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BEST PROXIMITY RESULTS FOR KKM_0-MULTIMAPS

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Abstract

We establish the existence of theorems of best proximity pairs for KKM_0 multimap (resp. $\mathfrak{U}_c^{\mathfrak{K}}$-multimap) in the setting of Hausdorff locally convex topological vector space E, with a continuous seminorm p which generalizes the previous best proximity theorems of Al-Thagafi and Shahzad [2].

1. Introduction

The best approximation theorem due to Fan [7] states that if K is a nonempty compact convex subset of a locally convex Hausdorff topological vector space E with a continuous seminorm p and $f : K \to E$ is a single valued continuous function, then there exists an element $x \in K$ such that

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$$p(f(x)-x) = d_p(f(x), K) = \inf\{p(f(x)-y)/y \in K\}.$$

Since then, a number of generalizations of this theorem have been obtained in various directions by several authors (e.g., see [8, 16, 18, 19]). Indeed, Reich [15] has shown that even if K is a nonempty approximately p-compact convex subset of a locally convex Hausdorff topological vector space E with a relatively compact image $f(K)$, then the same conclusion holds. Also, there is no guarantee that such an approximate solution is optimal. For suitable subsets A and B of E and a multimap $T : A \to 2^B$ Sadiq Basha and Veeramani [16] provided sufficient conditions for the existence of an optimal solution $(a, T(a))$, (called a *best proximity pair*) such that

$$d_p(a, T(a)) = d_p(A, B) = \inf\{p(x-y) : x \in A,\ y \in B\}.$$

Srinivasan and Veeramani [18, 19] extended these results and obtained existence theorems of equilibrium pairs for constrained generalized games. Kim and Lee [8, 9] generalized the results [18, 19] and obtained existence theorems of equilibrium pairs for free n-person games. Al-Thagafi and Shahzad [1] generalized and extended the above results to Kakutani multimaps.

In this paper, we establish the existence theorems of best proximity pairs for KKM_0-multimaps (resp. $\mathfrak{U}_c^{\mathfrak{K}}$-multimaps) in Hausdorff locally convex topological vector space E with a continuous seminorm p. As applications, we obtain existence theorems of equilibrium pairs for free n-person games as well as free 1-person games. We consider A_i^0 as approximately p-compact and convex for each $i \in I_n$ and one of A_i^0's contained in some compact subset of $A \subset E$ instead of A_i^0 is compact for each $i \in I_n$.

Lemma 1.1. *If X is a nonempty compact and convex subset of a locally convex Hausdorff topological vector space E, then any generalized Kakutani factorizable multifunction $T : X \to 2^X$ has a fixed point.*

2. Preliminaries

Throughout E is a Hausdorff locally convex topological vector space with a continuous seminorm p, A and B are nonempty subsets of E, 2^A is the family of all subsets of A, C_0A is the convex shell of A in E, $\operatorname{int} A$ is the interior of A in E, $C(A, B)$ is the set of all continuous single-valued maps, $d_p(x, A) = \inf\{d_p(x, a) : a \in A\}$ and $d_p(A, B) = \inf\{p(a - b) : a \in A \text{ and } b \in B\}$. A map $T : A \to 2^B$ is called a *multimap* (*multifunction* or *correspondence*) if $T(x)$ is nonempty for each $x \in A$. A multimap $T : A \to 2^A$ is said to have a *fixed point* $a \in A$ if $a \in T(a)$; the set of all fixed points of T is denoted by $F(T)$. A multimap $T : A \to 2^B$ is said to be a

(a) *upper semicontinuous* if $T^{-1}(D) = \{x \in A : T(x) \cap D \neq \varnothing\}$ is closed in A whenever D is closed in B;

(b) *compact* if $\overline{T(A)}$ is compact in B;

(c) *closed* if its graph $\operatorname{Gr}(T) = \{(x, y) : x \in A \text{ and } y \in T(x)\}$ is closed in $A \times B$; and

(d) *compact valued* (resp. *convex*) if $T(x)$ is compact (resp. convex) in B for every $x \in A$.

A map $f : A \to B$ is proper if $f^{-1}(K)$ is compact in A whenever K is compact in B. A map $f : A \to E$ is quasi p-affine if the set $Q(x) = \{a \in A : p(f(a) - x) \leq r\}$ is convex for every $x \in E$ and $r \in [0, \infty)$.

Definition 2.1 ([4])**.** Given a convex subset C of a topological vector space E with a seminorm p. A single valued function $g : C \to E$ is said to be *almost p-affine* if

$$p((\lambda u + (1 - \lambda)v) - x) \leq p(u - x) + (1 - \lambda)\, p(v - x),$$

for all $u, v \in C$ and $x \in E$.

Clearly, any almost p-affine mapping is quasi p-affine but the converse is not true.

Example 2.2. Let $E = \mathbb{R}^2$ and the seminorm p on E be defined as $p(x, y) = \sqrt{x^2 + y^2}$. Let $g : E \to E$ be defined as follows:

$$g(x, y) = \begin{cases} (2, e^x), & \text{if } u \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is clear that g is quasi p-affine but not almost p-affine.

Definition 2.3. Let A be a nonempty subset of a topological vector space E with a continuous seminorm p. Then a single valued function $g : A \to E$ is said to be *p-continuous* if $p[g(x_a) - g(x)] \to 0$ for each x in A and every net $\{x_\alpha\}$ in A converging to x.

It is apparent that p-continuity is, in general, weaker than continuity.

