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STUDY ABOUT GRADED RINGS BY COMPLETELY REGULAR SEMIGROUP AND COMPLETELY SIMPLE SEMIGROUP

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Abstract

Let S be a semigroup and $R = \bigoplus_{s \in S} R_s$ be an S-graded ring with an identity element. We study some properties of the components of the identity element of R and the support of R when S is a completely regular semigroup and when S is a completely regular semigroup with a neutral element. We also study some of these properties when R is a commutative ring and S is a completely simple semigroup, and when R is a non-commutative ring and S is a completely regular semigroup and the component R_s $(\forall s \in S)$ is an ideal in R.

1. Preface

Remark 1.1. Throughout this paper the word "semigroup" refers to multiplicative semigroup if not mentioned otherwise.

Definition 1.2 [2, 10]**.** Let R be a ring, and S be a semigroup. Then we say that R is a graded ring by the semigroup S, or R is an S-graded ring if and only if there

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exist additive subgroups $\{R_s\}_{s\in S}$ of *R* satisfying the following:

(1) $R = \bigoplus_{s\in S} R_s$;

(2) $R_g R_h \subseteq R_{gh}, \quad \forall g, h \in S.$

Definition 1.3. Let *R* be a ring and *S* be a semigroup.

(1) [3, 11] Let *o* be a binary operation on *R*, defined as:

$$aob = a + b - ab, \quad \forall a, b \in R.$$

Then *o* is an associative operation on *R*, so (R, o) is a monoid with the zero of *R* as the identity element, we say that (R, o) is the monoid induced of *R*.

An element $a \in R$ is called *left* (*right*) *quasi-regular* if *a* has a left (right) inverse in the monoid (R, o) with identity, i.e., if there exists an element *b* of *R* such that

$$boa = 0 \quad (aob = 0).$$

- If *R* has an identity 1, then an element *a* of *R* is *left* (*right*) *quasi-regular* if $1-a$ has a left (right) inverse with respect to ring multiplication.

- If *a* is both left and right quasi-regular, then we say that *a* is *quasi-regular*.

(2) [3, 11] Let *I* be a non-empty set of *R*. Then we say that *I* is (*left*, *right*) *quasi-regular* if every element of *I* is (left, right) quasi-regular.

(3) [4] Let *e* be an idempotent of a semigroup *S*. Then we say that *e* is a *primitive idempotent* if *e* is minimal in the set of non-zero idempotent. Thus, a primitive idempotent *e* has the following property:

$$ef = fe = f \neq 0 \Rightarrow e = f,$$

where *f* is an idempotent of *S*.

4. [4] Let *S* be a semigroup without zero. Then we say that *S* is *simple* if it has no proper ideals.

5. [4] Let *S* be a semigroup without zero. Then we say that *S* is *completely simple* if it is simple and if it contains a primitive idempotent.

6. [4] A semigroup *S* will be called *completely regular* if there exists a unary

operation $a \to a^{-1}$ on S with the properties:

$$(a^{-1})^{-1} = a \quad \text{and} \quad aa^{-1}a = a \quad \text{and} \quad aa^{-1} = a^{-1}a.$$

- An equivalent definition:

Let S be a semigroup without zero. Then we say that S is *completely regular* if every element of S lies in a subgroup of S.

Lemma 1.4 [4]. *Let S be a semigroup without zero. Then the following conditions are equivalent*:

(1) *S is completely simple*.

(2) *S is regular, and has* "*weak cancellation*" *property*:

For all a, b, c in S,

$$ca = cb \quad \text{and} \quad ac = bc \;\Rightarrow\; a = b.$$

(3) *S is regular, and for all a in S,*

$$aba = a \Rightarrow bab = b.$$

(4) *S is regular and every idempotent is primitive*.

2. Results

Theorem 2.1. *Let R be a ring and S be a completely regular semigroup. Suppose that* $R = \bigoplus_{s \in S} R_s$ *is a graduation of R by S. Then let* $\{G_i\}_{i \in I}$ *be the family of all the maximal subgroups of S. So*:

(1) R_{G_i} $(i \in I)$ *is a subring of R, and* $R = \bigoplus_{i \in I} R_{G_i}$.

