



THE DISCRETE AGGLOMERATION MODEL: SOLUTION OF THE FUNDAMENTAL AGGLOMERATION PROBLEM WITH A TIME-VARYING KERNEL

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Abstract

Agglomeration of particles in a fluid environment is an integral part of many industrial processes and has been the subject of scientific investigation. The fundamental mathematical problem is the determination of the number of particles of each particle-type as a function of time for a system of particles that may agglutinate during two particle collisions. In this paper, we document a complete solution to the Fundamental Agglomeration Problem (FAP) with a time-varying kernel.

1. Introduction

Agglomeration of particles in a fluid environment (e.g., a chemical reactor or the atmosphere) is an integral part of many industrial processes (e.g., Goldberger [2]) and has been the subject of scientific investigation (e.g., Siegel [15]). The fundamental mathematical problem is the determination of the number of particles of each *particle-type* as a function of time for a system of particles that may agglutinate during two particle collisions. Little or no work has been done for systems where particle-type requires several variables. Efforts have been focused on a particle-type list with only one variable, size (or mass). This allows use of what is often referred to as the *coagulation equation* which has been well studied in aerosol research

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(Drake [1]). Original work on this equation was done by Smoluchowski [16] and it is also referred to as *Smoluchowski's equation*. The *agglomeration equation* is perhaps more descriptive since the term coagulation implies a process carried out until solidification whereas we focus on the agglomeration process; that is, on the determination of a time-varying *particle-size distribution* even if coagulation is never reached.

In his original work, Smoluchowski considered the agglomeration equation in a discrete form. Later it was considered in a continuous form by Müller [14]. In either case, an *initial particle-size distribution* to specify the initial number of particles for each size is needed to complete the *initial value problem* (IVP). We refer to these as the *Discrete Agglomeration Model* and the *Continuum Agglomeration Model*. Since both models have an infinite number of sizes, the *state* (or *phase*) *space* is infinite dimensional. Solution of either model yields an updated particle-size distribution giving *number densities* as time progresses. For various conditions, studies of these models include Morganstern [9], Melzak [8], Marcus [5], Spouge [17], McLeod [7], White [19], Treat [18], McLaughlin et al. [6] and Moseley [13]. Moseley divided the Discrete Agglomeration Model up into several problems to be considered separately. This allows individual progress on the separate problems.

Let \mathbf{R} be the real numbers, $Int_o = \{I \subseteq \mathbf{R} : I \text{ is a finite, infinite or semi-infinite open interval}\}$, and for $I \in Int_o$, $C(I, \mathbf{R}) = \{f : I \rightarrow \mathbf{R} : f \text{ is continuous on } I\}$. Under certain conditions, a reasonably complicated change of (both the independent and dependent) variables transforms the Discrete Agglomeration Model (see Moseley [13]) into an IVP consisting of an infinite system of nonlinear Ordinary Differential Equations (ODEs) each with an Initial Condition (IC) that may be written in scalar form as:

$$\text{System of ODE's: } \frac{d\tilde{x}_i}{d\tau} = \frac{1}{2} \sum_{j=1}^{i-1} \tilde{A}(\tau) \tilde{x}_j \tilde{x}_{i-j}, \quad \tau \in \mathcal{J}_p. \quad (1.1)$$

$$\text{IVP} \quad i \in \mathbf{N} = \{1, 2, 3, \dots\}.$$

$$\text{IC's} \quad \tilde{x}_i(0) = n_i^0, \quad (1.2)$$

where $\tilde{A}(\tau) \in C(\mathcal{J}_p, \mathbf{R})$ is the *kernel*, τ is the *scaled time*, and for $i = 1$, the empty sum on the right hand side of (1.1) is assumed to be zero. We refer to this IVP as the *Fundamental Agglomeration Problem (FAP)* with a time-varying kernel. A solution

is a time-varying “vector” $\tilde{x} = \{\tilde{x}(t)\}_{i=1}^\infty \in \tilde{C}(\mathcal{J}_p, \mathbf{R}^\infty) = \{\tilde{x} = \{\tilde{x}(t)\}_{i=1}^\infty : \tilde{x}(t) \in C(\mathcal{J}_p, \mathbf{R})\}$ that satisfies (1.1) on \mathcal{J}_p and has an initial number density given by

$$\bar{n}_0 = \{n_i^0\}_{i=1}^\infty \in \mathbf{R}^\infty = \{\{a_i\}_{i=1}^\infty : a_i \in \mathbf{R}\}.$$

Thus, we show that (even though (1.1) is nonlinear) the interval of validity is indeed \mathcal{J}_p , (i.e., exactly where $A(\tau)$ is defined). We have used the *extended interval notation*, $\mathcal{J}_p = (\tau_{0-}, 0, \tau_{0+})$ to indicate that $0 \in \mathcal{J}_p = (\tau_{0-}, \tau_{0+})$. Note that we can move $\tilde{A}(\tau)$ out of the summation as “stickiness” depends on time, but not on particle size. Unlike the Discrete Agglomeration Model, FAP does not contain an infinite series which needs to be shown to converge.

