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THE GROWTH ESTIMATES FOR DIRECTION DEPENDENT RANDOM FIELDS

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Abstract

In this paper, the growth estimates for direction dependent random fields with values in Banach spaces satisfying the general Kolmogorov's continuity test condition are mainly investigated. In particular, the estimates for Hilbert-valued Gaussian random fields which are also direction dependent, are investigated in detail. The main results are essentially established based on the application of a generalized version of the celebrated Garsia, Rodemich and Rumsey lemma.

1. Introduction

The main purpose of the paper is to investigate the growth estimates for some general random fields taking values in Banach spaces which are direction dependent and satisfy the general Kolmogorov's continuity test condition. The regularities of random fields, especially, the continuity and the growth (as the parameter tends to

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infinity) are very interesting and important in many fields, such as neurology, quantum field theory, population genetics and so on, we refer the reader to [1], [3], [5], [9], [12] and [13], and the references therein. In particular, the growth estimates for random fields are very important to determine an appropriate state space of solutions to some stochastic partial differential equations, see [10] and [11] for examples.

It is well known that the classical Garsia, Rodemich and Rumsey lemma, see [7], [8] and [12], is an effective method to study the modulus continuity of trajectories of a stochastic process and weak convergence in probability theory ([1], [2], [4] and [12]). A generalized version of Garsia, Rodemich and Rumsey lemma has been obtained in [6] (see also Lemma 3.1 below), which is aimed to study the sharp bounds on the modulus of continuity of solutions of stochastic partial differential equations driven by a direction dependent random field. In this article, instead of bounded parameter set, we will study the growth estimates for Banach valued and Hilbert valued random fields indexed by $\mathbb{R}^d$ with direction dependent moment estimates, using the generalized Garsia, Rodemich and Rumsey lemma by appropriate choices of functions. We are going to offer an effective method to study of the growth for random fields which admits a direction dependent moment estimate. As a special case, we will also study the growth estimates for Hilbert valued Gaussian random fields and the estimates are more accurate than the general case.

The paper is arranged as follows: In Section 2, the main results of the growth estimates for direction dependent random fields with values in Banach spaces and Gaussian direction dependent random fields with values in Hilbert spaces will be formulated. In Section 3, a generalized Garsia, Rodemich and Rumsey lemma obtained in [6] is first cited and then based on it, the proofs of main results are given. At the end of the paper, we provide an application of our main results to a Brownian sheet.

2. Main Results

Let $(B, \|\cdot\|)$ be a Banach space endowed with the Borel σ-field $\mathcal{B}(B)$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Then we recall that a random field $X = \{X(x); x \in \mathbb{R}^d\}$ with values in B is the collection of B-valued random variables (or elements) indexed by $\mathbb{R}^d$.

As we introduced in Section 1, our main goal in this paper is to study the growth for the general random fields with values in the Banach space B as the parameter tending to infinity. To state the main assertion precisely, let us first set

$$q_{\alpha}(x) := \max_{1 \le i \le d} |x_i|^{\alpha_i},$$

where $\alpha_i > 1$ such that $\alpha_0^{-1} := \sum_{1 \le i \le d} \alpha_i^{-1} < 1$ and $x = (x_1, ..., x_d) \in \mathbb{R}^d$. Then we have the following theorem:

Theorem 2.1. *Assume that* $\{X(x); x \in \mathbb{R}^d\}$ *is a B-valued random field which admits a direction dependent moment estimate as follows*: *there are some positive constants* γ, *K and k*, *such that*

$$\mathbb{E}[\| X(x) - X(y) \|^{\gamma}] \le Kn^k q_{\alpha}(x - y) \tag{2.1}$$

holds for every x, *y in the hypercube* $[-n, n]^d$, $n \in \mathbb{N}$. *Then the following results hold*:

(1) *X has a locally Hölder continuous version Y*.