Example 2.4. Let $E = \mathbb{R}^2$ with the seminorm $p : E \to [0, \infty)$ be defined as $p(x, y) = |x|,$ for all $(x, y) \in E.$ Let $g : E \to E$ be defined as

$$g(x, y) = \begin{cases} (0, 1), & \text{if } (x, y) \neq (0, 0), \\ (0, 0), & \text{otherwise.} \end{cases}$$

Then g is p-continuous but not continuous.

Definition 2.5 ([15])**.** Let A be a nonempty subset of a Hausdorff locally convex topological vector space E with a continuous seminorm p. Then the set K is said to be *approximately p-compact* if for each $y \in E$ and each net $\{x_\alpha\}$ in A satisfying the condition that $p(x_\alpha - y) \to d_p(y, A)$, there exists a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ converging to an element in A.

It is remarked that approximately compact subsets are closed.

Evidently, any compact subset of a Hausdorff locally convex topological vector space with a continuous seminorm p is approximately p-compact. However, approximately compact sets, which are not compact, are available in great profusion. Indeed, any nonempty closed and convex subset of a uniformly convex Banach space is approximately compact.

The set $P_A(x) = \{a \in A : p(a - x) = d_p(x, A)\}$ is called the *set of p-best approximation* in A to $x \in E$. Let A and B be nonempty subsets of E. Then a polytope P in A is any convex hull of a nonempty finite subset D of A. Whenever $\mathfrak{X}$ is a class of maps, denote the set of all finite compositions of maps in $\mathfrak{X}$ by $\mathfrak{X}_c$ and denote the set of all multimaps $T: A \to 2^B$ in $\mathfrak{X}$ by $\mathfrak{X}(A, B)$. Let $\mathfrak{U}$ be an abstract class of maps [14] satisfying the following properties:

1. $\mathfrak{U}$ contains the class $\mathbb{C}$ of continuous single valued maps.

2. Each $T \in \mathfrak{U}_c$ is upper semicontinuous with compact values.

3. For any polytope P, each $T \in \mathfrak{U}_c(P, P)$ has a fixed point.

Definition 2.6. Let $T: A \to 2^B$. Then we say that (a) T is an $\mathfrak{U}_c^{\mathfrak{K}}$*-multimap* [14] if for every compact set K in A, there exists an $\mathfrak{U}_c$-multimap $f: K \to 2^B$ such that $f(x) \subseteq T(x)$ for each $x \in K$, (b) T is a K-multimap (or Kakutani multimap) [10] if T is upper semicontinuous with compact and convex values (c) $S: A \to 2^B$ is a generalized KKM-multimap with respect to T [5] if $T(\mathrm{CoD}) \subseteq S(D)$ for each finite subset D of A (d) T has the KKM property [5] if whenever $S: A \to 2^B$ is a generalized KKM multimap w.r.t. T, the family $\{\overline{S(x)} : x \in A\}$ has the finite intersection property (e) T is a PK-multimap [13] if there exists a multimap $g: A \to 2^B$ satisfying $A = \bigcup\{\operatorname{int} g^{-1}(y) : y \in A\}$ and $C_0(g(x)) \subseteq T(x)$ for every $x \in A$. Note that each $\mathfrak{U}_c^{\mathfrak{K}}$-multimap has the KKM property and each K-multimap (resp. $\mathfrak{U}_c$-multimap has the KKM property and each K-multimap (resp. $\mathfrak{U}_c$-multimap, PK-multimap) is an $\mathfrak{U}_c^{\mathfrak{K}}$-multimap (see [11, 12, 14]).

Let A and B_i be nonempty subsets of a topological vector space with a seminorm p for each $i \in I_n = \{1, 2, 3, \ldots, n\}$. Define

$$d_p(A, B) = \inf\{p(a - b) : a \in A \text{ and } b \in B_i\},$$

$$Prox(A, B_i) = \{(a, b) \in A \times B_i : p(a - b) = d_p(A, B_i)\},$$

$$A_i^0 = \{a \in A : p(a - b) = d_p(A, B_i) \text{ for some } b \in B_i\},$$

$$B_i^0 = \{b \in B_i : p(a - b) = d_p(A, B_i) \text{ for some } a \in A\}.$$

$A^0 = \bigcap_{i \in I_n} A_i^0$. For $n = 1$, let $A_0 = A_1^0 = A^0$ and $B_0 = B_1^0$ (see [16]). Notice that for each $i \in I_n$, A_i^0 is nonempty if and only if B_i^0 is nonempty.

The following are essential in proving our results in this sequel.

Lemma 2.7. *Let A and for each $i \in I_n$, B_i be nonempty subset of E. Then the following statements hold for each $i \in I_n$:*

(a) *If A_i^0 (resp. A) and B_i are convex, then B_i is (resp. A_i^0 and B_i^0) convex.*

(b) *If A_i^0 (resp. A) and B_i are compact, then B_i^0 is (resp. A_i^0 and B_i^0) compact.*

(c) $P_A(B_i^0) = P_{A_i^0}(B_i^0) = A_i^0$.

