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STECHKIN-MARCHAUD-TYPE INEQUALITIES IN L_p FOR LINEAR COMBINATION OF BERNSTEIN-DURRMEYER OPERATORS

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Abstract

In this paper, we use the equivalence relation between K-functional and modulus of smoothness, and give the Stechkin-Marchaud-type inequalities for linear combination of Bernstein-Durrmeyer operators. Moreover, we obtain the inverse result of approximation for linear combination of Bernstein-Durrmeyer operators with $\omega_{\varphi^\lambda}^{2r}(f;\, x)$. Meanwhile we unify and extend some previous results.

1. Introduction and Main Results

Let $f \in L_p[0, 1]$, $(1 \leq p \leq \infty)$. Then the Bernstein-Durrmeyer operator $D_n(f;\, x)$ $(n \in \mathbb{N} :=$ set of naturals$)$ is defined as follows:

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$$D_n(f;\, x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_0^1 p_{n,k}(t) f(t)dt, \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

which was first introduced and investigated by Derrieinnic [1] in 1985. The linear combination of Bernstein-Durrmeyer operators is given by

$$O_{n,r}(f;\, x) = \sum_{i=0}^{2r-1} c_i(n) D_{n_i}(f;\, x), \tag{1.2}$$

where n_i and $c_i(n)$ satisfy:

(i) $n \le n_0 \le n_1 \le \cdots \le n_{2r-1} \le c_n$,

(ii) $\sum_{i=0}^{2r-1} c_i(n) = 1$,

(iii) $\sum_{i=0}^{2r-1} |c_i(n)| \le M$,

(iv) $\sum_{i=0}^{2r-1} c_i(n) D_{n_i}((t-x)^m;\, x) = 0, \quad m = 1, 2, \ldots, 2r-1.$ (1.3)

Ditzian and Ivanov [2], Zhou [3] and Guo and Li [4] studied the linear combination of Bernstein-Durrmeyer operators, and obtained the characterization of approximation, the relationship of differential and modulus of smoothness for $O_{n,r}(f;\, x)$.

In this paper, we first establish Bernstein-type inequality with parameter λ for $O_{n,r}(f;\, x)$. After that, we use the equivalence relation between K-functional and modulus of smoothness, and give the Stechkin-Marchaud-type inequalities in $L_p[0, 1]$ for linear combination of Bernstein-Durrmeyer operators. Moreover, we obtain the inverse result of approximation for linear combination of Bernstein-Durrmeyer operators with $\omega_{\varphi^\lambda}^{2r}(f;\, x)$. Meanwhile we unify and extend [2-4] results.

First, we introduce some useful definitions and notations.

Definition 1.1. Let $\varphi^2(x) = x(1-x)$, $0 \le \lambda \le 1$, $1 \le p \le \infty$.

The modulus of smoothness by

$$\omega_{\varphi^\lambda}^{2r}(f;t)_p = \sup_{0 \le h < t} \| \Delta_{h\varphi^\lambda}^{2r} f \|_p,$$

where

$$\Delta_h^r f(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k f(x + (r/2 - k)h), \quad \left[x - \frac{rh}{2}, x + \frac{rh}{2}\right] \subseteq [0, 1],$$

otherwise $\Delta_h^r f(x) = 0$.

The K-functional by

$$K_{\varphi^\lambda}^{2r}(f; t^{2r})_p = \inf_{g \in G}\{\| f - g \|_p + t^{2r} \| \varphi^{2r\lambda} g^{(2r)} \|_p\},$$

where

$$G = \{g \mid g \in L_p[0,1], g^{(2r-1)} \in A.C._{loc}, \varphi^{2r\lambda} g^{(2r)} \in L_p[0,1]\}.$$

By [5, p. 10-11], there exists $M > 0$ such that

$$M^{-1} K_{\varphi^\lambda}^{2r}(f; t^{2r})_p \le \omega_{\varphi^\lambda}^{2r}(f;t)_p \le M K_{\varphi^\lambda}^{2r}(f; t^{2r})_p.$$

We are now in a position to state our main results.

