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WHICH ROLE PLAYS THE CONDITION OF CONTINUITY (C) IN THE REPRESENTATION OF INVOLUTIVE θ-VALUED M-£ ALGEBRA

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Abstract

In this paper, we show that in an involutive θ-valued £ukasiewicz-Moisil algebra $(£, J, (\varphi_\alpha)_{\alpha\in J^0}, (\psi_\alpha)_{\alpha\in J^1}, n, N)$ monomorphic with an involutive fuzzy algebra $\widetilde{P}(E)$, the condition of continuity (for each $\alpha \in J^0$, $\underset{\beta<\alpha}{\wedge} \psi_\beta(x) = \varphi_\alpha(x)$) ...(C) is necessary.

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1. Introduction

Since the inception of fuzzy set theory [4, 10, 15, 18], various mathematicians became interested in the problem "To what are the relationships existing between the θ-valued £ukasiewicz-Moisil algebras ($£ - M_\theta$ algebra for short) and fuzzy algebras". In particular, those working in the representation of involutive $£ - M_\theta$ algebra by means of fuzzy algebra [1, 8, 9, 15] (see also [16]), were facing two important results:

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(1) Every involutive θ-valued fuzzy algebra $(\widetilde{P}(U), J, (N_\alpha)_{\alpha\in J^0}, (N'_\alpha)_{\alpha\in J^1}, n, N')$ is an involutive $£ - M_\theta$ algebra.

(2) In [9], Coulon and Coulon gave a condition (C) that guarantees that an involutive $£ - M_\theta$ algebra $(£, J, (\varphi_\alpha)_{\alpha\in J^0}, (\psi_\alpha)_{\alpha\in J^1}, n, N)$ is embeddable in an involutive fuzzy algebra $(\widetilde{P}(U), K, (N_\alpha)_{\alpha\in K^0}, (N'_\alpha)_{\alpha\in K^1}, m, N')$, where U is the set of prime filters of £ and K is the MacNeille completion of J. They also showed that this condition (C) is satisfied by all involutive fuzzy algebras, and that condition (C) is also necessary for the embedding in case J is a dense chain.

Our aim in this paper is to show that condition (C) is always necessary. More precisely, we shall prove the following:

Theorem 1.1. *Every involutive* $£ - M_\theta$ *algebra* $(£, J, (\varphi_\alpha)_{\alpha\in J^0}, (\psi_\alpha)_{\alpha\in J^1}, n, N)$ *embedded in an involutive fuzzy algebra satisfies the condition of continuity* (C).

This paper is organized as follows: In Section 1, we introduce the necessary background material, and we prove Theorem 1.1 in Section 2.

2. Notation and Preliminaries

In order to keep the paper brief, we refer the reader to [1, 3, 9, 13, 15] for results on $£ - M_\theta$ algebras and fuzzy algebras, and to [4, 10, 18] for results on fuzzy algebras. We note by $J^0 = J - \{0\}$, $J^1 = J - \{1\}$, $J^{01} = J - \{0, 1\}$.

2.1. £ukasiewicz-Moisil algebra

We start by recalling the definition of involutive $£ - M_\theta$ algebra as a system $(£, J, (\varphi_\alpha)_{\alpha\in J^0}, (\psi_\alpha)_{\alpha\in J^1}, n, N)$, where

(1) $(£, \vee, \wedge, 0, 1)$ is a distributive lattice with the smallest element 0 and the greatest element 1.

The Boolean sublattice of complemented elements of £ is denoted by C(£).

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(2) J is a chain with the smallest element 0 and the greatest element 1 and whose order type is θ.

(3) n is an order reversing involution in J.

(4) N is an order reversing involution in £ such that: $N(x) = \overline{x}$, $\forall x \in \mathrm{C}(£)$.

(5) $(\varphi_\alpha)_{\alpha \in J^0}$ is a family of morphisms $£ \to \mathrm{C}(£)$ such that: $\forall \alpha, \beta \in J^0$,

(i) $\varphi_\alpha(0) = 0$, $\varphi_\alpha(1) = 1$.

(ii) If $\alpha \leq \beta$, then $\varphi_\beta \leq \varphi_\alpha$.

(iii) $\varphi_\alpha \circ \varphi_\beta = \varphi_\beta$.