(d) *If A_i^0 is nonempty, compact and convex and B_i^0 is convex, then $P_{A_i^0}/B_i^0$ is a K-multimap.*

Proof. (a) Let $b_1, b_2 \in B_i$. Then there exist $a_1, a_2 \in A_i$ such that $p(a_K - b_K) = d_p(A, B_i)$ for $K = 1, 2$. Let $\lambda \in (0, 1)$, $x = \lambda a_1 + (1 - \lambda) a_2$ and $y = \lambda b_1 + (1 - \lambda) b_2$. If A_i^0 and B_i are convex, then it follows that $x \in A_i^0$, $y \in B_i$ and

$$
\begin{aligned}
p(x - y) &= p((\lambda a_1 + (1 - \lambda) a_2) - (\lambda b_1 + (1 - \lambda) b_2)) \\
&= p(\lambda(a_1 - b_1) + (1 - \lambda)(a_2 - b_2)) \\
&\leq p(\lambda(a_1 - b_1)) + p((1 - \lambda)(a_2 - b_2)) \\
&= \lambda p(a_1 - b_1) + (1 - \lambda) p(a_2 - b_2) \\
\therefore\ p(x - y) &= \lambda d_p(A, B_i) + (1 - \lambda) d_p(A, B_i) \\
&= d_p(A, B_i).
\end{aligned}
$$

(b) Suppose A_i^0 and B_i are compact. Let $\{b_n\}$ be a sequence in B_i^0 such that $b_n \to b \in B_i$. Then there exists a sequence $\{a_n\}$ in A_i^0 such that $p(a_n - b_n) = d_p(A, B_i)$. Since A_i^0 is compact, we may assume that $a_n \to a \in A_i^0$. It follows from

$$
\begin{aligned}
d(A, B_i) &\leq p(a - b) \\
&= p(a - a_n + a_n - b_n + b_n - b)
\end{aligned}
$$

$$\leq p(a - a_n) + d_p(A, B_i) + p(b_n - b)$$

$$\to d_p(A, B_i).$$

Therefore, $d_p(A, B_i) = p(a - b) \Rightarrow b \in B_i^0$. Therefore, B_i^0 is closed and hence compact. The rest follows similarly.

(c) To show that $P_A(B_i^0) \subseteq A_i^0$, let $a \in P_A(B_i^0)$. Then there exists $b \in B_i^0$ and $y \in A$ such that $a \in P_A(b)$ and $p(y - b) = d_p(A, B_i)$. Since

$$d_p(A, B_i) \leq p(a - b) = d_p(b, A) \leq p(y - b) = d_p(A, B_i),$$

hence $d_p(A, B_i) = p(a - b)$ and $a \in A_i^0$. Therefore, $P_A(B_i^0) \subseteq A_i^0$.

To show that $A_i^0 \subseteq P_A(B_i^0)$, let $a' \in A_i^0$. Then there exists $b' \in B_i$ such that $p(a' - b') = d_p(A, B_i) \leq d_p(b', A)$. Therefore, $a' \in P_A(b') \leq P_A(B_i)$.

$\therefore \; A_i^0 \subseteq P_A(B_i^0)$.

$\therefore \; P_A(B_i^0) = A_i^0$. By a similar argument, we can show that $P_{A_i^0}(B_i^0) = A_i^0$.

(d) Since A_i^0 is nonempty, compact and convex, $P_{A_i^0} : E \to 2^{A_i^0}$ is a K-multimap. Since B_i^0 is convex and from part (c), B_i^0 is compact.

$P_{A_i^0 / B_i^0}$ is a K-multimap. □

Remark 2.8. We note from part (c) of the above theorem and the definitions of A^0, A_i^0 and B_i^0 that

(c1) A_i^0 is nonempty if and only if B_i^0 is nonempty.

(c2) $A^0 \neq \varnothing$ is equivalent to $\bigcap_{i=1}^n P_A(b_i) \neq \varnothing$ for some $(b_1, b_2, ..., b_n) \in \prod_{i=1}^n B_i^0$.

(c3) $P_A(B_i^0) = A_i^0$ if and only if $A_i^0 = A^0$; so by Kim and Lee [8, 9, Theorems 1, 2 and 4] are valid only whenever $A_i^0 = A^0$.

(c4) $\bigcap_{i=1}^n P_A(B_i^0) = \bigcap_{i=1}^n P_{A_i^0}(B_i^0) = \bigcap_{i=1}^n A_i^0 = A^0$. So $A^0 \neq \varnothing$ if and only if $\bigcap_{i=1}^n P_{A_i^0}(y_i) \neq \varnothing$ for some $(y_1, y_2, ..., y_n) \in \prod_{i=1}^n B_i^0$.

Lemma 2.9 ([15])**.** *Let A be a nonempty, approximately p-compact and convex subset of a Hausdorff locally convex topological vector space E with a continuous seminorm p.*

For every element $y \in E$, *let* $P_A(y) = \{x \in A : p(x - y) = d_p(y, A)\}$. *Then the following statements hold good*:

(a) *The set* $P_A(y)$ *is a nonempty, compact and convex subset of A.*

(*Every element of* $P_A(y)$ *is called a p-best approximation in A to* $y \in E$.)

(b) *The multifunctions* $P_A : E \to 2^A$ *is upper semicontinuous.*

(*The multifunction* P_A *is called a projection map.*)

Definition 2.10 ([2])**.** Let $T : A \to 2^B$ be a multimap. Then we say that *T* is a KKM_0*-multimap* if *T* and so $T : A \to 2^A$ are closed and have the *KKM* property for each *K*-multimap $S : B \to 2^A$.

Lemma 2.11 ([17])**.** *Let A and B be nonempty subsets of a normed space E. If* $A \to 2^B$ *is an upper semicontinuous multimap with compact values, then T is closed.*

Lemma 2.12 ([5, 12])**.** *Let A be nonempty convex subset of a normed space E. If* $T : A \to 2^A$ *is closed and compact multimap having the KKM property, then T has a fixed point.*

Lemma 2.13 ([6])**.** *For each* $i \in I_n$, *let* B_i *be a nonempty, compact and convex subset of a normed space E and* $P_i : \prod_{i=1}^n B_j \to 2^{B_i}$ *be a map such that*

(a) $x_i \notin C_0 P_i(x)$ *for each* $u = (x_1, x_2, ..., x_n) \in B = \prod_{j=1}^{n} B_j$.