(2) *If* R_{G_i} $(\forall i \in I)$ *is an ideal in R and right quasi-regular in itself, then R is a right quasi-regular ring*.

Proof. (1) R_{G_i} $(i \in I)$ is a subring of R because on the one hand R_{G_i} $(i \in I)$ is additive subgroup of R and on the other hand:

$$\forall g, h \in R_{G_i} \Rightarrow g = \sum_{t \in G_i} r_t \quad \text{and} \quad h = \sum_{t \in G_i} r_t';$$

$$r_t, r_t' \in R_t \Rightarrow gh = \left(\sum_{t \in G_i} r_t\right)\left(\sum_{t \in G_i} r_t'\right).$$

Since G_i is a subgroup of S, thus

$$t_1 \cdot t_2 \in G_i, \quad \forall t_1, t_2 \in G_i.$$

It follows that

$$gh = \left(\sum_{t \in G_i} r_t\right)\left(\sum_{t \in G_i} r'_t\right) \in R_{G_i}.$$

Since S is a completely regular semigroup, so S is a union of groups, i.e.,

$$S = \bigcup_{\substack{H \text{ subgroup} \\ \text{of } S}} H.$$

Moreover, since every subring of S is contained in a maximal subgroup of S, so for every subgroup H of S, there exists a maximal subgroup G of S such that $H \subseteq G$. Thus

$$S = \bigcup_{\substack{G \text{ maximal} \\ \text{subgroup of } S}} G.$$

Since $\{G_i\}_{i \in I}$ is the family of all the maximal subgroup of S, so

$$G_{i_1} \cap G_{i_2} = \varnothing, \quad \forall i_1, i_2 \in I, \; i_1 \neq i_2 \quad \text{and} \quad S = \bigcup_{i \in I} G_i.$$

Remark that $R = \bigoplus_{s \in S} R_s$, we deduce

$$R = \bigoplus_{i \in I} R_{G_i}.$$

(2) Suppose that $R_{G_i} (i \in I)$ is an ideal in R and right quasi-regular in itself. As $R_{G_i} (i \in I)$ is an ideal in R and right quasi-regular in itself and $R = \bigoplus_{i \in I} R_{G_i}$, then R is a right quasi-regular ring (see the proof of the Propositions (2-6) in [1]).

Proposition 2.2. *Let R be a ring with an identity element* 1 *such that R does not have any divisor of zero, and let S be a completely regular semigroup. Suppose that $R = \bigoplus_{s \in S} R_s$ is a graduation of R by S. Also suppose that $R = \bigoplus_{i \in I} R_{G_i}$ such that*

$\{G_i\}_{i\in I}$ *is the family of all the maximal subgroups of S. If* 1 *is a homogenous element in R, then* $\text{supp}(R, S)$ *is a submonoid of S, and* $1 \in R_e$ *such that e is neutral element in* $\text{supp}(R, S)$.

Proof. Suppose that 1 is a homogenous element in *R*. Since *R* does not have any divisor of zero, so $\text{supp}(R, S)$ is a subsemigroup of *S*.

As 1 is a homogenous element in *R*, so there exists an element *s* of *S* such that $1 \in R_s$. Since $S = \bigcup_{i\in I} G_i$, so there exists $\alpha \in I$ such that $S \in G_\alpha$. Thus

$$1 \in R_s \subseteq R_{G_\alpha}.$$

Since R_{G_α} is a ring and $R_{G_\alpha} = \bigoplus_{g_\alpha \in G_\alpha} R_{g_\alpha}$, so R_{G_α} is a graded ring by the group G_α, and since $1 \in R_{G_\alpha}$, $1 \in R_e$ such that *e* is the neutral element in G_α. Since $1 \in R_e$, so $R_e \neq \{0\}$, thus $e \in \text{supp}(R, S)$. Furthermore, if s_1 is an element of $\text{supp}(R, S)$, and *a* is an element of $R_{s_1} - \{0\}$, then

$$0 \neq a = 1 \cdot a \in R_e R_{s_1} \subseteq R_{es_1}$$

and

$$a = 1 \cdot a \in R_{s_1} \Rightarrow R_{es_1} = R_{s_1} \Rightarrow es_1 = s_1.$$