(2) *If there exists* $\tilde{x} \in [-1, 1]^d$, *such that* γ*th moment of the random variable* $X(\tilde{x})$ *exists*, *i.e.*, $\mathbb{E}[\| X(\tilde{x})\|^{\gamma}] < \infty$, *then for every* $\delta > 1$, *there exists a real valued random variable* $\Xi_{\delta} \in (0, \infty)$ *a.s. such that*

$$\| Y(x) \| \le \Xi_{\delta}\left(1 + |x|^{\frac{k+\delta+\alpha_M}{\gamma}} (|x|^{-\alpha_m} q_x(x))^{\frac{\alpha_0 - 1}{\alpha_0 \gamma}}\right), \quad x \in \mathbb{R}^d \quad a.s., \tag{2.2}$$

where

$$\alpha_M = \max_{1 \le i \le d}\{\alpha_i\} \quad \textit{and} \quad \alpha_m = \min_{1 \le i \le d}\{\alpha_i\}.$$

Remark 2.1. The meaning of the word "version" appeared in the above theorem is that the constructed random field Y differs from X only on a set of $\mathbb{P}$-measure zero, i.e., $\mathbb{P}(X = Y) = 1$. In the proof, we will not make any distinction between such random fields.

According to this theorem, we can give the growth estimates for B-valued random fields satisfying the classical Kolmogorov's test condition.

Corollary 2.2. *Let* $\{X(x);\ x \in \mathbb{R}^d\}$ *be a random field taking values in the Banach space B. Suppose that there are positive constants* γ, *K and* $\sigma > d$ *such that for each* $n \in \mathbb{N}$,

$$\mathbb{E}[\|X(x) - X(y)\|^{\gamma}] \leq Kn^k |x - y|^{\sigma}, \quad x, y \in [-n, n]^d$$

and $\mathbb{E}[\|X(\tilde{x})\|^{\gamma}]$ *is bounded for some* $\tilde{x} \in [-1, 1]^d$. *Then, for each* $\delta > 1$, *there exists a locally Hölder continuous version Y of X satisfying*

$$\|Y(x)\| \leq \Xi_{\delta}\left(1 + |x|^{\frac{k+\delta+\sigma}{\gamma}}\right), \quad x \in \mathbb{R}^d \quad a.s., \tag{2.3}$$

where Ξ_{δ} *is a random variable with* $\mathbb{P}[0 < \Xi_{\delta} < \infty] = 1$.

In the above, we have studied the growth estimate for a general random field under Condition (2.1). However, we can expect that a better estimate can be obtained for a special random field. In the following, we investigate the growth corresponding to Gaussian random fields. Let us first assume that H is a separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{B}(H)$ denote the Borel σ-field of H. Let $\{X(x);\ x \in \mathbb{R}^d\}$ be an H-valued mean zero Gaussion random field defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Before stating our result for Gaussian random fields, let us set

$$\psi_{\zeta} := \max_{1 \leq i \leq d} |x_i|^{\zeta_i}, \quad \zeta_i > 0, \quad i = 1, \ldots, d$$

and

$$\zeta_0^{-1} := \sum_{i=1}^{d} \zeta_i^{-1}.$$

Theorem 2.3. *Let* $\{X(x); x \in \mathbb{R}^d\}$ *be an H-valued mean zero Gaussian random field. Assume that the following direction dependent moment estimate is satisfied: there exist positive constants K and k such that*

$$\mathbb{E}[|X(x) - X(y)|^2] \leq Kn^k \psi_{\zeta}(x - y) \tag{2.4}$$

holds for each $x, y \in [-n, n]^d$, $n \in \mathbb{N}$. *Then, for each* $\delta > 1$, *there exists a locally Hölder continuous version Z of X satisfying*

$$|Z(x)| \le \Theta_\delta\left(1 + |x|^{\frac{k+\zeta_M-\zeta_m}{2}}\sqrt{\psi_\zeta(x)\log(1+|x|)}\right), \quad x \in \mathbb{R}^d \quad a.s.,$$

where Θ_δ *is a real valued random variable with* $\mathbb{P}[0 < \Theta_\delta < \infty] = 1$, $\zeta_M = \max_{1\le i\le d}\zeta_i$ *and* $\zeta_m = \min_{1\le i\le d}\zeta_i$ *and* $|\cdot|$ *stands for the norm of the Hilbert space H hereafter*.

Remark 2.2. In this theorem, we do not require $\zeta_0^{-1} < 1$; recalling that the similar condition is necessary in Theorem 2.1.