As continuity of $A(\tau)$ is sufficient for a unique solution of FAP in $\tilde{C}(\mathcal{J}_p, \mathbf{R}^\infty)$, this leads to a larger function space for uniqueness for the Discrete Agglomeration Model than was claimed by Moseley. To accomplish this and to provide details not provided by Moseley, in this paper we provide complete documentation of a solution of FAP for a general (physical and non-physical) time-varying kernel $\tilde{A}(\tau)$. We begin with a recursive solution which provides existence and uniqueness and then derive the explicit solution:

$$\tilde{x}_i(\tau) = n_i^0 + \sum_{n=1}^{i-1} \frac{k_i^{(n+1)}}{2^n} [\mathcal{A}(\tau)]^n, \quad (1.3)$$

where

$$\mathcal{A}(\tau) = \int_{\sigma=0}^{\sigma=\tau} \tilde{A}(\sigma) d\sigma$$

and

$$k_i^{(n+1)} = \sum_{i_1+i_2+\dots+i_{n+1}=i} n_{i_1}^0 n_{i_2}^0 \dots n_{i_{n+1}}^0.$$

In later work, we will investigate further the other problems defined by Moseley and detail the change of variables which leads to FAP. Recall that $k_i^{(n+1)}$ is the

coefficient of s^i in $\left[\sum_{i=1}^\infty n_i^0 s^i \right]^{n+1}$ using the Henrici notation (Henrici [3]) so that

$k_i^{(n+1)} = 0$ for $n > i - 1$. Note that

$$\begin{aligned} k_i^{(2)} &= \sum_{i_1+i_2=i} n_{i_1}^0 n_{i_2}^0 = \sum_{j=1}^{i-1} [n_j^0 n_{i-j}^0], \\ k_i^{(3)} &= \sum_{j=1}^{i-2} [n_j^0 k_{i-j}^{(2)}], \quad k_i^{(4)} = \sum_{j=2}^{i-2} [k_j^{(2)} k_{i-j}^{(2)}], \\ k_i^{(5)} &= \sum_{j=3}^{i-2} [k_j^{(3)} k_{i-j}^{(2)}], \quad k_i^{(6)} = \sum_{j=3}^{i-3} [k_j^{(3)} k_{i-j}^{(3)}] \end{aligned}$$

and in general

$$k_i^{(n+2)} = \sum_{j=1}^{i-n-1} n_j^0 k_{i-j}^{(n+1)} = \sum_{j=m+1}^{i-n-1+m} k_j^{(m+1)} k_{i-j}^{(n+1-m)}. \quad (1.4)$$

To rearrange terms in finite and infinite series, we will need

$$\sum_{i=1}^{k-1} \sum_{j=1}^{k-i} a_{i,j} = \sum_{j=1}^{k-1} \sum_{i=1}^{k-j} a_{i,j}, \quad \sum_{i=1}^k \sum_{j=1}^{i-1} a_{i,j} = \sum_{i=2}^k \sum_{j=1}^{i-1} a_{i,j} = \sum_{j=1}^{k-1} \sum_{i=j+1}^k a_{i,j}, \quad (1.5)$$

$$\sum_{i=1}^{\infty} \sum_{j=1}^{i-1} a_{i-j,j} = \sum_{j=1}^{\infty} \sum_{i=j+1}^{\infty} a_{i,j}, \quad \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} a_{i,j} = \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=j+1}^{\infty} a_{i,j}. \quad (1.6)$$

By change of variables, we obtain

$$\sum_{j=1}^{i-m-1} \sum_{n=1}^{j-1} a_{j,n} = \sum_{j=2}^{i-m-1} \sum_{n=1}^{j-1} a_{j,n} = \sum_{n=1}^{i-m-2} \sum_{j=n+1}^{i-m-1} a_{j,n}, \quad \sum_{j=1}^{i-2} \sum_{n=1}^{i-1-j} a_{j,n} = \sum_{n=1}^{i-2} \sum_{j=1}^{i-1-n} a_{j,n}, \quad (1.7)$$

$$\sum_{m=1}^{i-3} \sum_{n=1}^{i-2-m} a_{m,n} = \sum_{n=1}^{i-3} \sum_{m=1}^{i-2-n} a_{m,n}, \quad \sum_{n=1}^{i-2} \sum_{m=1}^{n-1} a_{n,m} = \sum_{n=2}^{i-2} \sum_{m=1}^{n-1} a_{n,m} = \sum_{m=1}^{i-3} \sum_{n=m+1}^{i-2} a_{n,m}. \quad (1.8)$$