Similarly, with observation that $a \cdot 1 = a$, we find that

$$s_1 e = s_1, \quad \forall s_1 \in \text{supp}(R, S).$$

So

$$es_1 = s_1 e = s_1, \quad \forall s_1 \in \text{supp}(R, S).$$

Thus *e* is neutral element in $\text{supp}(R, S)$. Therefore, $\text{supp}(R, S)$ is a submonoid of *S* with a neutral element *e*, and $1 \in R_e$.

$*$ **-** In the special case when $S = \text{supp}(R, S)$, *S* is monoid and $1 \in R_e$ such that *e* is the neutral element in *S* whether *R* has divisors of zero or not.

Theorem 2.3. *Let R be a ring with an identity element* 1 *and let S be a completely regular semigroup with a neutral element e. Suppose that* $R = \bigoplus_{s\in S} R_s$ *is* *a graduation of R by S, and also* $R = \bigoplus_{i \in I} R_{G_i}$ *such that* $\{G_i\}_{i \in I}$ *is the family of all the maximal subgroups of S. Since* $1 \in R = \bigoplus_{s \in S} R_s$, *so* 1 *can be written with an only way, by the form*

$$1 = \sum_{i=1}^{n} a_{s_i}; \qquad a_{s_i} \in R_{s_i} - \{0\},$$

such that $s_1, s_2, ..., s_n$ *are distinct elements of S. If* $R_e \neq \{0\}$, *then* $e \in \{s_1, s_2, ... s_n\}$, *and if we suppose, for example, that* $e = s_1$ *and* G_β $(\beta \in I)$ *is the maximal subgroup of S which e belongs to, then* $a_{s_1} = a_e$ *is the identity element of the ring* R_{G_β}.

Proof. Suppose that $R_e \neq \{0\}$, $e \in \{s_1, s_2, ..., s_n\}$ because if it is not and if b is an element of $R_e \neq \{0\}$, then

$$b = b \cdot 1 = b\left(\sum_{i=1}^{n} a_{s_i}\right) = \sum_{i=1}^{n} b a_{s_i}.$$

Since

$$e \notin \{s_1, s_2, ..., s_n\} \text{ and } ba_{s_i} \in R_{es_i} = R_{s_i}, \quad \forall i = 1, 2, ..., n \text{ and } b \in R_e,$$

so

$$b = 0,$$

which is a contradiction.

Suppose, for example, that $e = s_1$. Then

$$a_e = a_e \cdot 1 = a_e(a_e + a_{s_2} + \cdots + a_{s_n}) = a_e a_e + a_e a_{s_2} + \cdots + a_e a_{s_n}.$$

Since $a_e a_{s_i} \in R_{es_i} = R_{s_i}$, $\forall i = 2, 3, ..., n$ and $e, s_2, s_3, ..., s_n$ are distinct elements of S, so

$$a_e a_{s_i} = 0, \qquad \forall i = 2, 3, ..., n.$$

If we suppose, for instance, that G_β $(\beta \in I)$ is the maximal subgroup of S which e

belongs to, then

$$s_i \notin G_\beta, \quad \forall i = 2, 3, ..., n,$$

because, if any element of the set $\{s_2, s_3, ..., s_n\}$, for example, s_2 belongs to G_β, then

$$a_{s_2} = 1 \cdot a_{s_2} = a_e a_{s_2} + a_{s_2} a_{s_2} + \cdots + a_{s_n} a_{s_2},$$

so if j is an element of the set $\{3, 4, ..., n\}$, such that $a_{s_j} a_{s_2} \neq 0$, then $a_{s_j} a_{s_2} \notin R_{s_2}$, because if $a_{s_j} a_{s_2} \in R_{s_2}$ it follows that

