follows that (2.2) in Definition 2.3 is true, so we are done.

Remark 3.2. In applications of Theorem 3.1, we need to determine the location of stable manifold $W^s(O)$. By the notations in Theorem 3.1, $\alpha(q)$ may be either bounded or unbounded, also $\alpha(q)$ may have equilibria (see [9]). Of course, $\alpha(q)$ can be a closed orbit in [12].

Theorem 3.3. *Suppose that* $W_1^u(O) \subset B(A_1)$ *and* $W_2^u(O) \subset B(A_2)$. *If the system* (2.1) *has the intertwining property, then the* α*-limit set* $\alpha(q)$ *of* $q \in W_1^s(O) \setminus \{O\}$ *or* $q \in W_2^s(O) \setminus \{O\}$ *is not empty*.

The proof of the above theorem is similar to the proof of Theorem 3 in [14]. We omit it.

4. An Example

To characterize the intertwining property of basins of attraction of the dynamical systems on a smooth manifold, we give the following example.

We now consider the dynamical system defined by the following differential equations in the disc $D = \{(x, y) | x^2 + y^2 \leq 3\}$:

$$\dot{x} = y \cdot (x^2 + y^2 - 3), \quad \dot{y} = (x - x^3 - \phi(x^2 + y^2) \cdot y)(x^2 + y^2 - 3), \tag{4.1}$$

where ϕ is a sufficiently smooth function satisfying $\phi(x^2 + y^2) > 0$ for $0 \leq x^2 + y^2 < 3$ and $\phi(x^2 + y^2) = 0$ for $x^2 + y^2 = 3$. By the Gluing lemma in [6], we consider a dynamical system defined on the sphere S^2 with the center $O(o, o)$ and the radius $r = 3$ as follows: the flow of the system (4.1) vertically projects to the northern hemisphere; similarly, the flow of the system (4.1) vertically projects to the flow on the southern; on the equator the flow homeomorphism the flow on $x^2 + y^2 = 3$ of the system (4.1), so on the sphere S^2 we define a dynamical system. It is easy to verify that the system (4.1) has a saddle point and two sinks $(1, 0)$, $(-1, 0)$. Take $H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$ and note that $H(x, y)$ is the first integral of unforced and undamped Duffing oscillator. But now we have $\dot{H}(x, y)|_{(4.1)} = -\phi(x^2 + y^2) \cdot y^2(x^2 + y^2 - 3)$. Thus it follows that if an orbit lies in the open disc $\{(x, y) \in R^2 \mid x^2 + y^2 < 3\}$ at some time t_0, then $p \cdot t$ goes to one of the three equilibria of the system (4.1). Also, the stable manifold of $O = (0, 0)$ is bounded, hence in this case for $q \in W^s(O) \setminus \{O\}$ the α-limit set $\alpha(q)$ must be compact and connected. Actually, $\alpha(q)$ is the simple closed curve $\{(x, y) \in R^2 \mid x^2 + y^2 = 3\}$, which consists of singular points. So the equator of S^2 is the limit set $\alpha(q)$ of the projection q of $p \in W^s(O) \setminus \{O\}$. From Theorem 3.1, we immediately assert that the system on the sphere S^2 has the intertwining property by two sides of the equator of S^2.

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