Far East Journal of Mathematical Sciences (FJMS)
Volume 45, Number 1, 2010, Pages 41-47
Published Online: November 15, 2010
This paper is available online at http://pphmj.com/journals/fjms.htm

ON THE ORLICZ-PETTIS THEOREM FOR INTEGRALS

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Abstract

In this note, we propose a more efficient approach to the definition of the integral of Banach-valued functions that enables us to give an improved version of the Orlicz-Pettis theorem for integrals.

1. Introduction

The Orlicz-Pettis theorem asserts that weakly subseries convergence implies norm subseries convergence in Banach spaces. In [4], the author gave a continuous version of this theorem by replacing weakly subseries convergence by Pettis integrability for measurable functions defined on closed bounded intervals and norm series convergence by the Henstock-Kurzweil integrable functions. In this note, we propose an approach that will extend the notion of integral to functions defined on arbitrary measurable space. Such an approach will enable us to show that in fact Pettis integrability condition for measurable functions can be replaced by a "weaker" integrability condition in the continuous version of the Orlicz-Pettis theorem.

Unless stated otherwise, the notation and conventions used in the present note

2010 Mathematics Subject Classification: Primary 46G10, 28B05; Secondary 28C20, 46G12.

Keywords and phrases: weak integrability, Dunford integral, Pettis integral, Orlicz-Pettis theorem.

Received June 9, 2010

are standard. Throughout the paper, Σ will denote a σ-algebra of subsets of a set Ω. A measure on Σ is a countably additive function $\mu : \Sigma \to [0, \infty)$.

Recall that a function $f : [a, b] \to X$ is called *scalarly measurable* if $x^* f$ is measurable for each $x^* \in X^*$. f is called *scalarly absolutely integrable* if the function $\omega \mapsto |x^* f(\omega)|$ is integrable, i.e., $x^* f \in L^1(\mu)$. A scalarly absolutely integrable function is said to be *Pettis integrable* if there exists an element $\int_{[a,b]}^{P} fd\mu \in X$ such that for all $x^* \in X^*$,

$$x^*\left(\int_{[a,b]}^{P} fd\mu\right) = \int_{[a,b]} x^* fd\mu.$$

For details on Pettis integral, we refer the reader to [2].

Let $(I)_{i=1}^n$ be a finite partition of the interval $[a, b]$, $c_i \in I_i$ for each i, and let δ be a positive function on $[a, b]$. Then the collection of pairs $(I_i, c_i)_{i=1}^n$ is said to be *gauged* by δ if for each i, $c_i \in I_i \subset (c_i - \delta(c_i), c_i + \delta(c_i))$. A function $f : [a, b] \to X$ is said to be *Henstock-Kurzweil integrable* on $[a, b]$ if there exists an element $\int_{[a,b]}^{HK} fd\mu \in X$ with the property: for $\varepsilon > 0$, there is a function $\delta : [a, b] \to [0, \infty)$ such that the inequality

$$\left\| \sum_{i=1}^{n} f(c_i) | I_i | - \int_{[a,\, b]}^{HK} fd\mu \right\| < \varepsilon$$

holds for every partition $(I)_{i=1}^n$ gauged by δ. The book of Bartle [1] is a good source on scalar-valued Henstock-Kurzweil integrals. For vector valued integrals, the reader is referred to [2] and [7].

The main result in [4] can be stated as follows:

Theorem 1. *If* $[a, b] \to X$ *is measurable and Pettis integrable*, *then* f *is Henstock-Kurzweil integrable*.

We extend such a result to X-valued functions defined on an arbitrary measurable space (Ω, Σ), and for which the Pettis integrability condition is replaced with weaker integrability condition.

2. Strong and Weak Integrability

We fix a measure space (Ω, Σ, μ) and X be a Banach space. By a partition of Ω, we always mean a countable family $P := \{A_n : n \in \mathbb{N}\}$ consisting of disjoint measurable subsets of Ω such that $\Omega = \bigcup_{n\in\mathbb{N}} A_n$. For each partition $P = \{A_n : n \in \mathbb{N}\}$, we arbitrarily choose a sequence $c := (c_1, c_2, \ldots) \in A_1 \times A_2 \times \cdots$. The pair (P, c) is then called a *tagged partition* of Ω. We denote by $\prod$ the set of all the tagged partitions of Ω. Note that $\prod$ is naturally directed by $(P_1, c_1) \succ (P_2, c_2)$ if P_1 is a refinement of P_2, that is, if $P_1 \supset P_2$.

Definition 1. We say that a function $f : \Omega \to X$ is μ-*summable over a measurable set E* if there exists $(P_0, c_0) \in \Pi$ such that for every $((A_n), (c_n)) \succ (P_0, c_0)$ in $\prod$, the series $\sum_{n\in\mathbb{N}} \mu(A_n \cap E) f(c_n)$ converges.