3. Proofs and Application

In this section, we will devote to the proofs of the main results in Section 2. Before doing it, we first cite a generalization of Garsia, Rodemich and Rumsey lemma which is obtained in [6] as below and is an effective method to study the Hölder continuity of a random process, see [4] and [12].

To do it, the following hypotheses and notations will be introduced. Let q be a mapping from $\mathbb{R}^d \times \mathbb{R}^d$ to $[0, \infty)$ satisfying the following conditions:

H1. There exists a positive constant M_q, such that for all $x, y, z \in \mathbb{R}^d$,

$$q(x, y) \le M_q(q(x, z) + q(z, y)).$$

H2. The mapping q is symmetric, i.e., for any $x, y \in \mathbb{R}^d$, $q(x, y) = q(y, x)$.

H3. For any sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, if $\lim_{n\to\infty} q(x, x_n) = 0$, then the series x_n converges to x.

Let Φ and ϕ be positive, strictly increasing, and continuous functions on $[0, \infty)$ such that

$$\Phi(0) = \phi(0) = 0$$

and

$$\lim_{t\to\infty}\Phi(t) = \infty.$$

Lemma 3.1. *Suppose that* q, ϕ *and* Φ *are defined as above. Let* $f : \mathbb{R}^d \mapsto B$ *be a continuous function.*

Assume that

$$V := \int_{I_d} \int_{I_d} \Phi\left(\frac{\| f(x) - f(y) \|}{\phi(q(x, y))}\right) dxdy < \infty.$$

Then

$$\| f(x) - f(y) \| \leq 8 \max_{z \in \{x, y\}} \int_0^{q(x, y)} \Phi^{-1}\left(\frac{4V}{| B_z(u) \cap I_d |^2}\right) d\tilde{\phi}(u), \tag{3.1}$$

where $\tilde{\phi}(u) = \phi(4M_q^2 u)$, $B_z(u) = \{x \in \mathbb{R}^d ; q(x, z) \leq u\}$, $I_d = [0, 1]^d$ *stands for the unit hypercube in* $\mathbb{R}^d$ *and* M_q *is the constant appeared in* **H1**.

Remark 3.1. We have that $q(x, y) := \max_{1 \leq i \leq d} | x_i - y_i |^{\alpha_i}$ satisfies the conditions **H1-H3** in the above with $M_q = \max_{1 \leq i \leq d} \max\{2^{\alpha_i - 1}, 1\}$, for every $\alpha_i > 0$, $i = 1, \ldots, d$. In addition, it is obvious that if q is metric on $\mathbb{R}^d$, then Conditions **H1-H3** above are satisfied.

Although Lemma 3.1 is a purely deterministic result, it can often be applied to the random case, as below. In fact, everything hinges on the choices of the functions Φ and ϕ as we do below.

In the proofs, for brevity, we allow the positive constant denoted by C to vary from one appearance into another even within the same proof if there is no confusion.

Proof of Theorem 2.1. The Hölder continuity of the B-valued random field satisfying the generalized Kolmogorov test condition (2.1) can be verified by the standard arguments with some modification, we refer the reader to [9, p. 31].

In the following, we will prove that the estimate (2.2) for the random field X holds. At first, if there exists $\tilde{x} \in I_d$, such that the expectation of $\| X(\tilde{x}) \|^\gamma$ exists, then by Theorem 3.2 in [6], we know that

$$\| X(x) \| \leq \| X(\tilde{x}) \| + C\Theta, \quad x \in [-1, 1]^d \quad a.s.,$$

where Θ is a positive random variable with the existence of γth moment, i.e.,

$$\mathbb{E}[\Theta^{\gamma}] < \infty.$$

Hence,

$$\sup_{x \in [-1, 1]^d} \| X(x) \| < \infty \quad \text{a.s.} \tag{3.2}$$

Assume that $x_0 \in \mathbb{R}^d$ is the center of the unit hypercube I_d and $n \geq 2$ is an integer in the following.