(b) $P_i^{-1}(y)$ *is open in B for each* $y \in B_i$.

Then there exists $b \in B$ *such that* $P_i(b) = \varnothing$ *for each* $i \in I_n$.

Lemma 2.14 ([3, 6, 8, 9])**.** *Let B be a nonempty, compact and convex subset of a normed space E and* $P : B \to 2^B$ *be a map such that*

(a) $x \notin C_0 P(x)$ *for each* $x \in B$.

(b1) *If* $z \in P^{-1}(y)$, *then there exists some* $y' \in B$ *such that* $z \in \operatorname{int} P^{-1}(y')$.

(b2) $P^{-1}(y')$ *is open in B for each* $y \in B$.

Then there exists $b \in B$ *such that* $P(b) = \varnothing$.

3. Best Proximity Results for KKM_0-multimaps (resp. $\mathfrak{U}_c^{\mathfrak{K}}$-multimaps)

Lemma 3.1. *Let A and* B_i *be subsets of a Hausdorff locally convex topological vector space E with a continuous seminorm p such that* A_i^0 $($*resp.* $B_i^0)$ *are nonempty approximately p-compact* (*resp. closed*) *and convex for each* $i \in I_n$. *Assume that one of* A_i^0*'s is contained in some compact subset of A. Let* $f : A^0 \to A^0$ *be a p-continuous, proper, quasi p-affine and surjective self-map, and* $P : Y \to 2^{A^0}$ *be a multimap defined by* $P(y_1, y_2, ..., y_n) = \bigcap_{i=1}^{n} P_{A_i^0}(y_i)$ *for each* $(y_1, y_2, ..., y_n) \in Y$ $= \prod_{i=1}^{n} B_i^0$. *Then* $P : Y \to 2^{A^0}$ *is a K-multimap.*

Proof. Fix $i \in I_n$. As A_i^0 is an approximately p-compact and convex subset of E, by Lemma 2.9, it is known that the projection map $P_{A_i^0} : E \to 2^{A_i^0}$ is upper semicontinuous with compact and convex values. Therefore, $P_{A_i^0} : E \to 2^{A_i^0}$ is a K-multimap. In order to show B_i^0 is convex, let $x_1, x_2 \in B_i^0$ be arbitrary. Then there exist $y_1, y_2 \in A_i^0$ such that

$$d_p(x_1, y_1) = d_p(A, B_i), \quad d_p(x_2, y_2) = d_p(A, B_i).$$

Let $\lambda \in (0, 1)$, $y = \lambda y_1 + (1-\lambda) y_2$, $x = \lambda x_1 + (1-\lambda) x_2$. Since A_i^0 is convex, $y \in A_i^0$, we have

$$\begin{aligned} p(x-y) &= p(\lambda x_1 + (1-\lambda) x_2) - (\lambda y_1 + (1-\lambda) y_2) \\ &\le \lambda p(x_1 - y_1) + (1-\lambda) p(x_2 - y_2) \\ &= d_p(A, B_i). \end{aligned}$$

Therefore, $x \in B_i^0$ and hence B_i^0 is convex. Since A_i^0 is approximately p-compact, A_i^0 is closed for each $i \in I_n$ and hence A^0 is closed. Since one A_i^0's is contained in some compact subset of A, A_i^0 is compact and convex. Therefore, A_i^0 is nonempty, compact and convex, and B_i^0 is convex. By Lemma 2.7(d), $P_{A_i^0/B_i^0}$ is a K-multimap and hence $P : Y \to 2^{A^0}$ is a K-multimap. Let $S = f^{-1}P$. Since f is surjective and $S(Y) = f^{-1}P(Y) \subseteq f^{-1}(A^0) = A^0$, $S : Y \to 2^{A^0}$ is a multimap. To show that S is a convex valued function, it is enough to prove that $S(y)$ is convex for $y \in Y$. For $a_1, a_2 \in S(y)$, we have $f(a_1), f(a_2) \in P_{A_i^0}(y_i)$. This implies that

$$p(f(a_1) - y_i) = d_p(y_i, A_i^0) = p(f(a_2) - y_i).$$

Since f is quasi p-affine, the set

$$Q(y_i) = \{a \in A^0 / p(f(a) - y_i) \le d_p(y_i, A_i^0)\}$$

is convex.

Therefore, $a_1, a_2 \in Q(y_i)$ which is convex. $\Rightarrow \lambda a_1 + (1-\lambda) a_2 \in Q(y_i)$, for all $\lambda \in [0, 1]$.

Let $y_\lambda = \lambda a_1 + (1-\lambda) a_2$. Then

$y_\lambda \in Q(y_i)$.

$\Rightarrow p(f(y_\lambda) - y_i) = d_p(y_i, A_i^0)$

$\Rightarrow f(y_\lambda) \in P_{A_i^0}(y_i)$

$\Rightarrow f(y_\lambda) \in P(y)$

$\Rightarrow f(y_\lambda) \in f^{-1}P(y) = S(y)$ hence $S(y)$ is convex.