$a_{s_j} a_{s_2} \in R_{s_2}$ and

$a_{s_j} a_{s_2} \in R_{s_j s_2} \Rightarrow 0 \neq a_{s_j} a_{s_2} \in R_{s_2} \cap R_{s_j s_2} \Rightarrow s_2 = s_j s_2 \Rightarrow s_2 s_2^{-1} = s_j s_2 s_2^{-1};$

s_2^{-1} is the inverse of s_2 in $G_\beta \Rightarrow e = s_j e \Rightarrow s_j = e,$

and this contradicts $s_j \neq e$. Since $a_e a_{s_2} = 0$, so

$$a_{s_2} = a_{s_2} a_{s_2}.$$

Thus

$$0 \neq a_{s_2} \in R_{s_2} \cap R_{s_2 s_2}.$$

Hence

$$s_2 = s_2 s_2.$$

But $s_2 \in G_\beta$ and G_β is a group in which its identity is the only idempotent, so $e = s_2$, and this contradicts $e \neq s_2$. Since

$$s_i \notin G_\beta, \quad \forall i = 2, 3, ..., n, \tag{$*$}$$

so a_e is the identity element in R_{G_β}, because on the one hand a_e belongs to R_{G_β}, and on the other hand

$$\forall b \in R_{G_\beta} \Rightarrow b = \sum_{g \in G_\beta} b_g;$$

$$b_g \in R_g \Rightarrow b = b1 = \left(\sum_{g \in G_\beta} b_g\right)(a_e + a_{s_2} + \dots + a_{s_n})$$

$$\Rightarrow b = \sum_{g \in G_\beta} b_g a_e + \sum_{i=2}^{n}\left(\sum_{g \in G_\beta} b_g a_{s_i}\right).$$

Let t be an element of the set $\{2, 3, \dots, n\}$ and y be an element of G_β such that $b_y a_{s_t} \neq 0.$ Then $b_y a_{s_t} \notin R_{G_\beta}$ because if it is not so, we have

$$0 \neq b_y a_{s_t} \in R_{ys_t}$$

and

$$0 \neq b_y a_{s_t} \in R_{G_\beta} \Rightarrow g' \in G_\beta; \quad ys_t = g'.$$

Suppose that y^{-1} is the inverse of y in $G_\beta,$ it follows that

$$ys_t = g' \Rightarrow y^{-1}ys_t = y^{-1}g' \Rightarrow s_t = y^{-1}g'.$$

Since G_β is a group and $y^{-1},$ g' are two elements of $G_\beta,$ we have

$$s_t \in G_\beta,$$

and this contradicts $(*)$.

Thus

$$b = \sum_{g \in G_\beta} b_g a_e = ba_e.$$

In the same way, we can prove that

$$b = a_e b.$$

So

$$b = a_e b = ba_e, \quad \forall b \in R_{G_\beta}.$$

Thus $a_{s_1} = a_e$ is the identity element in $R_{G_\beta}.$

Theorem 2.4. *Let R be a commutative ring with an identity element* 1, *and S be a completely simple semigroup* (*S without zero*). *Suppose that* $R = \bigoplus_{s \in S} R_s$ *is a graduation of R by S, and also* $R = \bigoplus_{i \in I} R_{G_i}$ *such that* $\{G_i\}_{i \in I}$ *is the family of all the maximal subgroups of S. So we can write* 1 *in the form*

$$1 = \sum_{t=1}^{m} (a_{s_{t0}} + a_{s_{t1}} + \ldots + a_{s_{tn_t}}); \quad a_{s_{tj}} \in R_{s_{tj}} - \{0\}, \quad \forall j = 0, 1, \ldots, n_t; \; m \in Z^+,$$