2. Solution of FAP

We solve the Fundamental Agglomeration Problem (FAP) recursively and then explicitly. Recall that the problem parameters are $(\mathcal{J}_p, A(\tau), \bar{n}_0) \in \text{Int}_0 \times C(\mathcal{J}_p, \mathbf{R}) \times \mathbf{R}^\infty$. Since (1.1) is a sequentially linear system (i.e., only backward coupled), it is easy to see that FAP has a unique solution with interval of validity \mathcal{J}_p . It can be solved recursively by forward substitution, i.e., we first obtain $\tilde{x}_1(\tau) = n_1^0$ and then for $i \geq 2$, $\tilde{x}_i(t)$ can be obtained recursively from $\tilde{x}_j(t)$, $0 \leq j < i$, up to obtaining antiderivatives (i.e., up to quadrature) in a finite number of steps. Hence the solution $\tilde{\tilde{x}}(t) = \{\tilde{x}_i(t)\}_{i=1}^\infty$ of FAP can be computed recursively as:

$$\tilde{x}_i(\tau) = n_i^0 + \frac{1}{2} \int_{\sigma=0}^{\sigma=\tau} \tilde{A}(\sigma) \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\sigma) \tilde{x}_j(\sigma) d\sigma, \quad i \in \mathbf{N}, \quad (2.1)$$

where the empty sum for $i = 1$ means $\tilde{x}_1(\tau) = n_1^0$. Thus $\tilde{\tilde{x}}(\tau) = \{\tilde{x}_i(\tau)\}_{i=1}^\infty$ solves FAP in $\tilde{C}(\mathcal{J}_p, \mathbf{R}^\infty)$ with interval of validity \mathcal{J}_p . We have proved:

Theorem 2.1. *Let $(\mathcal{J}_p, A(\tau), \bar{n}_0) \in \text{Int}_0(t_0, I_0) \times C(\mathcal{J}_p, \mathbf{R}) \times \mathbf{R}^\infty$. Then FAP has a unique solution given recursively by (2.1) with interval of validity \mathcal{J}_p .*

Again, even though (1.1) is nonlinear, since it is sequentially linear, we have a unique solution with interval of validity \mathcal{J}_p .

Example. If $\tilde{A}(\tau) = \tilde{A}_0 = \text{constant}$, then

$$\tilde{x}_i(\tau) = n_i^0 + \frac{\tilde{A}_0}{2} \int_{\sigma=0}^{\sigma=\tau} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\sigma) \tilde{x}_j(\sigma) d\sigma,$$

so that

$$\tilde{x}_1(\tau) = n_1^0, \quad \tilde{x}_2(\tau) = n_2^0 + \frac{\tilde{A}_0}{2} \int_{\sigma=0}^{\sigma=\tau} (n_1^0)^2 d\sigma = n_2^0 + \frac{\tilde{A}_0}{2} (n_1^0)^2 \tau,$$

$$\tilde{x}_3(\tau) = n_3^0 + \frac{\tilde{A}_0}{2} \int_{\sigma=0}^{\sigma=\tau} \left(n_2^0 + \frac{\tilde{A}_0}{2} (n_1^0)^2 \tau \right) ((n_1^0)^2) + \left(n_2^0 + \frac{\tilde{A}_0}{2} (n_1^0)^2 \tau \right)^2 d\sigma, \dots$$

It is easy to see that $\tilde{x}_i(\tau)$ is a polynomial in τ and that \mathcal{J}_p can be taken to be \mathbf{R} . Also, it is clear that if $\tilde{A}(\tau)$ is a polynomial in τ , then $\tilde{x}_i(\tau)$ is a polynomial in τ and $\mathcal{J}_p = \mathbf{R}$.

To obtain an explicit formula for $\tilde{x}_i(\tau)$, we use two steps. We first derive the solution without worrying about rigor. In particular, we do not need to worry about convergence of infinite series. We then prove that our formula is correct on \mathcal{J}_p by checking the initial conditions and then substituting the formula into the (system of) ODEs (1.1). Since (1.3) contains no infinite series, this will not require proof of convergence.

To solve for $\tilde{x}_i(\tau)$ explicitly, we assume $\tilde{x}(\tau) = \{\tilde{x}_i(\tau)\}_{i=1}^{\infty}$ is a solution to (1.1) and (1.2) and use a generating function:

$$G(s, \tau) = \sum_{i=1}^{\infty} \tilde{x}_i(\tau) s^i. \quad (2.2)$$

The process of using a generating function may be considered to be a discrete analogue of the Laplace transform. $G(s, \tau)$ is a formal sum that “sums out” the discrete variable i , but maintains the integrity of $\tilde{x}_i(\tau)$ as the coefficient of s^i . The variable τ is treated as a parameter. Again, since we already have existence and uniqueness and will show that the formula is correct by substitution, we do not need to worry about rigor in the derivation. (Rigor may be useful for a better understanding.) Multiplying (each equation in) (1.1) by s^i and summing over i , we obtain formally

$$\begin{aligned} \sum_{i=1}^{\infty} s^i \frac{d\tilde{x}_i}{d\tau} &= \sum_{i=1}^{\infty} s^i \frac{1}{2} \sum_{j=1}^{i-1} [\tilde{A}(\tau)] \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) \\ &= \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} s^i [\tilde{A}(\tau)] \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) \\ &= \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} s^{i-j+j} [\tilde{A}(\tau)] \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \tilde{A}(\tau) \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} s^{i-j} \tilde{x}_{i-j}(\tau) s^j \tilde{x}_j(\tau) \\
&= \frac{1}{2} \tilde{A}(\tau) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} s^i \tilde{x}_i(\tau) s^j \tilde{x}_j(\tau) \\
&= \frac{1}{2} \left[\tilde{A}(\tau) \sum_{i=1}^{\infty} s^i \tilde{x}_i(\tau) \sum_{j=1}^{\infty} s^j \tilde{x}_j(\tau) \right] \\
&= \frac{\tilde{A}(\tau)}{2} G^2(s, \tau),
\end{aligned}$$

where we have used the first infinite sum in (1.6). Hence we have

$$\sum_{i=1}^{\infty} s^i \frac{d\tilde{x}_i}{d\tau} = \frac{\tilde{A}(\tau)}{2} G^2(s, \tau). \quad (2.3)$$