To show that S is upper semicontinuous, let D be a closed subset of A^0 and $\{y_\alpha\}$ be any net in $S^{-1}(D)$ such that $y_\alpha \to y = (y_1, \ldots, y_n)$. Then by definition of $S^{-1}(D)$ for each α, $S(y_\alpha) \bigcap D \neq \varnothing$. Choose a net $\{x_\alpha\}$ in (D) such that $x_\alpha \in S(_\alpha) \bigcap D$ so that

$$d_p(f(x_\alpha), y_\alpha) = d_p(y_\alpha, A_i^0)$$

$\Rightarrow x_\alpha \in f^{-1}P(y_\alpha)$ which is compact

$\Rightarrow \{x_\alpha\}$ has a convergent subnet.

Since D is closed subset of A^0, without loss of generality, we may assume that $\{x_\alpha\} \to x \in D$. Since f is p-continuous, $p(f(x_\alpha) - f(x)) \to 0$.

Now

$$p(f(x_\alpha) - f(x)) \leq p(f(x) - f(x_\alpha)) + p(f(x_\alpha) - y_\alpha) + p(y_\alpha - y)$$

$$\Rightarrow p(f(x) - y) \leq p(f(x_\alpha) - y_\alpha) + p(y_\alpha - y)$$

$$= d_p(y_\alpha, A_i^0) + p(y_\alpha - y) \quad (\because p(f(x_\alpha) - y_\alpha) = d_p(y_\alpha, A_i^0)).$$

Also, $d_p(y_\alpha, A_i^0) \to d_p(y, A_i^0)$. So

$$p(f(x) - y) = d_p(y, A_i^0)$$

$\Rightarrow f(x) \in P_{A_i^0}(y_i)$

and hence $f(x) \in P(y)$. Therefore, $x \in f^{-1}P(y) = S(y)$. So

$$x \in S(y) \bigcap D$$

$\Rightarrow y \in S^{-1}(D)$.

Therefore,

$S^{-1}(D)$ is closed

$\Rightarrow S$ is upper semicontinuous.

Notice, as $P : Y \to 2^{A^0}$ is K-multimap, P is compact valued map and $P(y)$ is compact in A^0 for each $y \in Y$. Since $f : A^0 \to A^0$ is proper, $f^{-1}(P(y))$ is compact in A^0, that is, $S(y)$ is compact in A^0. Therefore, $S = f^{-1}P : Y \to 2^{A_i^0}$ is upper semicontinuous with compact and convex valued map.

$$\Rightarrow f^{-1}P : Y \to 2^{A_i^0} \text{ is a } K\text{-multimap.} \qquad \square$$

Definition 3.2. Let A and B_i be nonempty subsets of a topological space E with a continuous seminorm p. Let $T_i : A \to 2^{B_i}$ be a multimap for each $i \in I_n$. Then $f : A' \to A'$ is a self-map of a nonempty subset A' of A and $a \in A$.

If $d_p(f(a), T_i(a)) = d_p(A, B_i)$, then we can say that $(f(a), T_i(a))$ is a *best proximity pair*. The best proximity set for the pair $(f(a), T_i(a))$ is given by

$$\mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f) = \{b \in T_i(a) : d_p(f(a), T_i(a)) = p(f(a) - b) = d_p(A, B_i)\}.$$

For $n = 1$, let $\mathfrak{T}_{\mathfrak{a}}(f) = T_a'(f)$. Whenever f is the identity map, we write T_a' instead of $\mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$.

Theorem 3.3. *Let A and B_i be subsets of a Hausdorff locally convex topological vector space E with a continuous seminorm p such that A_i^0 (resp. closed) and nonempty approximately p-compact (resp. closed) and convex for each $i \in I_n$. Assume that one of A_i^0's is contained in some compact subset of A. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, y_2, ..., y_n) \in Y = \prod_{i=1}^n B_i^0$ and $T : A^0 \to 2^Y$ is a KK_0-multimap (resp. $\mathfrak{U}_{\mathfrak{c}}^{\mathfrak{K}}$-multimap) where $T(x) = \prod_{i=1}^n T_i(x)$ for each $x \in A^0$. Then for each p-continuous, proper, quasi p-affine and surjective self-map $f : A^0 \to A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$ is nonempty and closed.*

Proof. Fix $i \in I_n$. Define $P : Y \to 2^{A^0}$ by $P(y_1, y_2, ..., y_n) = \bigcap_{i=1}^n P_{A_i^0}(y_i)$ for each $(y_1, y_2, ..., y_n) \in Y = \prod_{i=1}^n B_i^0$. Let $f : A^0 \to A^0$ be a self-map. As $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, y_2, ..., y_n) \in Y$, it follows from Lemma 3.1 that $f^{-1}P : Y \to 2^{A_i^0}$ is a *K*-multimap.

Now assume that $T : A^0 \to 2^Y$ is a KKM_0-multimap. It follows from the definition of KKM_0-multimap that $f^{-1}P \circ T : A^0 \to 2^{A^0}$ and *T* are closed multimap and have the *KKM*-property. Since A_i^0 is approximately *p*-compact for each $i \in I_n$, A_i^0 is closed for each $i \in I_n$. But $A^0 = \bigcap_{i=1}^n A_i^0$ and one of A_i^0's is contained in some compact subset of *A*. Therefore, A_i^0 is compact and hence A^0 is compact. Therefore, $f^{-1}P \circ T$ is a compact multimap. By Lemma 2.12, there exists $a \in A^0$ such that $a \in (f^{-1}P \circ T)(a)$ and hence $f(a) \in P(T(a))$. Thus there exists $(b_1, b_2, ..., b_n) \in T(a) = \prod_{i=1}^n T_i(a)$ such that $f(a) \in P(b_1, b_2, ..., b_n) = \bigcap_{i=1}^n P_{A_i^0}(b_i)$ $\subseteq A^0$. Hence $f(a) \in P_{A_i^0}(b_i) \subseteq A_i^0$ and $b_i \in T_i(a) \subseteq B_i^0$. This implies that there exists $a_i' \in A_i^0$ such that $p(a' - b_i) = d_p(A, B_i)$ and hence

$$d_p(A, B_i) \leq d_p(f(a), T_i(a)) \leq p(f(a) - b_i) = d_p(b_i, A_i^0) \leq p(a_i' - b_i) = d(A, B_i).$$