Theorem 1.1. *For* $f \in G$, $r \in \mathbb{N}$, $0 \le \lambda \le 1$, $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$, *we have the Stechkin-Marchaud inequality*

$$\omega_{\varphi^\lambda}^{2r}(f; n^{-\frac{r}{2}} \delta_n^{r(1-\lambda)}(x))_p \le M n^{-1} \sum_{k=1}^{n} \| O_{k,r}(f) - f \|_p. \tag{1.4}$$

Theorem 1.2. *Let* $f \in G$, $r \in \mathbb{N}$, $0 < \alpha < 2r$. *Then*

$$\| O_{n,r}(f) - f \|_p = O((n^{-\frac{1}{2}} \delta_n^{(1-\lambda)}(x))^\alpha) \Rightarrow \omega_{\varphi^\lambda}^{2r}(f;t)_p = O(t^\alpha). \tag{1.5}$$

Remark 1.1. For the inverse result, it is obvious that the result of [2] is a special case of (1.5) with $\lambda = 1,$ the result of [3] is a special case of (1.5) with $\lambda = 0,$ $p = \infty,$ and the result of [4] is a special case of (1.5) with $p = \infty.$

Throughout this paper, M denotes a positive constant independent of x, y, n and f which may be different in different places.

2. Auxiliary Lemmas

To prove the theorems, we need also the following lemmas:

Lemma 2.1. *If* $c < \frac{1}{2}, \; d < \frac{1}{2},$ *then*

$$\int_0^1 p_{n,k}(t) t^{-c}(1-t)^{-d}\, dt \le M n^{-1}\left(\frac{k+1}{n}\right)^{-c}\left(1 - \frac{k-1}{n}\right)^{-d}. \tag{2.1}$$

Proof. We notice [5, p. 164]

$$\int_0^1 p_{n,k}(t) t^{\eta} dt \le M n^{-1}\left(\frac{k+1}{n}\right)^{\eta}, \quad \eta > -1,$$

$$\int_0^1 p_{n,k}(t)(1-t)^{\zeta} dt \le M n^{-1}\left(1 - \frac{k-1}{n}\right)^{\zeta}, \quad \zeta > -1.$$

Using Hölder's inequality, we have

$$\int_0^1 p_{n,k}(t) t^{-c}(1-t)^{-d}\, dt \le \left(\int_0^1 p_{n,k}(t) t^{-2c} dt\right)^{\frac{1}{2}}\left(\int_0^1 p_{n,k}(t)(1-t)^{-2d}\, dt\right)^{\frac{1}{2}}$$

$$\le M n^{-1}\left(\frac{k+1}{n}\right)^{-c}\left(1 - \frac{k-1}{n}\right)^{-d}.$$

Lemma 2.1 has been proved. □

Lemma 2.2. *If* $c \ge 0, \; d \ge 0, \; x > 0,$ *then*

$$\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k+1}{n}\right)^{-c}\left(1 - \frac{k-1}{n}\right)^{-d} \le M x^{-c}(1-x)^{-d}. \tag{2.2}$$

Proof. We notice [5, p. 164]

$$\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{n}{k+1}\right)^{l} \le M x^{-l}, \quad \text{for} \quad l \in \mathbb{N},$$

$$\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{n}{n-k+1}\right)^{\zeta} \leq M(1-x)^{-\zeta}, \quad \text{for} \quad \zeta \in \mathbb{N}.$$

For $c = 0,\ d = 0,$ the result of (2.2) is obvious. For $c > 0,\ d > 0,$ using Hölder's inequality, we have

$$\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k+1}{n}\right)^{-c}\left(1-\frac{k-1}{n}\right)^{-d}$$

$$\leq \left(\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k+1}{n}\right)^{-2c}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n} p_{n,k}(x)\left(1-\frac{k-1}{n}\right)^{-2d}\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{n}{k+1}\right)^{[2c]+1}\right)^{\frac{c}{[2c]+1}}\left(\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{n}{n-k+1}\right)^{[2d]+1}\right)^{\frac{2d}{[2d]+1}}$$

$$\leq M\left(x^{-([2c]+1)}\right)^{\frac{c}{[2c]+1}}\left((1-x)^{-([2d]+1)}\right)^{\frac{d}{[2d]+1}} \leq Mx^{-c}(1-x)^{-d}.$$