We have an infinite sum in (2.3), but recall that we do not require rigor. Since (under certain conditions)

$$\sum_{i=1}^{\infty} s^i \frac{d\tilde{x}_i}{d\tau} = \frac{\partial \left[\sum_{i=1}^{\infty} s^i \tilde{x}_i(\tau) \right]}{\partial \tau} = \frac{\partial G(s, \tau)}{\partial \tau}, \quad (2.4)$$

we obtain the nonlinear (partial) differential equation

$$\frac{\partial G(s, \tau)}{\partial \tau} = \frac{\tilde{A}(\tau)}{2} G^2(s, \tau) \quad (2.5)$$

with the initial condition

$$G(s, 0) = \sum_{i=1}^{\infty} s^i \tilde{x}_i(0) = \sum_{i=1}^{\infty} s^i n_i^0. \quad (2.6)$$

Since there are no derivatives with respect to s , we may view (2.5) as a first order separable ODE in τ with s being a parameter and solve to obtain

$$G(s, \tau) = -\frac{1}{\frac{\tilde{\mathcal{A}}(\tau; t_0, p_g)}{2} + \phi(s)} = -\frac{2}{\tilde{\mathcal{A}}(\tau; t_0, p_g) + 2\phi(s)}, \quad (2.7)$$

where $\tilde{\mathcal{A}}(\tau) = \int_{\sigma=0}^{\sigma=\tau} \tilde{A}(\sigma) d\sigma$ and ϕ is an arbitrary function of s . Applying the initial condition (2.6), we obtain

$$\begin{aligned} G(s, 0) &= -\frac{2}{\tilde{\mathcal{A}}(0; t_0, p_g) + 2\phi(s)} = -\frac{2}{0 + 2\phi(s)} = -\frac{1}{\phi(s)} \\ &= \sum_{i=1}^{\infty} s^i x_i(0) = \sum_{i=1}^{\infty} s^i n_i^0 \end{aligned} \quad (2.8)$$

so that

$$\phi(s) = -\frac{1}{\sum_{i=1}^{\infty} s^i n_i^0}. \quad (2.9)$$

Hence, substituting ϕ given by (2.7) into (2.9), we have

$$G(s, \tau) = \frac{\sum_{i=1}^{\infty} s^i n_i^0}{1 - (1/2)\tilde{\mathcal{A}}(\tau; t_0, p_g) \sum_{i=1}^{\infty} s^i n_i^0} = \frac{G(s, 0)}{1 - (1/2)\tilde{\mathcal{A}}(\tau; t_0, p_g) G(s, 0)}. \quad (2.10)$$

Obtaining $x_i(\tau)$ is analogous to obtaining the inverse Laplace transform. But before obtaining $x_i(\tau)$ explicitly, we derive a recursion formula that gives the values of the integrals in (2.1) in terms of $\tilde{x}_j(\tau)$ for $1 \leq j < i$. Multiplying (2.10) by the denominator, we obtain

$$\begin{aligned} G(s, \tau)[1 - (1/2)\tilde{\mathcal{A}}(\tau; t_0, p_g)G(s, 0)] &= G(s, 0), \\ G(s, \tau) &= G(s, 0) + (1/2)\tilde{\mathcal{A}}(\tau)G(s, 0)G(s, \tau). \end{aligned} \quad (2.11)$$

Hence, substituting (2.2) and (2.6) into (2.11) and using the first infinite sum in (1.6), we have

$$\sum_{i=1}^{\infty} \tilde{x}_i(\tau) s^i = \sum_{i=1}^{\infty} n_i^0 s^i + \frac{1}{2} \tilde{\mathcal{A}}(\tau) \sum_{i=1}^{\infty} n_i^0 s^i \sum_{i=1}^{\infty} \tilde{x}_i(\tau) s^i$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty} n_i^0 s^i + \frac{1}{2} \mathcal{A}(\tau) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} n_i^0 s^i \tilde{x}_j(\tau) s^j \\
&= \sum_{i=1}^{\infty} n_i^0 s^i + \frac{1}{2} \mathcal{A}(\tau) \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} n_{i-j}^0 s^{i-j} \tilde{x}_j(\tau) s^j \\
&= \sum_{i=1}^{\infty} n_i^0 s^i + \frac{1}{2} \sum_{i=1}^{\infty} s^i \mathcal{A}(\tau) \sum_{j=1}^{i-1} n_{i-j}^0 \tilde{x}_j(\tau) \\
&= \sum_{i=1}^{\infty} s^i \left[n_i^0 + \frac{1}{2} \mathcal{A}(\tau) \sum_{j=1}^{i-1} n_{i-j}^0 \tilde{x}_j(\tau) \right]
\end{aligned}$$

so that

$$\tilde{x}_i(\tau) = n_i^0 + \frac{1}{2} \mathcal{A}(\tau) \sum_{j=1}^{i-1} n_{i-j}^0 \tilde{x}_j(\tau). \quad (2.12)$$