Thus $d_p(f(a), T_i(a)) = p(f(a) - b_i) = d_p(A, B_i)$. Therefore, $b_i \in \mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$.

Suppose $T : A^0 \to 2^Y$ is an $\mathfrak{U}_{\mathfrak{c}}^{\mathfrak{K}}$-multimap. Then by definition, there exists $\mathfrak{U}_{\mathfrak{c}}^{\mathfrak{K}}$-multimap $T' : A^0 \to 2^Y$ such that T' is upper semicontinuous with compact values and $T'(x) = \prod_{i=1}^n T_i'(x) \subseteq T(x)$ for each $x \in A^0$. Now $f^{-1}P$ is a *K*-multimap implies $f^{-1}P$ is a $\mathfrak{U}_{\mathfrak{c}}^{\mathfrak{K}}$-multimap. Also, T' is a $\mathfrak{U}_{\mathfrak{c}}^{\mathfrak{K}}$-multimap and hence T' is a multimap having the *KKM* property.

Therefore, $f^{-1}P \circ T'$ and T' are closed and having the *KKM* property. This implies that $f' : A^0 \to 2^Y$ is a KKM_0-multimap. It follows from the previous paragraph that there exists $(a, b) \in A^0 \times Y$ such that $b = (b_1, b_2, ..., b_n)$, $b_i \in T_i'(a)$ and $d_p(f(a), T_i'(a)) = p(f(a) - b_i) = d_p(A, B_i)$. As $d_p(A, B_i) \leq d_p(f(a), T_i(a)) \leq d_p(f(a), T_i'(a))$, we conclude that

$$d_p(f(a), T_i(a)) = p(f(a) - b_i) = d_p(A, B_i).$$

This means that $b_i \in \mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$. Therefore, in both cases, the best proximity set $\mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$ is nonempty and its closedness follows from the continuity of the seminorm. □

Corollary 3.4. *Let A and B_i be subsets of a Hausdorff locally convex topological vector space E with a continuous seminorm p such that A_i^0* (resp. B_i^0) *are nonempty approximately p-compact* (*resp. closed*) *and convex for each $i \in I_n$. Assume that one of A_i^0's is contained in some compact subset of A. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, y_2, ..., y_n) \in Y = \prod_{i=1}^n B_i^0$ and $T_i : A^0 \to 2^{B_i^0}$ is an $\mathfrak{U}_{\mathfrak{c}}^{\mathfrak{K}}$-multimap for each $i \in I_n$. Then for each continuous*, *proper*, *quasi p-affine and surjective self-map $f : A^0 \to A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$ is nonempty and closed.*

Proof. Define $T : A^0 \to 2^Y$ by $T(x) = \prod_{i=1}^n T_i(x)$ for each $x \in A^0$. Since $T_i : A^0 \to 2^{B_i^0}$ is an $\mathfrak{U}_{\mathfrak{c}}^{\mathfrak{K}}$-multimap for each $i \in I_n$ for every compact set K in A^0, (that is, K may be treated as A^0) there exists an U_c-multimap $f : K(= A^0) \to 2^{B_i^0}$ such that $f(x) \subseteq T_i(x)$ for each $x \in A^0$ and for each $i \in I_n$. This implies that $f(x) \subset \prod_{i=1}^n T_i(x) = T(x)$ for each $x \in A^0$. Therefore, $T : A^0 \to 2^Y$ is an $\mathfrak{U}_{\mathfrak{c}}^{\mathfrak{K}}$-multimap. The result follows from Theorem 3.3. □

Theorem 3.5. *Let A and B_i be subsets of a Hausdorff locally convex topological vector space E with a continuous seminorm p, A_i^0* (*resp. B_i^0*) *be nonempty*, *approximately p-compact* (*resp. closed*) *and convex*, *$T_i : A^0 \to 2^{B_i}$ be an* *upper semicontinuous multimap with compact values and* $T_i(x) \cap B_i^0$ *be nonempty for each* $x \in A^0$, *for each* $i \in I_n$. *Suppose that* $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ *is nonempty for each* $(y_1, y_2, \ldots, y_n) \in Y = \prod_{i=1}^n B_i^0$. *Assume that one of* A_i^0*'s is contained in some compact subset of A. Then for each continuous, proper, quasi p-affine and surjective self-map* $f : A^0 \to A^0$, *there exists* $a \in A^0$ *such that the best proximity set* $\mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$ *is nonempty and closed.*