such that $s_{t0}, s_{t1}, \ldots, s_{tn_t}$ $(t = 1, 2, \ldots, m)$ *are distinct elements of* G_{r_t} $(r_t \in I)$, *then* $G_{r_1}, G_{r_2}, \ldots, G_{r_m}$ *are the all maximal subgroups of S, where*

$$R_{G_{r_1}} \neq \{0\} \text{ and } R_{G_{r_2}} \neq \{0\} \text{ and ... and } R_{G_{r_m}} \neq \{0\},$$

and if we suppose that e_t $(t = 1, 2, \ldots, m)$ *is the neutral of* G_{r_t}, *then* $\{e_1, e_2, \ldots, e_m\} \subseteq \{s_{10}, \ldots, s_{1n_1}, s_{20}, \ldots, s_{2n_2}, \ldots, s_{m0}, \ldots, s_{mn_m}\}$ *and* a_{e_t} $(t = 1, 2, \ldots, m)$ *is the identity element of the subring* $R_{G_{r_t}}$ *and* $e_1, e_2, \ldots, e_m$ *are the all idempotents in* $\text{supp}(R, S)$.

Proof. Since $1 \in R = \bigoplus_{s \in S} R_s = \bigoplus_{i \in I} R_{G_i}$, so we can write 1 in only way in the form

$$1 = \sum_{t=1}^{m} b_{G_{r_t}}; \qquad [b_{G_{r_t}} \in R_{G_{r_t}} - \{0\} \text{ and } r_t \in I, \; \forall t = 1, 2, \ldots, m].$$

Thus

$$b_{G_{r_t}} \in R_{G_{r_t}}, \quad \forall t = 1, 2, \ldots, m$$

$$\Rightarrow b_{G_{r_t}} = a_{s_{t0}} + \ldots + a_{s_{tn_t}}, \quad \forall t = 1, 2, \ldots, m,$$

such that

$$a_{s_{tj}} \in R_{s_{tj}} - \{0\}, \quad \forall j = 0, 1, \ldots, n_t, \quad \forall t = 1, 2, \ldots, m,$$

and

$$s_{t0},\ s_{t1},\ ...,\ s_{tn_t}\ \ (t = 1,\ 2,\ ...,\ m)\ \text{ are different elements of } G_{r_t}.$$

It follows that

$$1 = \sum_{t=1}^{m} (a_{s_{t0}} + a_{s_{t1}} + \dots + a_{s_{tn_t}}).$$

Since

$$a_{s_{t0}} \in R_{s_{t0}} - \{0\}, \quad \forall t = 1,\ 2,\ ...,\ m \quad \text{and} \quad s_{t0} \in G_{r_t}, \quad \forall t = 1,\ 2,\ ...,\ m,$$

so

$$0 \neq a_{s_{t0}} \in R_{G_{r_t}}, \quad \forall t = 1,\ 2,\ ...,\ m.$$

Thus

$$R_{G_{r_t}} \neq \{0\}, \quad \forall t = 1,\ 2,\ ...,\ m.$$

If k is an element of I such that $R_{G_{r_k}} \neq \{0\}$, then there exists in G_k an element y such that $R_y \neq \{0\}$, so if b_y is an element of $R_y - \{0\}$ and e is the neutral of G_k, we have

$$b_y = b_y \cdot 1 = \sum_{t=1}^{m} (b_y a_{s_{t0}} + b_y a_{st_1} + \cdots + b_y a_{s_{tn_t}}).$$

Let s_β be an element of the set

$$\{s_{10},\ ...,\ s_{1n_1},\ s_{20},\ ...,\ s_{2n_2},\ ...,\ s_{m0},\ ...,\ s_{mn_m}\} \backslash G_k$$

such that $b_y a_{s_\beta} \neq 0$. Then $b_y a_{s_\beta} \notin R_y$, because if $b_y a_{s_\beta} \in R_y$, then

$$0 \neq b_y a_{s_\beta} = a_{s_\beta} b_y \in R_y \Rightarrow y s_\beta = y \quad \text{and} \quad s_\beta y = y$$

$$(y^{-1} \text{ is the inverse of } y \text{ in } G_k)$$

$$\Rightarrow e s_\beta = e \text{ and } s_\beta e = e.$$

If we suppose that f is the neutral element of the maximal subgroup which s_β

belongs to, then

$$\left.\begin{array}{c} ef = (es_{\beta})f = e(s_{\beta}f) = es_{\beta} = e \\ \text{and} \\ fe = f(s_{\beta}e) = (fs_{\beta})e = s_{\beta}e = e \end{array}\right\} \Rightarrow ef = fe = e. \qquad (*)$$