For $c > 0,\ d = 0$ or $c = 0,\ d > 0,$ the proof is similar. Thus, this proof is completed. $\square$

Lemma 2.3. *For* $f \in L_p[0, 1],\ r \in \mathbb{N},\ 0 \leq \lambda \leq 1,\ \delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}},\ n \geq 2r,$ *we have the Bernstein-type inequality*

$$\| \varphi^{2r\lambda} O_{n,r}^{(2r)}(f) \|_p \leq Mn^r \delta_n^{2r(\lambda-1)}(x) \| f \|_p. \tag{2.3}$$

Proof. For $p = 1,$ if $x \in E_n = \left[\frac{1}{n}, 1-\frac{1}{n}\right],\ \varphi^{-\xi}(x) \leq n^{\frac{\xi}{2}},\ \xi > 0,$ then by simple computation, we have

$$D_n^{(2r)}(f;\ x) = (x(1-x))^{-2r}\sum_{i=0}^{2r} Q_i(x,\ n)n^i$$

$$\times \sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n}-x\right)^{i}(n+1)\int_0^1 p_{n,k}(u)\, f(u)du \tag{2.4}$$

with $Q_i(x, n)$ is a polynomial in $nx(1-x)$ of degree $[(2r-i)/2]$ with nonconstant bounded coefficients. Therefore,

$$|Q_i(x, n)n^i| \le M(x(1-x))^{r-\frac{i}{2}} n^{r+\frac{i}{2}}, \quad x \in E_n.$$

Thus,

$$\begin{aligned} &|\varphi^{2r\lambda}(x)D_n^{(2r)}(f; x)| \\ &\le Mn^{r(2-\lambda)}\left|\sum_{i=0}^{2r} n^{\frac{i}{2}}\varphi^{-i}(x)\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n}-x\right)^i (n+1)\int_0^1 p_{n,k}(u) f(u)du\right|. \end{aligned} \tag{2.5}$$

Note that [5, p. 129]

$$\int_{E_n} \varphi^{-2m}(x)\, p_{n,k}(x)\left(\frac{k}{n}-x\right)^{2m} dx \le Mn^{-m-1},$$

we can write

$$\begin{aligned} &\| \varphi^{2r\lambda} D_n^{(2r)}(f) \|_{1(E_n)} \\ &\le Mn^{r(2-\lambda)}\left|\sum_{i=0}^{2r} n^{\frac{i}{2}} \sum_{k=0}^{n} \int_{E_n} \varphi^{-i}(x)\, p_{n,k}(x)\left(\frac{k}{n}-x\right)^i dx (n+1)\int_0^1 p_{n,k}(u) f(u)du\right| \\ &\le Mn^{r(2-\lambda)}\sum_{k=0}^{n}\int_0^1 p_{n,k}(u)|f(u)|du \le Mn^{r(2-\lambda)}\| f \|_1. \end{aligned} \tag{2.6}$$

If $x \in E_n^c = \left[0, \frac{1}{n}\right) \cup \left(1-\frac{1}{n}, 1\right]$, then $\frac{n!}{(n-2r)!} \sim n^{2r}$, $\| \varphi^{2r\lambda} \|_\infty \sim n^{-r\lambda}$, $\int_0^1 p_{n,k}(x)dx = \frac{1}{n+1}$. By simple calculation, we have

$$\begin{aligned} &D_n^{(2r)}(f; x) \\ &= \frac{n!}{(n-2r)!}\sum_{k=0}^{n-2r} p_{n-2r,k}(x)(n+1)\int_0^1 \sum_{j=0}^{2r}(-1)^j \binom{2r}{j} p_{n,k+j}(u) f(u)du, \end{aligned} \tag{2.7}$$

we can write

$$\|\varphi^{2r\lambda}D_n^{(2r)}(f)\|_{1(E_n^c)}$$

$$\le Mn^{r(2-\lambda)}\sum_{k=0}^{n-2r}\int_0^1 p_{n-2r,k}(x)dx\sum_{j=0}^{2r}\binom{2r}{j}(n+1)\int_0^1 p_{n,k+j}(u)|f(u)|du$$

$$\le Mn^{r(2-\lambda)}\sum_{j=0}^{2r}\binom{2r}{j}\sum_{k=0}^{n-2r}\int_0^1 p_{n,k+j}(u)|f(u)|du \le Mn^{r(2-\lambda)}\|f\|_1. \tag{2.8}$$