This provides an explicit formula for the integral in (2.1) in terms of the previous $\tilde{x}_j(\tau)$ for $1 \leq j < i$ without having to find antiderivatives. We have

$$\int_{\sigma=0}^{\sigma=\tau} \tilde{A}(\sigma) \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\sigma) \tilde{x}_j(\sigma) d\sigma = \mathcal{A}(\tau) \sum_{j=1}^{i-1} n_{i-j}^0 \tilde{x}_j(\tau), \quad i \in \mathbf{N}. \quad (2.13)$$

We now compute $\tilde{x}_i(\tau)$ explicitly. From (2.11), using the geometric series and a property of $k_i^{(n+1)}$, we have

$$\begin{aligned}
G(s, \tau) &= \frac{G(s, 0)}{1 - \frac{1}{2} \mathcal{A}(\tau) G(s, 0)} \\
&= G(s, 0) \sum_{n=0}^{\infty} \left[\frac{1}{2} \mathcal{A}(\tau) G(s, 0) \right]^n \\
&= G(s, 0) + \sum_{n=1}^{\infty} \left[\frac{1}{2} \mathcal{A}(\tau) \right]^n [G(s, 0)]^{n+1}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty} n_i^0 s^i + \sum_{n=1}^{\infty} \left[\frac{1}{2} \tilde{\mathcal{A}}(\tau) \right]^n \left[\sum_{i=1}^{\infty} n_i^0 s^i \right]^{n+1} \\
&= \sum_{i=1}^{\infty} n_i^0 s^i + \sum_{n=1}^{\infty} \left[\frac{1}{2} \tilde{\mathcal{A}}(\tau) \right]^n \left[\sum_{i=n+1}^{\infty} k_i^{(n+1)} s^i \right].
\end{aligned}$$

Hence using the second infinite sum in (1.6), we obtain

$$\begin{aligned}
\sum_{i=1}^{\infty} \tilde{x}_i(\tau) s^i &= \sum_{i=1}^{\infty} n_i^0 s^i + \sum_{n=1}^{\infty} \left[\frac{1}{2} \tilde{\mathcal{A}}(\tau) \right]^n \left[\sum_{i=n+1}^{\infty} k_i^{(n+1)} s^i \right] \\
&= \sum_{i=1}^{\infty} n_i^0 s^i + \left[\sum_{n=1}^{\infty} \sum_{i=n+1}^{\infty} \left[\frac{1}{2} \tilde{\mathcal{A}}(\tau) \right]^n k_i^{(n+1)} s^i \right] \\
&= \sum_{i=1}^{\infty} n_i^0 s^i + \left[\sum_{i=1}^{\infty} \sum_{n=1}^{i-1} \frac{1}{2^n} [\tilde{\mathcal{A}}(\tau)]^n k_i^{(n+1)} s^i \right] \\
&= \sum_{i=1}^{\infty} \left[n_i^0 + \sum_{n=1}^{i-1} \frac{k_i^{(n+1)}}{2^n} [\tilde{\mathcal{A}}(\tau)]^n \right] s^i \tag{2.14}
\end{aligned}$$

so that $\tilde{x}_i(\tau)$ is given by (1.3).

3. Verification of the Explicit Solution

We will show that $\tilde{x}_i(\tau)$ given by (1.3) is the solution of the Fundamental Agglomeration Problem (FAP) with interval of validity \mathcal{I}_p by substituting it directly into (1.1). Note that the domains of $\tilde{A}(\tau)$ and $\tilde{\mathcal{A}}(\tau) = \int_{\sigma=0}^{\sigma=\tau} \tilde{A}(\sigma) d\sigma$ are both \mathcal{I}_p so that the domain of $\tilde{x}_i(\tau)$ is also \mathcal{I}_p . Clearly, $\tilde{x}_i(\tau)$ satisfies the initial condition in (1.2) as $\tilde{\mathcal{A}}(0) = 0$. To substitute into the differential equation in (1.1), we first compute the derivative of $\tilde{x}_i(\tau)$

$$\frac{d\tilde{x}_i}{d\tau} = \left[\sum_{n=1}^{i-1} \frac{k_i^{(n+1)}}{2^n} n [\tilde{\mathcal{A}}(\tau)]^{n-1} \right] \tilde{A}(\tau). \tag{3.1}$$

Equating this with the right hand side of (1.1), we obtain

$$\left[\sum_{n=1}^{i-1} \frac{k_i^{(n+1)}}{2^n} n [\mathcal{A}(\tau)]^{n-1} \right] \tilde{A}(\tau) = \frac{\tilde{A}(\tau)}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau). \quad (3.2)$$