We say that f is μ-*summable* if f is μ-summable over all measurable subsets of Ω.

We notice that in the above definition, we do not require measurability for the function f.

Given a function $f : \Omega \to X$ and a tagged partition $(P, c) = ((A_n), (c_n))$, we define the *Riemann sum* of f to be

$$R_E(f, P, c) = \sum_{n\in\mathbb{N}} \mu(A_n \cap E) f(c_n).$$

The collection $\{R_E(f, P, c) : (P, c) \in \Pi\}$ becomes a net with the partial ordering $\succ$ on $\prod$.

We define (strong) integrability as follows:

Definition 2. We say that a μ-summable function $f : \Omega \to X$ is μ-(*strong*) *integrable over a measurable set E* if the net $\{R_E(f, P, c) : (P, c) \in \Pi\}$ is Cauchy, that is, if for any given $\varepsilon > 0$, there exists (P_E, c_E) in Π such that

$$\| R_E(f, P, c) - R_E(f, Q, d) \|_X < \varepsilon$$

whenever $(P, c), (Q, d) \succ (P_E, c_E)$ in Π. The element of X defined by

$$\lim_{\Pi} R_E(f, P, c) =: \int_E f d\mu$$

is then called the (*strong*) *integral* of the function f over E with respect to the measure μ.

We say that $f : \Omega \to X$ is μ-(*strong*) *integrable* if f is μ-(strong) integrable over any measurable set $E \in \Sigma$.

We denote by $I_E(\mu, X)$, the set of all μ-integrable functions over the measurable set E. If $X = \mathbb{R}$ is the scalar field, then we write $I_E(\mu, \mathbb{R}) =: I_E(\mu)$. The usual properties of integrals such as the stability under sums, scalar multiples are quickly seen to apply.

Our definition of the integral is large enough to contain most of the classical notions of the integral. Clearly, every Bochner μ-integrable functions are μ-integrable; that is, $L^1_E(\mu, X) \subset I_E(\mu, X)$. When $\Omega = [a, b]$, the definition of the integral given here agrees with the definition of the usual Henstock-Kurtzweil integral of Banach valued functions defined on intervals ([4]).

We now define weak integrability.

Definition 3. A function $f : \Omega \to X$ is said to be *weakly integrable* or *weakly-*$I_E(\mu)$ (resp. *weakly absolutely integrable* or *weakly-*$L^1_E(\mu)$) if for every $x^* \in X^*$, $x^* f \in I_E(\mu)$ (resp. $x^* f \in L^1_E(\mu)$).

Definition 4. We say that a function $f : \Omega \to X$ is *Dunford-*$I_E(\mu)$ (resp. *Dunford-*$L^1_E(\mu)$) if for each x^*, $x^* f \in I_E(\mu)$ (resp. $L^1_E(\mu)$) and if there exists $\int_E^D f \in X^{**}$ such that for each all $x^* \in X^*$,

$$\left\langle x^*, \int_E^D f \right\rangle = \int_E x^* f.$$

If in addition $\int_E^D f \in X$, then f is called *Pettis-* $I_E(\mu)$ (resp. *Pettis-* $L_E^1(\mu)$), and we write

$$\int_E^D f =: \int_E^P f.$$

It is clear from the definitions that a Dunford- $I_E(\mu)$ is weakly- $I_E(\mu)$. Our next result shows that the converse of such a statement also holds. The following results can be deduced from [5] and [6] when Ω is a closed bounded interval. However, for the sake of completeness, we provide an easy and direct proof for the more general case of the set Ω.

Theorem 2. *A function* $f : \Omega \to X$ *is Dunford-* $I_E(\mu)$ (*resp. Dunford-* $L_E^1(\mu)$) *if and only if it is weakly-* $I_E(\mu)$ (*resp. weakly-* $L_E^1(\mu)$).

Proof. We only need to prove the sufficiency. Suppose $f : \Omega \to X$ is weakly- $I_E(\mu)$ (resp. weakly- $L_E^1(\mu)$), where E is a measurable set. For each $x^* \in X^*$, let μ_{x^*f} be defined by $\mu_{x^*f}(A) := \int_A x^* f$ for $A \in \Sigma$. Then, μ_{x^*f} belongs to $M(\Sigma)$, the Banach space of scalar measures with the semivariation norm. Let $T : X^* \to M(\Sigma)$ be defined by $Tx^* = \mu_{x^*f}$. Then the adjoint T^* of T maps $M(\Sigma)^*$ into X^{**}. The indicator function $\mathbf{1}_E$ can be considered as an element of $M(\Sigma)^*$ as follows: $\langle \lambda, \mathbf{1}_E \rangle = \int_E d\lambda$ for $\lambda \in M(\Sigma)$. Thus $T^*(\mathbf{1}_E) \in X^{**}$ and we have

$$\langle x^*, T^*(\mathbf{1}_E)\rangle = \langle Tx^*, \mathbf{1}_E \rangle = \langle \mu_{x^*f}, \mathbf{1}_E \rangle = \int_E x^* f.$$