Proof. Fix $i \in I_n$. Define $T_i' : A^0 \to 2^{B_i^0}$ by $T_i'(x) = T_i(x) \cap B_i^0$ for each $x \in A^0$. Thus $T_i' : A^0 \to 2^{B_i^0}$ is an upper semicontinuous multimap with compact values. Define $T : A^0 \to 2^Y$ by $T(x) = \prod_{i=1}^n T_i'(x)$ for each $x \in A^0$. As in Theorem 3.3, A^0 is compact. Therefore, $T : A^0 \to 2^Y$ is an upper semicontinuous multimap with compact values. This implies that T is K-multimap and hence $\mathfrak{U}_{\mathfrak{c}}^{\mathfrak{K}}$-multimap. Then the result follows from the second half of Theorem 3.3. □

Remark 3.6. Lemma 3.1, Theorem 3.3, Corollary 3.4 and Theorem 3.5 generalize Lemma 3.1, Theorem 3.4, Corollary 3.5 and Theorem 3.8 of Al-Thagafi and Shahzad [2] by relaxing the condition A_i^0 is compact for each $i \in I_n$ into the weaker condition A_i^0 is approximately p-compact for each $i \in I_n$ in the setting of Hausdorff locally convex topological vector space with a continuous seminorm p. Also, we relaxed the condition "A_i^0 is compact for each $i \in I_n$ by the weaker condition" one of A_i^0's is contained in some compact subset of $A \subset E$ (Hausdorff locally convex topological vector space with a continuous seminorm p).

4. Equilibrium Pair Results for Free *n*-person Games

A free n-person game is a family of ordered quadruples $(A, B_i, T_i, P_i)_{i \in I_n}$ such that A and B_i are nonempty subsets of a normed space E, $T_i : A \to 2^{B_i}$ is a constraint multimap, and $P_i : B \to 2^{B_i}$ is a preference map where $B := \prod_{j=1}^n B_j$

(see [9]). An equilibrium pair for $(A, B_i, T_i, P_i)_{i \in I_n}$ is a point $(a, b) \in A \times B$ such that $T_i(a) \cap P_i(b) = \emptyset$. For details on economic terminology, see [8, 9].

Theorem 4.1. *Let $(A, B_i, T_i, P_i)_{i \in I_n}$ be a free n-person game such that A and B_i are nonempty subsets of a Hausdorff locally convex topological vector space E with a continuous seminorm p. Then $T_i : A \to 2^{B_i}$ is a constraint multimap, and $P_i : B \to 2^{B_i}$ is a preference map where $B := \prod_{j=1}^{n} B_j$. Assume that A^0 is nonempty, $T(x) := \prod_{i=1}^{n} T_i(x)$ for each $x \in A^0$, $Y := \prod_{i=1}^{n} B_i^0$, and for each $i \in I_n$,*

(a) *A_i^0 is approximately p-compact and convex for each $i \in I_n$, B_i is compact and convex for each $i \in I_n$. Also, one of the A_i's is contained in some compact subset of A;*

(b) *$\bigcap_{i=1}^{n} P_{A_i^0}(y_i)$ is nonempty for each $(y_1, y_2, \ldots, y_n) \in \prod_{i=1}^{n} B_i^0$;*

(c) *$T : A^0 \to 2^Y$ is a KKM_0-multimap (resp. $\mathfrak{U}_{\mathfrak{c}}^{\mathfrak{K}}$-multimap);*

(d) *$x_i \notin \mathrm{co}P_i(x)$ for each $x = (x_1, x_2, \ldots, x_n) \in B$;*

(e) *$P^{-1}(y)$ is open for each $y \in B_i$.*

Then there exists $b \in B$ such that $P_i(b) = \emptyset$ and, for each continuous, proper, quasiaffine, and surjective self-map $f : A^0 \to A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$ is nonempty and compact. If, in addition, $P_i(z)$ is nonempty for each $z \notin \prod_{i=1}^{n} \mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$, then (a, b) is an equilibrium pair in $A^0 \times \prod_{i=1}^{n} \mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$.

Proof. Fix $i \in I_n$. Since A_i^0 is approximately p-compact, A_i^0 is closed for each $i \in I_n$ and hence A^0 is closed. Since one of A_i^0's is contained in some compact subset of A, A_i^0 is compact. As A_i^0 and B_i are compact and convex, it follows from Lemma 2.7(b) that B_i^0 is compact and convex. By Theorem 3.3, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$ is nonempty and closed. By Lemma 2.14, there exists $b = (b_1, b_2, ..., b_n) \in Y$ such that $P_i(b) = \varnothing$. As $P_i(z)$ is nonempty for each $z \in \prod_{i=1}^n \mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$, we conclude that $b = (b_1, b_2, ..., b_n) \in \prod_{i=1}^n \mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$. Thus $(a, b) \in A^0 \times Y$, $b = (b_1, b_2, ..., b_n) \in \prod_{i=1}^n T_i(a)$, $T_i(a) \cap P_i(b) = \varnothing$ and $d(f(a), T_i(a))$ $= p(f(a) - b_i) = d(A, B_i)$. Thus (a, b) is an equilibrium pair in $A^0 \times \prod_{i=1}^n \mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$.