Define the random field $\{X^n(x);\ x \in I_d\}$ for each n by

$$X^n(x) = X(2n(x - x_0)).$$

From the key Condition (2.1) and recalling the definition of q_α, we see that

$$\begin{aligned} E[\|X^n(x) - X^n(y)\|^{\gamma}] &= \mathbb{E}[\| X(2n(x - x_0)) - X(2n(y - x_0)) \|^{\gamma}] \\ &\leq Kn^k q_\alpha(2n(x - y)) \\ &\leq Cn^{k+\alpha_M} q_\alpha(x - y). \end{aligned} \tag{3.3}$$

For simplicity, we introduce the following notations:

$$\Phi(t) := |t|^{\gamma},$$

$$\phi(t) := \mathbf{L}_N\left(r't;\ \frac{\alpha_0 - 1}{\alpha_0 \gamma},\ \gamma^{-1},\ \beta\right), \quad N \in \mathbb{N},$$

$$V^n := \int_{I_d} \int_{I_d} \Phi\left(\frac{\| X^n(x) - X^n(y) \|}{\phi(q_\alpha(x - y))}\right) dxdy, \quad n \in \mathbb{N},$$

where

$$\mathbf{L}_N(t;\ a,\ b,\ c) := \begin{cases} t^a \prod_{n=1}^{N-1} \left(\left(\log^n \frac{1}{t}\right)^b\right) \left(\log^N \frac{1}{t}\right)^c, & N \geq 2, \\ t^a \left(\log \frac{1}{t}\right)^c, & N = 1, \end{cases}$$

for all strict positive t, and r' is taken such that ϕ is increasing in $[0,\ 4M_q^2]$.

By Fubini's theorem and (3.3), we can easily see that

$$\begin{aligned}\mathbb{E}[V^n] &\le \int_{I_d}\int_{I_d} \frac{Kn^{k+\alpha_M} q_\alpha(x-y)}{\phi(q_\alpha(x-y))\gamma}\,dxdy \\ &\le Kn^{k+\alpha_M}\int_0^1 \frac{2^d t^{(\alpha^{-1}-1)}}{\alpha_0}\,dt \\ &\le Kn^{k+\alpha_M}\int_0^1 \mathbf{L}_N(t, -1, -1, -\beta\gamma)\,dt \\ &\le Cn^{k+\alpha_M}. \end{aligned} \tag{3.4}$$

For each $\delta > 1$, let us now consider the random variable

$$V_\delta := \sum_{n=1}^{\infty} \frac{V^n}{n^{k+\delta+\alpha_M}}.$$

Then, from (3.4), it is obvious that

$$\mathbb{E}[V_\delta] \le C\sum_{n=1}^{\infty} n^{-\delta} < \infty,$$

which implies that there is a subset Ω_δ of Ω with $\mathbb{P}(\Omega_\delta) = 1$, such that $V_\delta(\omega)$ is finite and

$$V^n(\omega) \le n^{k+\delta+\alpha_M} V_\delta(\omega) < \infty, \quad \text{for all} \quad \omega \in \Omega_\delta. \tag{3.5}$$

Note that

$$|B_z(u) \cap I_d| \ge |B_0(u) \cap I_d| = u^{\alpha_0^{-1}}.$$

Applying Lemma 3.1 to X^n and by (3.1), we come to

$$\begin{aligned}\|X^n(x) - X^n(y)\| &\le 8\int_0^{q_\alpha(x-y)} \Phi^{-1}\left(\frac{4V^n}{u^{\alpha_0^{-2}}}\right) d\tilde{p}(u) \\ &\le C(V^n)^{1/\gamma}\int_0^{q_\alpha(x-y)} \mathbf{L}_N\left(rt; \frac{\alpha_0 - 1}{\alpha_0} - 1, \gamma^{-1}, \beta\right)dt \quad \text{a.s.,} \end{aligned} \tag{3.6}$$

where $r = 4M_q^2 r'$.

Note that for any $a > 0$, $b, c \in \mathbb{R}$ and each N,

$$\lim_{\varepsilon \to 0} \frac{\int_0^{\varepsilon} \mathbf{L}_N(t; a-1, b, c)}{\mathbf{L}_N(\varepsilon; a, b, c)} = a^{-1}.$$

Then, by (3.5), we obtain that there exists a constant C, such that for all $x, y \in I_d$, (3.6) is bounded above by

$$C(V^n)^{\frac{1}{\gamma}} \mathbf{L}_N\left(rq_\alpha(x-y); \frac{\alpha_0 - 1}{\alpha_0 \gamma}, \gamma^{-1}, \beta \right)$$

$$\leq Cn^{\frac{k+\delta+\alpha_M}{\gamma}} V_\delta^{\frac{1}{\gamma}} \mathbf{L}_N\left(rq_\alpha(x-y); \frac{\alpha_0 - 1}{\alpha_0 \gamma}, \gamma^{-1}, \beta \right) \quad \text{a.s.} \tag{3.7}$$