(iv) If $\varphi_\alpha(x) = \varphi_\alpha(y)$ for all $\alpha \in J^0$, then $x = y$ (Moisil's determination principle).

(6) $(\psi_\alpha)_{\alpha \in J^1}$ is a family of morphisms $£ \to \mathrm{C}(£)$ such that:

(i) $\psi_\alpha \leq \varphi_\alpha$, for all $\alpha \in J^{01}$,

(ii) $\varphi_\alpha N = N\psi_{n\alpha}$, for all $\alpha \in J^0$.

(We use the same symbols $\leq$, 0, 1 for the partial order, the least and the greatest elements of J, respectively, and for those of £.)

A list of useful results from [13] is given in the following proposition:

Proposition 2.1.1. *In every involutive* $£ - M_\theta$ *algebra, the following properties hold*:

(i) *for all* $\alpha, \beta \in J^1$, *if* $\alpha \leq \beta$, *then* $\psi_\beta \leq \psi_\alpha$,

(ii) *for all* $\alpha, \beta \in J^1$, $\psi_\alpha \circ \psi_\beta = \Psi_\beta$,

(iii) *for all* $\alpha \in J^0$, *if* α *has a predecessor* α^-, *then* $\psi_{\alpha^-} = \varphi_\alpha$.

Definition 2.1.2. Two involutive $£ - M_\theta$ algebras $(£, J, (\varphi_\alpha)_{\alpha \in J^0}, (\psi_\alpha)_{\alpha \in J^1}, n, N)$ and $(£', J, (\varphi'_\alpha)_{\alpha \in J^0}, (\psi'_\alpha)_{\alpha \in J^1}, n, N')$ are said to be

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homomorphic if there exists a morphism $f : \pounds \to \pounds'$ such that:

(i) $f \circ \varphi_{\alpha} = \varphi'_{\alpha} \circ f$, for all $\alpha \in J^0$,

(ii) $f \circ N = N' \circ f$.

If f is a monomorphism, then we say that £ can be *embedded* into £'.

We note that (i) and (ii) imply: $f \circ \psi_{\alpha} = \psi'_{\alpha} \circ f$ for all $\alpha \in J^1$.

2.2. Fuzzy algebra and related topics

The clue of fuzzy set is to replace the binary $\{0, 1\}$-range of traditional indicator functions with the continuous I-range (I is a chain with the smallest element 0 and the greatest element 1 and whose order type is θ). A fuzzy set can be defined in the mathematical terms as follows. If U the universe of discourse, is an ordinary set, and $x \in U$ is a generic element of U, then a fuzzy set A on U is a function $A : U \to I$.

The value $A(x)$ is called the *membership degree* of $x \in U$ in A, and is commonly denoted by the notation $\mu_A(x)$.

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2.2.1. Basic definitions

A fuzzy set A defined in the universe of discourse U is empty if and only if $\mu_A(x) = 0, \forall x \in U$.

If A and B are two fuzzy sets in the universe of discourse U, then A is a fuzzy subset of B if and only if

$$\mu_A(x) \leq \mu_B(x), \quad \forall x \in U,$$

A and B are equal fuzzy sets if and only if

$$\mu_A(x) = \mu_B(x), \quad \forall x \in U.$$

The union of A and B is a fuzzy set such that

$$\mu_{A \cup B}(x) = \mu_A(x) \vee \mu_B(x), \quad \forall x \in U.$$

The intersection of A and B is a fuzzy set such that

$$\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x), \quad \forall x \in U.$$

The collection $(\widetilde{P}(U), \cup, \cap)$ of all fuzzy subsets is a complete and distributive lattice with the smallest element ϕ and the greatest element U.

If I is equipped with an order-reversing involution m, then the fuzzy complementation is the order-reversing involution N' on $\widetilde{P}(U)$ defined, for all $A \in \widetilde{P}(U)$ by

$$\mu_{N'A}(\cdot) = m\mu_A(\cdot).$$

The weak α-cut is the mapping $N_\alpha : \widetilde{P}(U) \to P(U)$ defined, for all $\alpha \in I^0$ by

$$N_\alpha(A) = \{x \in U : \mu_A(x) \geq \alpha\}.$$

The strong α-cut is the mapping $N'_\alpha : \widetilde{P}(U) \to P(U)$ defined, for all $\alpha \in I^1$ by

$$N'_\alpha(A) = \{x \in U : \mu_A(x) > \alpha\}.$$

For any sublattice L of $\widetilde{P}(U)$, containing ϕ and U, and closed under the fuzzy complementation N', under the weak α-cut N_α, for all $\alpha \in I^0$, and under the strong α-cut N'_α, for all $\alpha \in I^1$, the 6-tuple $(L, I, (N_\alpha)_{\alpha \in I^0}, (N'_\alpha)_{\alpha \in I^1}, m, N')$ is called, if $\overline{\theta}$ is the order type of I, an *involutive* $\overline{\theta}$*-valued fuzzy algebra*.