Cancelling $\tilde{A}(\tau)$, we see that to show that $\tilde{x}_i(\tau)$ satisfies (1.1), we must show for $i \in \mathbf{N}$ that

$$\begin{aligned} \frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) &= \sum_{n=1}^{i-1} \frac{k_i^{(n+1)}}{2^n} n [\mathcal{A}(\tau)]^{n-1} \\ &= \frac{k_i^{(2)}}{2} + \frac{k_i^{(3)}}{2^2} 2 [\mathcal{A}(\tau)] + \cdots + \frac{k_i^{(i)}}{2^{i-1}} (i-1) [\mathcal{A}(\tau)]^{i-2}. \end{aligned} \quad (3.3)$$

For $i = 1$, both sums in (3.3) are empty so that both sides are zero. For $i = 2$ and 3, it is straightforward to compute both sides of (3.3) to see that they are equal. For $i \geq 4$, we substitute $\tilde{x}_i(\tau)$ into the left hand side of (3.3) and obtain the right hand side:

$$\begin{aligned} &\frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) \\ &= \frac{1}{2} \sum_{j=1}^{i-1} \left\{ \left[n_j^0 + \sum_{n=1}^{j-1} \frac{k_j^{(n+1)}}{2^n} [\mathcal{A}(\tau)]^n \right] \left[n_{i-j}^0 + \sum_{n=1}^{i-j-1} \frac{k_{i-j}^{(n+1)}}{2^n} [\mathcal{A}(\tau)]^n \right] \right\} \\ &= \frac{1}{2} \sum_{j=1}^{i-1} [n_j^0 n_{i-j}^0] + \frac{1}{2} \sum_{j=1}^{i-1} n_j^0 \sum_{n=1}^{i-j-1} \frac{k_{i-j}^{(n+1)}}{2^n} [\mathcal{A}(\tau)]^n + \frac{1}{2} \sum_{j=1}^{i-1} n_{i-j}^0 \sum_{n=1}^{j-1} \frac{k_j^{(n+1)}}{2^n} [\mathcal{A}(\tau)]^n \\ &\quad + \frac{1}{2} \sum_{j=1}^{i-1} \left\{ \left[\sum_{n=1}^{j-1} \frac{k_j^{(n+1)}}{2^n} [\mathcal{A}(\tau)]^n \right] \left[\sum_{m=1}^{i-j-1} \frac{k_{i-j}^{(m+1)}}{2^m} [\mathcal{A}(\tau)]^m \right] \right\}. \end{aligned} \quad (3.4)$$

Since $k_i^{(2)} = \sum_{i_1+i_2=i} n_{i_1}^0 n_{i_2}^0 = \sum_{j=1}^{i-1} n_j^0 n_{i-j}^0$, the first term on the right hand side in (3.4)

is $\frac{1}{2} k_i^{(2)}$. In the second term, when $j = i - 1$, the inside sum is empty. In the third

sum, when $j = 1$, the inside sum is empty. In the fourth term, when $j = i - 1$, the second inside sum is empty. Hence (3.4) becomes

$$\begin{aligned}
& \frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) \\
&= \frac{1}{2} k_i^{(2)} + \frac{1}{2} \sum_{j=1}^{i-2} n_j^0 \sum_{n=1}^{i-j-1} \frac{k_{i-j}^{(n+1)}}{2^n} [\tilde{\mathcal{A}}(\tau)]^n + \frac{1}{2} \sum_{j=2}^{i-1} n_{i-j}^0 \sum_{n=1}^{j-1} \frac{k_j^{(n+1)}}{2^n} [\tilde{\mathcal{A}}(\tau)]^n \\
&+ \frac{1}{2} \sum_{j=1}^{i-2} \left\{ \left[\sum_{n=1}^{j-1} \frac{k_j^{(n+1)}}{2^n} [\tilde{\mathcal{A}}(t)]^n \right] \left[\sum_{m=1}^{i-j-1} \frac{k_{i-j}^{(m+1)}}{2^m} [\tilde{\mathcal{A}}(t)]^m \right] \right\}. \tag{3.5}
\end{aligned}$$

In the third term on the right hand side of (3.5), we let $\ell = i - j$ so that it becomes

$$\begin{aligned}
\frac{1}{2} \sum_{j=2}^{i-1} \left[n_{i-j}^0 \sum_{n=1}^{j-1} \frac{k_j^{(n+1)}}{2^n} [\tilde{\mathcal{A}}(\tau)]^n \right] &= \frac{1}{2} \sum_{\ell=2}^{i-1} \left[n_\ell^0 \sum_{n=1}^{i-\ell-1} \frac{k_{i-\ell}^{(n+1)}}{2^n} [\tilde{\mathcal{A}}(\tau)]^n \right] \\
&= \frac{1}{2} \sum_{j=2}^{i-1} \left[n_j^0 \sum_{n=1}^{i-j-1} \frac{k_{i-j}^{(n+1)}}{2^n} [\tilde{\mathcal{A}}(\tau)]^n \right] \\
&= \frac{1}{2} \sum_{j=2}^{i-2} \left[n_j^0 \sum_{n=1}^{i-j-1} \frac{k_{i-j}^{(n+1)}}{2^n} [\tilde{\mathcal{A}}(\tau)]^n \right], \tag{3.6}
\end{aligned}$$