Since S is completely simple semigroup, so by Lemma 1.4, we find that

$$e = f,$$

and this is a contradiction.

Since $b_y \neq \{0\}$, so there exists an element s_{α} in $\{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\} \cap G_k$ such that $b_y a_{s_{\alpha}} \in R_y - \{0\}$. Hence $ys_{\alpha} = y$. Thus

$$ys_{\alpha} = y \Rightarrow y^{-1}ys_{\alpha} = y^{-1}y \Rightarrow es_{\alpha} = e \Rightarrow s_{\alpha} = e.$$

It follows that the neutral element of G_k is an element of the set

$$\{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\} \cap G_k.$$

Thus G_k is one of the maximal subgroups

$$G_{r_1}, G_{r_2}, ..., G_{r_m},$$

so $G_{r_1}, G_{r_2}, ..., G_{r_m}$ are the all maximal subgroups of S, where

$$R_{G_{r_1}} \neq \{0\} \text{ and } R_{G_{r_2}} \neq \{0\} \text{ and ... and } R_{G_{r_m}} \neq \{0\}.$$

We also deduce that if s_{δ} is an element of $\{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\} \cap G_k$ such that $b_y a_{s_{\delta}} \in R_y - \{0\}$, then $s_{\delta} = e$, and since the elements of the set $\{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\} \cap G_k$ are all different, so

$$b_y = b_y a_e. \qquad (**)$$

Let us refer to the neutral element in G_{r_t} as e_t $(t = 1, 2, ..., m)$.

We see that a_{e_t} $(t = 1, 2, ..., m)$ is the identity of $R_{G_{r_t}}$, because on the one hand a_{e_t} is an element of $R_{G_{r_t}}$ and on the other hand

$$\forall b \in R_{G_{r_t}} \Rightarrow b = \sum_{g \in G_{r_t}} b_g \Rightarrow b = \sum_{g \in G_{r_t}} b_g a_{e_t} \text{ (that is from (**))}$$

$$\Rightarrow b = \left(\sum_{g \in G_{r_t}} b_g \right) a_{e_t} = b a_{e_t}.$$

Since e_t $(t = 1, 2, ..., m)$ is the neutral element in G_{r_t}, so $\{e_1, e_2, ..., e_m\} \subseteq \{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\}$, and since S is a completely regular semigroup so any idempotent of S is the neutral of a maximal subgroup of S, thus if e is an idempotent of $\operatorname{supp}(R, S)$, then $R_G \neq \{0\}$ such that G is the maximal subgroup that e belongs to, hence $e \in \{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\}$, and since $e_1, e_2, ..., e_m$ are all the idempotents in the set $\{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\}$ as well as also each one of them belongs to $\operatorname{supp}(R, S)$, so $e_1, e_2, ..., e_m$ are the all idempotents in $\operatorname{supp}(R, S)$.

Proposition 2.5. *Let R be a ring with an identity element 1, and S be a completely regular semigroup. Suppose that $R = \bigoplus_{s \in S} R_s$ is a graduation of R by S, and also $R = \bigoplus_{i \in I} R_{G_i}$ such that $\{G_i\}_{i \in I}$ is the family of all the maximal subgroups of S. If $R_{G_i} (i \in I)$ is an ideal in R, then we can find a subset $\{r_1, r_2, ..., r_m\}$ of I (m is a positive integer) such that $G_{r_1}, G_{r_2}, ..., G_{r_m}$ are the all maximal subgroups of S, where*