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We recall some results that we shall need in what follows.

Lemma 2.2.2. (i) *For all* $\alpha \in J^{01}$, $N'_\alpha = mN_{n\alpha}m$.

(ii) *For all* $\alpha, \beta \in J^{01}$, $N_\alpha = \bigcap_{\beta<\alpha} N'_\beta$.

3. Proof of Theorem 1.1

Let $(£, J, (\varphi_\alpha)_{\alpha \in J^0}, (\psi_\alpha)_{\alpha \in J^1}, n, N)$ be an involutive $£ - M_\theta$ algebra which is embedded in the involutive $\overline{\theta}$-valued fuzzy algebra $(\widetilde{P}(U), K, (N_\alpha)_{\alpha \in K^0}, (N'_\alpha)_{\alpha \in K^1}, m, N')$, and let f be the monomorphism from £ into $\widetilde{P}(U)$.

On the other hand, J can be embedded in K, and we are going to identify it by its image in K. Observe that from Definition 2.1.2 and Lemma 2.2.2(ii) it follows that

$$f(\varphi_{\alpha}(x)) = N_{\alpha}(f(x)) = \bigcap_{\substack{\beta \in K \\ \beta < \alpha}} N'_{\beta}(f(x)).$$

It is easy to verify that $\bigcap_{\substack{\beta \in K \\ \beta < \alpha}} N'_{\beta}(f(x)) \subseteq \bigcap_{\substack{\beta \in J \\ \beta < \alpha}} N'_{\beta}(f(x))$.

This implies that $f(\varphi_{\alpha}(x)) \subseteq N'_{\beta}(f(x))$, for all $\beta < \alpha$.

Then $f(\varphi_{\alpha}(x)) \subseteq f(\psi_{\beta}(x))$, for all $\beta < \alpha$.

Hence $\varphi_{\alpha}(x) \leq \psi_{\beta}(x)$, for all $\beta < \alpha$.

This means that

$$\varphi_{\alpha}(x) \leq \bigwedge_{\beta < \alpha} \psi_{\beta}(x). \tag{3.1}$$

Passing now to the second inequality we can distinguish at least two extremely different cases. However, before doing this we need the following:

Definition 3.1. A chain J is *dense* if for any $\alpha < \beta$ there exists a c such that: $\alpha < c < \beta$ (a non-dense chain is defined *dually*).

In fact, we have the following remark:

Remark 3.2. If there is no dense set $D \subseteq J$ (such that: $\alpha \in D$), then α has a predecessor α^{-}.

In the first case, if J is a dense chain, then $\bigvee_{\beta < \alpha} \beta = \alpha$. Consequently by Proposition 2.1.1(ii) we obtain that: $\bigwedge_{\beta < \alpha} \psi_{\beta}(x) = \psi_{\bigvee_{\beta < \alpha} \beta}(x) = \psi_{\alpha}(x)$. This means that

$$\bigwedge_{\beta < \alpha} \psi_{\beta}(x) \leq \varphi_{\alpha}(x). \tag{3.2}$$

We note that (3.1) and (3.2) imply $\bigwedge_{\beta < \alpha} \psi_{\beta}(x) = \varphi_{\alpha}(x)$ ...(C).

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In the second case, let J be a non-dense chain. Then if there exists a dense set $D \subseteq J$ (such that $\alpha \in D$), then we obtain the same result as above. Else by Remark 3.2, α has a predecessor α^-, and consequently $\bigwedge_{\beta<\alpha} \psi_\beta(x) = \psi_{\alpha^-}(x)$. Therefore by Proposition 2.1.1(iii), we obtain that $\bigwedge_{\beta<\alpha} \psi_\beta(x) = \varphi_\alpha(x)$ ...(C).

Hence, our proof is complete.

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