where we have now replaced ℓ with j . The last step follows since when $j = i - 1$, the sum is empty. Hence, the second and third terms in (3.5) are the same and add to become

$$\sum_{n=1}^{i-2} \left[\sum_{j=1}^{i-j-1} n_j^0 \frac{k_{i-j}^{(n+1)}}{2^n} [\tilde{\mathcal{A}}(\tau)]^n \right] = \sum_{n=1}^{i-2} \left\{ \frac{1}{2^n} k_i^{(n+2)} [\tilde{\mathcal{A}}(\tau)]^n \right\}, \tag{3.7}$$

where we have used the second equation in (1.7) and then (1.4) for the last step. Substituting (3.7) into (3.5), we obtain

$$\frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau)$$

$$\begin{aligned}
&= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n \\
&\quad + \frac{1}{2} \sum_{j=1}^{i-2} \left[\sum_{n=1}^{j-1} \sum_{m=1}^{i-j-1} k_j^{(n+1)} k_{i-j}^{(m+1)} \frac{1}{2^{n+m}} [\mathcal{A}(\tau)]^{n+m} \right] \\
&= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n \\
&\quad + \frac{1}{2} \sum_{m=1}^{i-2} \left[\sum_{j=2}^{i-m-1} \sum_{n=1}^{j-1} k_j^{(n+1)} k_{i-j}^{(m+1)} \frac{1}{2^{n+m}} [\mathcal{A}(\tau)]^{n+m} \right], \tag{3.8}
\end{aligned}$$

where we have used the second equation in (1.7) with $n = m$ and the fact that the inside sum is empty, when $j = 1$. Now using the first equation in (1.7), we obtain

$$\begin{aligned}
&\frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) \\
&= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n \\
&\quad + \sum_{m=1}^{i-2} \left[\sum_{n=1}^{i-m-2} \sum_{j=n+1}^{i-m-1} k_j^{(n+1)} k_{i-j}^{(m+1)} \frac{1}{2^{n+m+1}} [\mathcal{A}(\tau)]^{n+m} \right]. \tag{3.9}
\end{aligned}$$

Noting that the inside sum is now empty when $m = i - 2$, we have

$$\begin{aligned}
\frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) &= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n \\
&\quad + \sum_{m=1}^{i-3} \sum_{n=1}^{i-m-2} \left[\sum_{j=n+1}^{i-m-1} k_j^{(n+1)} k_{i-j}^{(m+1)} \frac{1}{2^{n+m+1}} [\mathcal{A}(\tau)]^{n+m} \right]. \tag{3.10}
\end{aligned}$$

Now using the first equation in (1.8), we obtain

$$\frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) = \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n$$

$$+ \sum_{n=1}^{i-3} \left[\sum_{m=1}^{i-n-2} \sum_{j=n+1}^{i-m-1} k_j^{(n+1)} k_{i-j}^{(m+1)} \frac{1}{2^{n+m+1}} [\mathcal{A}(\tau)]^{n+m} \right]. \quad (3.11)$$

Now, we let $\ell = n + m$ to obtain

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) \\ &= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n \\ &+ \sum_{n=1}^{i-3} \left[\sum_{\ell=n+1}^{i-2} \sum_{j=n+1}^{i-(\ell-n)-1} \frac{1}{2^{\ell+1}} [\mathcal{A}(\tau)]^\ell k_j^{(n+1)} k_{i-j}^{((\ell-n)+1)} \right] m = \ell - n \\ &= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n \\ &+ \sum_{n=1}^{i-3} \left[\sum_{\ell=n+1}^{i-2} \frac{1}{2^{\ell+1}} [\mathcal{A}(\tau)]^\ell \sum_{j=n+1}^{i-(\ell-n)-1} k_j^{(n+1)} k_{i-j}^{((\ell-n)+1)} \right] \\ &= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n \\ &+ \sum_{n=1}^{i-3} \left[\sum_{\ell=n+1}^{i-2} \frac{1}{2^{\ell+1}} [\mathcal{A}(\tau)]^\ell \sum_{j=n+1}^{i+n-\ell-1} k_j^{(n+1)} k_{i-j}^{(\ell-n+1)} \right]. \quad (3.12) \end{aligned}$$

Replacing n with m , we obtain

$$\begin{aligned} \frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) &= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n \\ &+ \sum_{m=1}^{i-3} \sum_{\ell=m+1}^{i-2} \frac{1}{2^{\ell+1}} [\mathcal{A}(\tau)]^\ell \left[\sum_{j=m+1}^{i+m-\ell-1} k_j^{(m+1)} k_{i-j}^{(\ell-m+1)} \right] \quad (3.13) \end{aligned}$$

and then ℓ with n , we obtain

$$\begin{aligned}
\frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) &= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n \\
&\quad + \sum_{m=1}^{i-3} \sum_{n=m+1}^{i-2} \frac{1}{2^{n+1}} [\mathcal{A}(\tau)]^n \left[\sum_{j=m+1}^{i+m-n-1} k_j^{(m+1)} k_{i-j}^{(n-m+1)} \right]. \quad (3.14)
\end{aligned}$$