$$R_{G_{r_1}} \neq \{0\} \text{ and } R_{G_{r_2}} \neq \{0\} \text{ and ... and } R_{G_{r_m}} \neq \{0\},$$

and then we can write 1 in the form

$$1 = \sum_{t=1}^{m} a_{e_t}; \quad a_{e_t} \in R_{e_t} - \{0\}, \ \forall t = 1, 2, ..., m,$$

such that e_t $(t = 1, 2, ..., m)$ is the neutral of G_{r_t}. We also find that $e_1, e_2, ..., e_m$ are the all idempotents in $\operatorname{supp}(R, S)$ and a_{e_t} $(t = 1, 2, ..., m)$ is the identity element of the subring $R_{G_{r_t}}$.

Proof. Since $1 \in R = \bigoplus_{s \in S} R_s = \bigoplus_{i \in I} R_{G_i}$, so we can write 1 in only way in the form

$$1 = \sum_{t=1}^{m} (a_{s_{t0}} + a_{s_{t1}} + \ldots + a_{s_{tn_t}}); \quad a_{s_{tj}} \in R_{s_{tj}} - \{0\}, \quad \forall j = 0, 1, \ldots, n_t$$

such that $s_{t0}, s_{t1}, \ldots, s_{tn_t}$ $(t = 1, 2, \ldots, m)$ are different elements of G_{r_t} $(r_t \in I)$ (see the proof of the last proposition).

Since

$$a_{s_{t0}} \in R_{s_{t0}} - \{0\} \text{ and } s_{t0} \in G_{r_t}, \quad \forall t = 1, 2, \ldots, m,$$

so

$$0 \neq a_{s_{t0}} \in R_{G_{r_t}}, \quad \forall t = 1, 2, \ldots, m.$$

Hence

$$R_{G_{r_t}} \neq \{0\}, \quad \forall t = 1, 2, \ldots, m.$$

If k is an element of I such that $R_{G_{r_k}} \neq \{0\}$, then there exists in G_k an element y such that $R_y \neq \{0\}$, so if b_y is an element of $R_y - \{0\}$ and e is the neutral of G_k, then

$$b_y = b_y \cdot 1 = \sum_{t=1}^{m} (b_y a_{s_{t0}} + b_y a_{s_{t1}} + \cdots + b_y a_{s_{tn_t}}).$$

Since

$$R = \bigoplus_{i \in I} R_{G_i} \text{ and } R_{G_i} \ (\forall i \in I) \text{ is an ideal in } R,$$

therefore

$$R_{G_{i_1}} R_{G_{i_2}} = \{0\}, \quad \forall i_1, i_2 \in I, \ i_1 \neq i_2.$$

Hence

$$b_y a_{s_\beta} = 0, \quad \forall s_\beta \in \{s_{10}, \ldots, s_{1n_1}, s_{20}, \ldots, s_{2n_2}, \ldots, s_{m0}, \ldots, s_{mn_m}\} \backslash G_k.$$

Since $b_y \neq \{0\}$, so there exists an element s_α in $\{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\} \cap G_k$ such that $b_y a_{s_\alpha} \in R_y - \{0\}$. Hence $ys_\alpha = y$, thus $s_\alpha = e$, so the neutral element of G_k is an element of the set

$$\{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\} \cap G_k.$$

It follows that G_k is one of the maximal subgroups

$$G_{r_1}, G_{r_2}, ..., G_{r_m},$$

so $G_{r_1}, G_{r_2}, ..., G_{r_m}$ are the all maximal subgroups of S, where

$$R_{G_{r_1}} \neq \{0\} \text{ and } R_{G_{r_2}} \neq \{0\} \text{ and ... and } R_{G_{r_m}} \neq \{0\}.$$