Therefore, for each $x, y \in [-n, n]^d$, we get that

$$\| X(x) - X(y) \| \leq \left\| X^n\left(\frac{x}{2n} + x_0 \right) - X^n\left(\frac{y}{2n} + x_0 \right) \right\|$$

$$\leq Cn^{\frac{k+\delta+\alpha_M}{\gamma}} V_\delta^{\frac{1}{\gamma}} \mathbf{L}_N\left(rq_\alpha\left(\frac{x-y}{2n} \right); \frac{\alpha_0 - 1}{\alpha_0 \gamma}, \gamma^{-1}, \beta \right) \quad \text{a.s.}$$

From the above inequality, we see that, for each $n > 1$,

$$\| X(x) \|$$

$$\leq \| X(0) \| + Cn^{\frac{k+\delta+\alpha_M}{\gamma}} V_\delta^{\frac{1}{\gamma}} \mathbf{L}_N\left(rq_\alpha\left(\frac{x}{2n} \right); \frac{\alpha_0 - 1}{\alpha_0 \gamma}, \gamma^{-1}, \beta \right), \quad x \in [-n, n]^d \quad \text{a.s.} \tag{3.8}$$

Since, for each $x \in \mathbb{R}^d \setminus [-1, 1]^d$, there exists $n \in \mathbb{N}$ such that

$$x \in [-n, n]^d \setminus [-n+1, n-1]^d,$$

we have that $n - 1 \leq |x| \leq \sqrt{d}n$ and $n \leq 2|x|$.

Hence, from (3.8), for $x \in [-n, n]^d$,

$$\| X(x) \| \leq \| X(0) \| + C|x|^{\frac{k+\delta+\alpha_M}{\gamma}} V_\delta^{\frac{1}{\gamma}} \mathbf{L}_N\left(rq_\alpha\left(\frac{x}{2n} \right); \frac{\alpha_0 - 1}{\alpha_0 \gamma}, \gamma^{-1}, \beta \right), \quad \text{a.s.} \tag{3.9}$$

Noticing that $\max_{1\le i\le d}\left(\frac{n-1}{2n}\right)^{\alpha_i} \le q_\alpha\left(\frac{x}{2n}\right) \le \max_{1\le i\le d}\left(\frac{1}{2}\right)^{\alpha_i}$ and by the property of $\mathbf{L}_N$, we see that, there is a constant C, such that

$$\mathbf{L}_N\left(rq_\alpha\left(\frac{x}{2n}\right); \frac{\alpha_0-1}{\alpha_0\gamma}, \gamma^{-1}, \beta\right) \le C(n^{-\alpha_m}q_\alpha(x))^{\frac{\alpha_0-1}{\alpha_0\gamma}}. \tag{3.10}$$

Therefore, form (3.9) and (3.10),

$$\| X(x)\| \le \| X(0)\| + C|x|^{\frac{k+\delta+\alpha_M}{\gamma}} V_\delta^{\frac{1}{\gamma}}(|x|^{-\alpha_m} q_\alpha(x))^{\frac{\alpha_0-1}{\alpha_0\gamma}}, \quad x \in [-n, n]^d \quad \text{a.s.}$$

Combining the above estimate with (3.2), we can conclude that there exists a strict positive real valued random variable Ξ_δ, i.e., $0 < \Xi_\delta < \infty$ a.s., such that

$$\| X(x)\| \le \Xi_\delta\left(1 + |x|^{\frac{k+\alpha_M}{\gamma}}(|x|^{-\alpha_m} q_\alpha(x))\right)^{\frac{\alpha_0-1}{\alpha_0\gamma}} \quad \text{a.s.}$$

Thus, the proof is completed.

Now let us give the proof of Corollary 2.2 which can be considered as a simple application of Theorem 2.1.