For $p=\infty$, if $x\in E_n$, then by (2.5), we can now write

$$|\varphi^{2r\lambda}(x)D_n^{(2r)}(f;x)|$$

$$\le Mn^{r(2-\lambda)}\left|\sum_{i=0}^{2r}n^{\frac{i}{2}}\varphi^{-i}(x)\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k}{n}-x\right)^i(n+1)\int_0^1 p_{n,k}(u)f(u)du\right|$$

$$\le Mn^{r(2-\lambda)}\|f\|_\infty\sum_{i=1}^{2r}n^{\frac{i}{2}}\varphi^{-i}(x)\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k}{n}-x\right)^i(n+1)\int_0^1 p_{n,k}(u)du$$

$$\le Mn^{r(2-\lambda)}\|f\|_\infty. \tag{2.9}$$

If $x\in E_n^c$, then by (2.7), the proof is similar to that (2.9), it is enough to show that

$$|\varphi^{2r\lambda}(x)D_n^{(2r)}(f;x)|\le Mn^{r(2-\lambda)}\|f\|_\infty. \tag{2.10}$$

By (2.6), (2.8), (2.9) and (2.10), applying Riesz-Thorin theorem, we get

$$\|\varphi^{2r\lambda}D_n^{(2r)}(f)\|_p\le Mn^{r(2-\lambda)}\|f\|_p\le Mn^r\delta_n^{2r(\lambda-1)}(x)\|f\|_p.$$

Combining (iii) of (1.3), we obtain

$$\|\varphi^{2r\lambda}O_{n,r}^{(2r)}(f)\|_p\le Mn^r\delta_n^{2r(\lambda-1)}(x)\|f\|_p.$$

Lemma 2.3 has been proved. □

Lemma 2.4. *If* $f\in G$, $r\in\mathbb{N}$, $0\le\lambda\le 1$, $n>2r$, *then*

$$\|\varphi^{2r\lambda}O_{n,r}^{(2r)}(f)\|_p\le M\|\varphi^{2r\lambda}f^{(2r)}\|_p. \tag{2.11}$$

Proof. By calculation, we have

$$D_n^{(2r)}(f;\,x) = (n+1)\sum_{k=0}^{n-2r} p_{n-2r,\,k}(x)\frac{(n!)^2(n-k-2r)!}{((n-2r)!)^2(k+2r)!(n-k)!}$$
$$\times\int_0^1 p_{n-2r,\,k}(u)\varphi^{4r}(u)f^{(2r)}(u)du. \tag{2.12}$$

For $p = 1$, by Lemmas 2.1 and 2.2, we can write

$$\|\varphi^{2r\lambda}D_n^{(2r)}(f)\|_1$$
$$\le M(n+1)\sum_{k=0}^{n-2r}\int_0^1 p_{n-2r,\,k}(x)\varphi^{2r\lambda}(x)dx\frac{(n!)^2(n-k-2r)!}{((n-2r)!)^2(k+2r)!(n-k)!}$$
$$\times\int_0^1 p_{n-2r,\,k}(u)\varphi^{4r}(u)|f^{(2r)}(u)|du$$
$$\le M\sum_{k=0}^{n-2r}\left(\frac{k+1}{n-2r}\right)^{r\lambda}\left(\frac{n-2r-k+1}{n-2r}\right)^{r\lambda}\frac{(n!)^2(n-k-2r)!}{((n-2r)!)^2(k+2r)!(n-k)!}$$
$$\times\int_0^1 p_{n-2r,\,k}(u)\varphi^{4r}(u)|f^{(2r)}(u)|du$$
$$\le M\int_0^1\sum_{k=0}^{n-2r}p_{n-2r,\,k}(u)\left(\frac{k+1}{n-2r}\right)^{r(\lambda-2)}\left(\frac{n-2r-k+1}{n-2r}\right)^{r(\lambda-2)}\varphi^{4r}(u)|f^{(2r)}(u)|du$$
$$\le M\int_0^1\varphi^{2r\lambda}(u)|f^{(2r)}(u)|du = M\|\varphi^{2r\lambda}f^{(2r)}\|_1. \tag{2.13}$$