Now using (1.4) and the last sum in (1.8), we obtain

$$\begin{aligned}
&\frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) \\
&= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n + \sum_{m=1}^{i-3} \sum_{n=m+1}^{i-2} \frac{1}{2^{n+1}} [\mathcal{A}(\tau)]^n k_i^{(n+2)} \\
&= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n + \sum_{n=2}^{i-2} \sum_{m=1}^{n-1} \frac{1}{2^{n+1}} [\mathcal{A}(\tau)]^n k_i^{(n+2)} \\
&= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n + \sum_{n=2}^{i-2} \frac{1}{2^{n+1}} [\mathcal{A}(\tau)]^n k_i^{(n+2)} \left[\sum_{m=1}^{n-1} 1 \right] \\
&= \frac{1}{2} k_i^{(2)} + \sum_{n=1}^{i-2} \frac{1}{2^n} k_i^{(n+2)} [\mathcal{A}(\tau)]^n + \sum_{n=2}^{i-2} \frac{1}{2^{n+1}} [\mathcal{A}(\tau)]^n k_i^{(n+2)} (n-1).
\end{aligned}$$

Now divide and multiply the first sum on the right hand side by 2 and then write out its first term to obtain

$$\begin{aligned}
&\frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) \\
&= \frac{1}{2} k_i^{(2)} + \frac{1}{2^2} k_i^{(1+2)} 2[\mathcal{A}(\tau)]^1 + \sum_{n=2}^{i-2} \frac{1}{2^{n+1}} k_i^{(n+2)} 2[\mathcal{A}(\tau)]^n \\
&\quad + \sum_{n=2}^{i-2} \frac{(n-1)}{2^{n+1}} [\mathcal{A}(\tau)]^n k_i^{(n+2)}.
\end{aligned}$$

Now add the two sums on the right hand side together to obtain

$$\frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) = \frac{1}{2} k_i^{(2)} + \frac{1}{2^2} k_i^{(3)} 2[\tilde{\mathcal{A}}(\tau)] + \sum_{n=2}^{i-2} \frac{n+1}{2^{n+1}} [\tilde{\mathcal{A}}(\tau)]^n k_i^{(n+2)}. \quad (3.15)$$

Letting $n = m - 1$, we obtain

$$\begin{aligned} \frac{1}{2} \sum_{j=1}^{i-1} \tilde{x}_{i-j}(\tau) \tilde{x}_j(\tau) &= \frac{1}{2} k_i^{(2)} + \frac{1}{2^2} k_i^{(3)} 2[\tilde{\mathcal{A}}(\tau)] + \sum_{m=3}^{i-1} \frac{m-1+1}{2^{m-1+1}} [\tilde{\mathcal{A}}(\tau)]^{m-1} k_i^{(m-1+2)} \\ &= \frac{1}{2} k_i^{(2)} + \frac{1}{2^2} k_i^{(3)} 2[\tilde{\mathcal{A}}(\tau)] + \sum_{m=3}^{i-1} \frac{m}{2^m} [\tilde{\mathcal{A}}(\tau)]^{m-1} k_i^{(m+1)} \\ &= \frac{1}{2} k_i^{(2)} + \frac{1}{2^2} k_i^{(3)} 2[\tilde{\mathcal{A}}(\tau)] + \sum_{n=3}^{i-1} \frac{n}{2^n} [\tilde{\mathcal{A}}(\tau)]^{n-1} k_i^{(n+1)} \\ &= \sum_{n=1}^{i-1} \frac{n}{2^n} [\tilde{\mathcal{A}}(\tau)]^{n-1} k_i^{(n+1)}, \end{aligned} \quad (3.16)$$

where we have now let $m = n$. Hence, we have shown that $\tilde{x}_i(\tau)$ given by (1.3) is an explicit formula for the solution of FAP.

4. Summary

Moseley [13] divided the Discrete Agglomeration Model up into several problems which can be considered separately. This allows progress on the separate problems individually. Under certain conditions, a reasonably complicated change of (both the independent and dependent) variables transforms the Discrete Agglomeration Model into an IVP consisting of an infinite system of nonlinear Ordinary Differential Equations (ODEs) each with an Initial Condition (IC) that may be written in scalar forms as (1.1) and (1.2), where $\tilde{A}(\tau) \in C(\mathcal{I}_p, \mathbf{R})$ is the kernel. We refer to this IVP as the Fundamental Agglomeration Problem (FAP) with a time-varying kernel. We have shown that since (1.1) is sequentially linear, the unique solution to FAP is given recursively by (2.1) and (2.12) and explicitly by (1.3). In future work, we will investigate other problems defined by Moseley [13] and show the relation of these and FAP to the Discrete Agglomeration Model.

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