Proof of Corollary 2.2. Let $\alpha_i = \sigma$, for $i = 1, ..., d$ in Theorem 2.1. Noticing that

$$q_\sigma(x) \le |x|^\sigma \le d^{\sigma/2} q_\sigma(x), \quad x \in \mathbb{R}^d$$

and by (2.3), we have that there exists a constant C such that

$$\mathbb{E}[\| X(x) - X(y)\|^\gamma] < Cn^k q_\sigma(x-y),$$

which is just the Kolmogorov's continuity test condition (2.1). Therefore, the result follows from Theorem 2.1 immediately. □

Now we are in the position to show Theorem 2.3. Recalling that X is an H-valued Gaussian random field, from the celebrated Fernique's theorem, we have that $\mathbb{E}[|X(x)|^\iota] < \infty$, $\iota \ge 1$, $x \in \mathbb{R}^d$. More accurately, we have the following assertion:

Lemma 3.2. *Let X be an H-valued Gaussian random variable with mean zero. Then, for arbitrary $m \in \mathbb{N}$, there exists a constant C_m such that*

$$\mathbb{E}[|X|^{2m}] \le C_m \mathbb{E}[|X|^2]^m.$$

Proof. The claim follows immediately from Corollary 2.17 in [4]. Here the concrete proof will be omitted. □

Proof of Theorem 2.3. From Lemma 3.2, it follows obviously that for every $m \in \mathbb{N}$,

$$\mathbb{E}[| X(x) - X(y)|^{2m}] \leq Cn^{km}\psi_\zeta^m(x - y).$$

Thus the claim of the existence of a locally Hölder continuous version follows immediately as in Theorem 2.1.

For simplicity, we will still denote this version by X hereafter. Let $\{X^n(x); x \in I_d\}_{n\in\mathbb{N}}$ be the random fields defined as in the proof of Theorem 2.1. From (2.4) and the relation

$$\psi_\zeta(2nx) \leq (2n)^{\zeta_M}\,\psi_\zeta(x),$$

we see that

$$\mathbb{E}[|X^n(x) - X^n(y)|^2] \leq C_0 n^{k+\zeta_M}\,\psi_\zeta(x - y), \tag{3.11}$$

where C_0 is a positive constant.

Define

$$Z^n(x, y) := \frac{X^n(x) - X^n(y)}{\sqrt{C_0 n^{k+\zeta_M}\,\psi_\zeta(x - y)}}, \quad x, y \in I_d, \quad n \in \mathbb{N}.$$

Then we have that $Z^n(x, y)$ is an H-valued Gaussian random variable with mean zero and covariance matrix $\Gamma^n(x, y)$, which is a trace class operator. By (3.11), we obtain that

$$\mathrm{Tr}(\Gamma^n(x, y)) = \mathbb{E}[|Z^n(x, y)|^2]$$

is uniformly bounded above by one, where Tr denotes the trace of $\Gamma^n(x, y)$. Hence, from Proposition 2.16 [4], it follows that for each $s > 2$,

$$\mathbb{E}\left[\exp\left(\frac{|Z^n(x, y)|^2}{s}\right)\right] \leq \exp\left(\sum_{j=1}^{\infty}\frac{2^{j-1}}{js^j}\right) < \infty. \tag{3.12}$$

In order to apply Lemma 3.1, we put

$$\phi(t) = t^{\frac{1}{2}},$$

$$\Phi_n(t) = \exp\left(\frac{t^2}{sC_0 n^{k+\zeta_M}}\right),$$

$$Q^n = \int_{I_d}\int_{I_d} \Phi_n\left(\frac{|X^n(x) - X^n(y)|}{\phi(\psi_\zeta(x-y))}\right)dxdy,$$

where $t \geq 0,\ s > 2$ and C_0 is the constant in (3.11). Then the inverse function Φ_n^{-1} of Φ_n equals to