For $p = \infty$, by (2.12) and Lemmas 2.1 and 2.2, using the method similar to that (2.9) and (2.13), it is enough to show that

$$|\varphi^{2r\lambda}(x)D_n^{(2r)}(f;\,x)| \le M\|\varphi^{2r\lambda}f^{(2r)}\|_\infty,$$

which implies

$$\|\varphi^{2r\lambda}D_n^{(2r)}(f)\|_\infty \le M\|\varphi^{2r\lambda}f^{(2r)}\|_\infty. \tag{2.14}$$

By (2.13) and (2.14), using Riesz-Thorin theorem, we get

$$\| \varphi^{2r\lambda} D_n^{(2r)}(f) \|_p \leq M \| \varphi^{2r\lambda} f^{(2r)} \|_p .$$

Combining (iii) of (1.3), we obtain

$$\| \varphi^{2r\lambda} O_{n,r}^{(2r)}(f) \|_p \leq M \| \varphi^{2r\lambda} f^{(2r)} \|_p ,$$

which completes the proof. □

Lemma 2.5. *If* $f \in G,\ \ r \in \mathbb{N},\ \ 0 \leq \lambda \leq 1,$ *then*

$$\| \varphi^{2r\lambda} O_{n,r}^{(2r)}(f) \|_p \leq M n^{r-1} \delta_n^{2r(\lambda-1)}(x) \sum_{k=1}^{n} \| O_{k,r}(f) - f \|_p . \tag{2.15}$$

Proof. By Lemmas 2.3 and 2.4, noting that $O_{1,r}^{(2r)}(f;\ x) = 0,$ we have

$$\begin{aligned} & n^{-r} \| \varphi^{2r\lambda} O_{n,r}^{(2r)}(f) \|_p \\ & \leq n^{-r} \| \varphi^{2r\lambda} O_{n,r}^{(2r)}(O_{k,r}(f)) \|_p + n^{-r} \| \varphi^{2r\lambda} O_{n,r}^{(2r)}(O_{k,r}(f) - f) \|_p \\ & \leq M_2 n^{-r} \| \varphi^{2r\lambda} O_{k,r}^{(2r)}(f) \|_p + M_1 \delta_n^{2r(\lambda-1)}(x) \| O_{k,r}(f) - f \|_p . \end{aligned} \tag{2.16}$$

We write $\| O_{q,r}(f) - f \|_p = \max_{1 \leq k \leq n} \| O_{k,r}(f) - f \|_p .$ For $\| O_{q,r}(f) - f \|_p ,$ $1 \leq k \leq n,$ $\| O_{k,r}(f) - f \|_p \neq 0,$ there exists $M_3 > 0,$ such that $\| O_{q,r}(f) - f \|_p \leq$ $M_3 \| O_{k,r}(f) - f \|_p .$ Therefore

$$\begin{aligned} & M_2 n^{-r} \| \varphi^{2r\lambda} O_{k,r}^{(2r)}(f) \|_p \\ & \leq M_2 n^{-r} \| \varphi^{2r\lambda} O_{k,r}^{(2r)}(O_{1,r}(f) - f) \|_p + M_2 n^{-r} \| \varphi^{2r\lambda} O_{k,r}^{(2r)}(O_{1,r}(f)) \|_p \\ & \leq M_1 M_2 \delta_k^{2r(\lambda-1)}(x) \| O_{1,r}(f) - f \|_p + M_2^2 \delta_k^{2r(\lambda-1)} \| O_{1,r}^{(2r)}(f) \|_p \\ & \leq M_1 M_2 \delta_k^{2r(\lambda-1)}(x) \| O_{q,r}(f) - f \|_p \\ & \leq M_1 M_2 M_3 \delta_k^{2r(\lambda-1)}(x) \| O_{k,r}(f) - f \|_p . \end{aligned} \tag{2.17}$$