Also if s_δ is an element of $\{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\} \cap G_k$ such that $b_y a_{s_\delta} \in R_y - \{0\}$, then $s_\delta = e$, and since the elements of the set $\{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\} \cap G_k$ are all different, so $b_y = b_y a_e$. By a similar argument, since $b_y = 1.b_y$, we can prove that

$$b_y = a_e b_y.$$

It follows that

$$b_y = a_e b_y = b_y a_e. \tag{$*$}$$

Denote the neutral element in G_{r_t} by e_t $(t = 1, 2, ..., m)$. Then

$$\{e_1, e_2, ..., e_m\} \subseteq \{s_{10}, ..., s_{1n_1}, s_{20}, ..., s_{2n_2}, ..., s_{m0}, ..., s_{mn_m}\}.$$

Suppose, for example, that

$$e_t = s_{td_t}, \quad \forall t = 1, 2, ..., m, \quad d_t \in \{0, 1, ..., n_t\}.$$

Then

$$a_{e_t} = a_{e_t} \cdot 1 = a_{e_t} a_{s_{10}} + \cdots + a_{e_t} a_{s_{1n_1}} + \cdots + a_{e_t} a_{s_{m0}} + \cdots + a_{e_t} a_{s_{mn_m}}$$

for all t in the set $\{1, 2, ..., m\}$.

Since $R_{G_{i_1}} R_{G_{i_2}} = \{0\}, \forall i_1, i_2 \in I, i_1 \neq i_2$, so

$$a_{e_t} = a_{e_t} a_{s_{t0}} + a_{e_t} a_{s_{t1}} + \cdots + a_{e_t} a_{s_{tn_t}} = \sum_{r=0}^{n_t} a_{e_t} a_{s_{tr}}$$

$$= a_{e_t} a_{s_{td_t}} + \sum_{\substack{r=0 \\ r \neq d_t}}^{n_t} a_{e_t} a_{s_{tr}}, \quad \forall t = 1, 2, \ldots, m.$$

And since

$$a_{e_t} a_{s_{tr}} \in R_{e_t} R_{s_{tr}} \subseteq R_{e_t s_{tr}} = R_{s_{tr}}, \quad \forall t = 1, 2, \ldots, m, \quad \forall r = 0, 1, \ldots, n_t$$

and

$$s_{t0}, s_{t1}, \ldots, s_{tn_t} \ (t = 1, 2, \ldots, m) \text{ are different elements of } G_{r_t},$$

we deduce that

$$a_{e_t} a_{s_{tr}} = 0, \quad \forall t = 1, 2, \ldots, m \text{ and } \forall r \in \{0, 1, \ldots, n_t\} \backslash \{d_t\}.$$

Thus by $(*)$ we find that

$$a_{s_{tr}} = 0, \quad \forall t = 1, 2, \ldots, m \text{ and } \forall r \in \{0, 1, \ldots, n_t\} \backslash \{d_t\}.$$

It follows that

$$\left.\begin{array}{l} 1 = \sum_{t=1}^{m} (a_{s_{t0}} + a_{s_{t1}} + \cdots + a_{s_{tn_t}}), \\ a_{s_{tr}} = 0, \quad \forall t = 1, 2, \ldots, m \text{ and } \forall r \in \{0, 1, \ldots, n_t\} \backslash \{d_t\}, \\ e_t = s_{td_t}, \quad \forall t = 1, 2, \ldots, m \end{array}\right\} \Rightarrow 1 = \sum_{t=1}^{m} a_{e_t}.$$

The proof that $e_1, e_2, \ldots, e_m$ are all the idempotents in $\operatorname{supp}(R, S)$ is similar to the proof that in the last proposition.

a_{e_t} $(t = 1, 2, \ldots, m)$ is the identity of $R_{G_{r_t}}$, because on the one hand a_{e_t} is an element of $R_{G_{r_t}}$ and on the other hand

$$\forall b \in R_{G_{r_t}} \Rightarrow b = \sum_{g \in G_{r_t}} b_g \Rightarrow \begin{cases} b = \sum_{g \in G_{r_t}} b_g a_{e_t} = b a_{e_t}, \\ b = \sum_{g \in G_{r_t}} a_{e_t} b_g = a_{e_t} b. \end{cases}$$

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