This shows that f is Dunford- $I_E(\mu)$ (resp. Dunford- $L_E^1(\mu)$) and $\int_E^D f = T^*(\mathbf{1}_E)$. □

3. The Orlicz-Pettis Theorem for Integrals

We now state and prove a continuous version of the Orlicz-Pettis theorem.

Theorem 3. μ-*measurable weakly integrable functions are strongly integrable*.

Proof. Suppose $f : \Omega \to X$ is μ-measurable and weakly-$I_E(\mu)$, where E is a measurable set. Then by Theorem 2, f is Dunford-$I_E(\mu)$. Let $A_0 := \{\omega \in \Omega : f(\omega) = 0\}$ and $A_n := \{\omega \in \Omega : n-1 < \| f(\omega) \| \leq n\}$. Then $(A_n)_{n=1}^{\infty}$ is a countable measurable partition of Ω. For each n, we consider the measurable function $g_n := \mathbf{1}_{A_n} f$. Clearly, the function $\omega \mapsto \| g_n(\omega) \|$ is μ-measurable and bounded, therefore it is Lebesgue μ-integrable. That is, g_n is Bochner μ-integrable; and therefore μ-integrable. It follows that for each n and for any given $\varepsilon > 0$, there exists (P_{n_0}, c_{n_0}) in Π such that

$$\| R_E(g_n, P, c) - R_E(g_n, Q, d) \| < \varepsilon/2^n$$

whenever $(P, c), (Q, d) \succ (P_{n_0}, c_{n_0})$ in Π.

Let $(P, c) = ((P_i)_{i \in \mathbb{N}}, (c_i)_{i \in \mathbb{N}})$ and $(Q, d) = ((Q_i)_{i \in \mathbb{N}}, (d_i)_{i \in \mathbb{N}})$ in Π. Then we note that

$$R_E(g_n, P, c) = \sum_{j \in S_n(c)} \mu(P_j \cap E) f(c_j),$$

$$R_E(g_n, Q, d) = \sum_{j \in S_n(d)} \mu(Q_j \cap E) f(d_j),$$

where $S_n(c) := \{j : c_j \in A_n\}$ and $S_n(d) := \{j : d_j \in A_n\}$ for $n \geq 0$. It follows that if $(P, c), (Q, d) \succ (P_{n_0}, c_{n_0})$ in Π,

$$\begin{aligned}
&\| R_E(f, (P, c) - R_E(f, Q, d)) \|_X \\
&= \left\| \sum_{n \in \mathbb{N}} \sum_{j \in S_n(c)} \mu(P_j \cap E) f(c_j) - \sum_{n \in \mathbb{N}} \sum_{j \in S_n(d)} \mu(Q_j \cap E) f(d_j) \right\|_X \\
&= \left\| \sum_{n \in N} [R_E(g_n, P, c) - R_E(g_n, Q, d)] \right\|_X \\
&\leq \sum_{n \in \mathbb{N}} \| R_E(g_n, (P, c)) - R_E(g_n, Q, d) \|_X \leq \varepsilon.
\end{aligned}$$

It follows that f is μ-integrable over E.

Since Pettis integrable is weakly integrable, Theorem 1 is seen as a direct corollary of the above result.

References

[1] R. G. Bartle, A modern theory of integration, AMS Graduate Studies in Mathematics, Vol. 32, Providence, RI, 2001.

[2] J. Diestel and J. Uhl, Jr., Vector measures, AMS Mathematical Survey, Vol. 15, Providence, RI, 1977.

[3] D. H. Fremlin, The Henstock and McShane integrals of vector-valued functions, Illinois J. Math. 38 (1994), 471-479.

[4] C. K. Fong, A continuous version of Orlicz-Pettis Theorem via vector-valued Henstock-Kurtzweil integrals, Czechoslovak Math. J. 52(127) (2002), 531-536.

[5] J. L. Gámez and J. Mendoza, On Denjoy-Dunford and Denjoy-Pettis integrals, Studia Math. 130(2) (1998), 115-133.

[6] Ye Guoju and An Tianqing, On Henstock-Dunford and Henstock-Pettis integrals, IJMMS 25(7) (2001), 467-478.

[7] R. A. Gordon, The Integral of Lebesgue, Denjoy, Perron, and Henstock, AMS Graduate Studies in Mathematics, Vol. 4, 1994.