$$(C_0 s n^{k+\zeta_M} \log t)^{\frac{1}{2}}.$$

By (3.12) and Fubini's theorem, $\mathbb{E}[Q^n]$ is uniformly bounded, i.e.,

$$\sup_{n\in\mathbb{N}} \mathbb{E}[Q^n] < \infty.$$

Define

$$Q_\delta := \sum_{n=1}^{\infty} n^{-\delta} Q^n, \quad \delta > 1.$$

It is obvious that the expectation of Q_δ is bounded and

$$Q^n \leq n^\delta Q_\delta \quad \text{a.s.}$$

Thus we can apply Lemma 3.1 for $\Phi_n,\ \phi$ and Q^n. Notice that

$$|B_z(u) \cap I_d| \geq |B_0(u) \cap I_d| = u^{\frac{1}{\zeta_0}}$$

and

$$(|a| + |b|)^{1/2} \leq |a|^{1/2} + |b|^{1/2}.$$

Lemma 3.1 now states that for any $x, y \in [-1, 1]^d$,

$$
\begin{aligned}
|X^n(x) - X^n(y)| &\leq C \max_{z \in \{x, y\}} \int_0^{\psi_\zeta(x-y)} \Phi^{-1}\left(\frac{4Q^n}{|B_z(u) \cap I_d|^2}\right) u^{-\frac{1}{2}} du \\
&\leq C n^{\frac{k+\zeta_M}{2}} \int_0^{\psi_\zeta(x-y)} \left(\log \frac{4Q^n}{u^{2/\zeta_0}}\right)^{\frac{1}{2}} u^{-\frac{1}{2}} du \\
&\leq C n^{\frac{k+\zeta_M}{2}} \psi_\zeta^{\frac{1}{2}}(x-y)\left(|\log 4Q_\delta|^{\frac{1}{2}} + (\delta \log n)^{\frac{1}{2}}\right) \\
&\quad + \int_0^{\psi_\zeta(x-y)} \left(\frac{2}{\zeta_0} \log \frac{1}{u}\right)^{\frac{1}{2}} u^{-\frac{1}{2}} du \quad \text{a.s.}
\end{aligned}
$$

On the other hand, it is easy to know that the integral of the right hand side of the above the inequality is bounded by

$$
C\left((\psi_\zeta(x-y)|\log \psi_\zeta(x-y)|)^{\frac{1}{2}} + 1\right).
$$

Therefore, we conclude that

$$
\begin{aligned}
&|X^n(x) - X^n(y)| \\
&\leq C n^{\frac{k+\zeta_M}{2}}\left(1 + \psi_\zeta^{\frac{1}{2}}(x-y)\left(|\log 4Q_\delta|^{\frac{1}{2}} + (\delta \log n)^{\frac{1}{2}} + |\log \psi_\zeta(x-y)|^{\frac{1}{2}}\right)\right) \quad \text{a.s.}
\end{aligned}
$$

Then the desired estimate can be verified by similar arguments as in the proof of Theorem 2.1. We omit the detail here. □

In the following, a simple example is given to illustrate the subtle difference between Theorem 2.1 and Theorem 2.3.

Example. Let $W := \{W(x), x \in \mathbb{R}^d\}$ be a real valued and centered Gaussian random field with covariance function given by

$$
\begin{aligned}
&\mathbb{E}[W(x)W(y)] \\
&= \begin{cases} 0, & \text{if } \exists\ x_i, y_i \text{ s.t. } \operatorname{sgn}(x_i) + \operatorname{sgn}(y_i) = 0, \text{ for } i = 1, \ldots, d, \\ \prod_{i=1}^d |x_i \wedge y_i|, & \text{otherwise,} \end{cases}
\end{aligned}
$$

where sgn is the signal function. In other words, W is in fact composed of independent Brownian sheets, see [12] or [13] for the detailed definition of Brownian sheets. Then, for each $n \in \mathbb{N}$, and $x, y \in [-n, n]^d$, we have

$$\mathbb{E}[|W(x) - W(y)|^2] \le dn \max_{1 \le i \le d} |x_i - y_i|.$$

From Theorem 2.1, it follows that for each $\delta > \frac{1}{2}$, there is an almost surely bounded and positive random variable Ξ_δ such that

$$W(x) \le \Xi_\delta (1 + |x|^\delta) \quad \text{a.s.} \tag{3.13}$$

However, Theorem 2.3 leads to

$$|W(x)| \le \Theta\left(1 + (|x| \log(1 + |x|))^{\frac{1}{2}}\right) \quad \text{a.s.}, \tag{3.14}$$

which is very close to the exactly asymptotic behavior comparing with the law of the iterated logarithm (see Theorem 1.5 in [12], for example). In addition, it is obvious that the estimate (3.14) is more accurate than the estimate (3.13).

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