□

Theorem 4.2. *Let $(A, B_i, T_i, P_i)_{i \in I_n}$ be a free n-person game such that A and B_i are subsets of a Hausdorff locally convex topological vector space E with continuous seminorm p. Then $T_i : A \to 2^{B_i}$ is a constraint multimap, and $P_i : B \to 2^{B_i}$ is a preference map where $B := \prod_{j=1}^n B_j$. Assume that A^0 is nonempty, $Y := \prod_{i=1}^n B_i^0$, and for each $i \in I_n$,*

(a) *A_i^0 is approximately p-compact and convex for each $i \in I_n$, B_i is compact and convex for each $i \in I_n$. Also, one of the A_i's is contained in some compact subset of A*;

(b) *$\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, y_2, ..., y_n) \in \prod_{i=1}^n B_i^0$*;

(c) *$T_i \mid A^0$ is an upper semicontinuous multimap with compact values and $T_i(x) \cap B_i^0$ is nonempty for each $x \in A^0$*;

(d) *$x_i \notin \mathrm{co}P_i(x)$ for each $x = (x_1, x_2, ..., x_n) \in B$*;

(e) *$P_i^{-1}(y)$ is open for each $y \in B_i$.*

Then, there exists $b \in B$ such that $P_i(b) = \varnothing$ and, for each continuous, proper, quasiaffine and surjective self-map $f : A^0 \to A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$ is nonempty and compact. If, in addition, $P_i(z)$ is nonempty for each $z \notin \prod_{i=1}^n \mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$, then (a, b) is an equilibrium pair in $A^0 \times \prod_{i=1}^n \mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$.

Proof. Use Theorem 3.5 instead of Theorem 3.3 in the proof of Theorem 4.1. □

Theorem 4.3. *Let (A, B, T, P) be a free 1-person game such that A and B are subsets of a Hausdorff locally convex topological vector space E. Then $T : A \to 2^B$ is a constraint multimap, and $P : B \to 2^B$ is a preference map. Assume that*

(a) *A_i^0 is approximately p-compact and convex for each $i \in I_n$, B is compact and convex and one of A_i^0's is contained in some compact subset of A*;

(b) *$T : A^0 \to 2^{B_0}$ is a KKM_0-multimap (resp. $\mathfrak{U}_{\mathfrak{c}}^{\mathfrak{K}}$-multimap)*;

(c) *$x_i \notin \mathrm{co}P(x)$ for each $x \in B$*;

(d) *one of the following conditions is satisfied*;

(d1) *if $z \in P^{-1}$ for some $y \in B$, then there exists some $y' \in B$ such that $z \in \mathrm{int}\, P^{-1}(y')$*;

(d2) *for each $y \in B$, $P^{-1}(y)$ is open in B.*

Then, there exists $b \in B$ such that $P(b) = \varnothing$ and, for each continuous, proper, quasiaffine and surjective self-map $f : A_0 \to A_0$, there exists $a \in A_0$ such that the best proximity set $\mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$ is nonempty and compact. If, in addition, $P(z)$ is nonempty for each $z \notin \mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$, then (a, b) is an equilibrium pair in $A_0 \times \mathfrak{T}_{\mathfrak{a}}^{\mathfrak{i}}(f)$.

Proof. Since A_i^0 is approximately p-compact for each $i \in I_n$, A_i^0 is closed for each $i \in I_n$. Hence $A^0 = \bigcap_{i=1}^n A_i^0$ is closed. But $A^0 \subset A_i^0$. Therefore, A^0 is compact, that is, A_0 is compact.

Since A_0 and B_0 are nonempty, compact and convex, it follows from Theorem 3.3 that there exists $(a, c) \in A_0 \times B_0$ such that $c \in T(a)$ and $d(f(a), T(a)) = p(f(a) - c) = d(A, B)$ and so $\mathfrak{T}_{\mathfrak{a}}(f)$ is nonempty. By Theorem 3.5, there exists $b \in B_0$ such that $P(b) = \varnothing$. As $P(z)$ is nonempty whenever $z \in B \backslash \mathfrak{T}_{\mathfrak{a}}(f)$, we conclude that $b \in \mathfrak{T}_{\mathfrak{a}}(f)$. So $(a, b) \in A_0 \times b_0$, $b \in T(a)$ and $d(f(a), T(a)) = p(f(a) - b) = d(A, B)$. Thus (a, b) is an equilibrium pair in $A_0 \times \mathfrak{T}_{\mathfrak{a}}(f)$. □

Theorem 4.4. *Let* (A, B, T, P) *be a free* 1*-person game such that A and B are subsets of a Hausdorff locally convex topological vector space E. Then* $T : A \to 2^B$ *is a constraint multimap, and* $P : B \to 2^B$ *is a preference map. Assume that*

(a) A_i^0 *is approximately p-compact and convex for each* $i \in I_n$, *B is compact and convex and one of* A_i^0*'s is contained in some compact subset of A*;

(b) $T | A_0$ *is an upper semicontinuous multimap with compact values and* $T(x) \cap B_0$ *is nonempty for each* $x \in A_0$;

(c) $x_i \notin \mathrm{co}P(x)$ *for each* $x \in B$;

(d) *one of the following conditions is satisfied*;

(d1) *if* $z \in P^{-1}$ *for some* $y \in B$, *then there exists some* $y' \in B$ *such that* $z \in \mathrm{int}\, P^{-1}(y')$;

(d2) *for each* $y \in B$, $P^{-1}(y)$ *is open in B*.

Then, there exists $b \in B$ *such that* $P(b) = \varnothing$ *and, for each continuous, proper, quasiaffine and surjective self-map* $f : A_0 \to A_0$, *there exists* $a \in A_0$ *such that the best proximity set* $\mathfrak{T}_{\mathfrak{a}}(f)$ *is nonempty and compact. If, in addition,* $P(z)$ *is nonempty for each* $z \notin \mathfrak{T}_{\mathfrak{a}}(f)$, *then* (a, b) *is an equilibrium pair in* $A_0 \times \mathfrak{T}_{\mathfrak{a}}(f)$.

Proof. Use Theorem 3.5 instead of Theorem 3.3 in the proof of Theorem 4.2. □

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