Noting that $\delta_k^{2r(\lambda-1)}(x) \le \delta_n^{2r(\lambda-1)}(x)$, by (2.16) and (2.17), we have

$$\| \varphi^{2r\lambda} O_{n,r}^{(2r)}(f) \|_p \le M n^{r-1} \delta_n^{2r(\lambda-1)}(x) \sum_{k=1}^{n} \| O_{k,r}(f) - f \|_p,$$

where $M = M_1 + M_1 M_2 M_3$. Lemma 2.5 has been proved. □

3. Proofs of Theorems

Proof of Theorem 1.1. For $n > 2$, there exists $m \in \mathbb{N}$, such that $\frac{n}{2} \le m \le n$, and

$$\| O_{m,r}(f) - f \|_p = \min_{\frac{n}{2} \le k \le n} \| O_{k,r}(f) - f \|_p,$$

$$\| O_{m,r}(f) - f \|_p \le 2n^{-1} \sum_{\frac{n}{2} \le k \le n} \| O_{k,r}(f) - f \|_p.$$

Therefore, using the definition of $K_{\varphi^\lambda}^{2r}(f; x)$, and Lemma 2.5, noting that $\delta_m^{2r(\lambda-1)}(x) \le \delta_n^{2r(\lambda-1)}(x)$, we have

$$\begin{aligned}
& K_{\varphi^\lambda}^{2r}(f; n^{-r} \delta_n^{2r(1-\lambda)}(x))_p \\
& \le \| O_{m,r}(f) - f \|_p + n^{-r} \delta_n^{2r(1-\lambda)}(x) \| \varphi^{2r\lambda} O_{m,r}^{(2r)}(f) \|_p \\
& \le 2n^{-1} \sum_{\frac{n}{2} \le k \le n} \| O_{k,r}(f) - f \|_p \\
& \quad + M n^{-r} \delta_n^{2r(1-\lambda)}(x) m^{r-1} \delta_m^{2r(\lambda-1)}(x) \\
& \quad \times \sum_{k=1}^{m} \| O_{k,r}(f) - f \|_p \\
& \le M n^{-1} \sum_{k=1}^{n} \| O_{k,r}(f) - f \|_p.
\end{aligned}$$

By relationship of K-functional and modulus of smoothness, we get

$$\omega_{\varphi^\lambda}^{2r}(f;\, n^{-\frac{r}{2}}\delta_n^{r(1-\lambda)}(x))_p \le Mn^{-1}\sum_{k=1}^{n}\| O_{k,r}(f) - f\|_p.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. By $\| O_{n,r}(f) - f\|_p \le M(n^{-\frac{1}{2}}\delta_n^{(1-\lambda)}(x))^\alpha$, according to the definition of $K_{\varphi^\lambda}^{2r}(f;\, t^{2r})$, we have

$$K_{\varphi^\lambda}^{2r}(f;\, t^{2r})_p$$

$$\le \| f - O_{n,r}(f)\|_p + t^{2r}\|\varphi^{2r\lambda}O_{n,r}^{(2r)}(f)\|_p$$

$$\le M[(n^{-\frac{1}{2}}\delta_n^{(1-\lambda)}(x))^\alpha + t^{2r}(\|\varphi^{2r\lambda}O_{n,r}^{(2r)}(f-g)\|_p + \|\varphi^{2r\lambda}O_{n,r}^{(2r)}(g)\|_p)]$$

$$\le M[(n^{-\frac{1}{2}}\delta_n^{(1-\lambda)}(x))^\alpha + t^{2r}(n^r\delta_n^{2r(\lambda-1)}(x)\| f - g\|_p + \|\varphi^{2r\lambda}g^{(2r)}\|_p)]$$

$$\le M\left((n^{-\frac{1}{2}}\delta_n^{(1-\lambda)}(x))^\alpha + \frac{t^{2r}}{n^{-r}\delta_n^{2r(1-\lambda)}}K_{\varphi^\lambda}^{2r}(f;\, n^{-r}\varphi^{2r(1-\lambda)})_p\right).$$

By Berens-Lorens theorem, and relationship of K-functional and modulus of smoothness, we have

$$\omega_{\varphi^\lambda}^{2r}(f;\, t)_p \le Mt^\alpha.$$

This completes the proof of Theorem 1.2. □

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