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GALOIS MODULE STRUCTURE OF A FAMILY OF GENERALIZED JACOBIANS

FAUSTO JARQUÍN ZÁRATE and GABRIEL VILLA SALVADOR

Universidad Autónoma de la Ciudad de México
Academia de Matemáticas
Plantel San Lorenzo Tezonco
Prolongación San Isidro No. 151 Col. San Lorenzo
Iztapalapa, C. P. 09790, D. F., México
e-mail: fausto.jarquin@uacm.edu.mx

Centro de Investigación y de Estudios Avanzados del I.P.N.
Departamento de Control Automático
Apartado Postal 14-740, 07000 México
D. F., México
e-mail: gvilla@ctrl.cinvestav.mx

Abstract

Let ℓ be a prime number and L/K be an arbitrary finite Galois ℓ-extension of function fields of one variable with field of constants k, an algebraically closed field of characteristic $p \geq 0$. In the wildly ramified case, i.e., $p = \ell$, we obtain the Galois module structure of the incomplete generalized Jacobian $\mathscr{C}_{0\mathfrak{B}}(p)$ and of ${}_p\mathscr{C}_{0\mathfrak{B}}$, the elements of order dividing p of $\mathscr{C}_{0\mathfrak{B}}(p)$, associated with the modulus $\mathfrak{B}$ in L which is induced by a modulus $\mathfrak{A}$ in K, where $\mathfrak{A}$ not necessarily contains in its support all the prime divisors of K ramified in L. That is, we obtain explicitly the decomposition of $\mathscr{C}_{0\mathfrak{B}}(p)({}_p\mathscr{C}_{0\mathfrak{B}})$ as direct sum of indecomposable

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$\mathbb{Z}_p[G]$-modules ($\mathbb{F}_p[G]$-modules). For the tamely ramified case, i.e., $p \neq \ell$, when the modulus $\mathfrak{A}$ in K contains in its support all except one of the prime divisors of K ramified in L, we obtain explicitly the decomposition of $\mathscr{C}_{0\mathfrak{B}}(\ell)$ (and of ${}_{\ell}\mathscr{C}_{0\mathfrak{B}}$, the ℓ-part of $\mathscr{C}_{0\mathfrak{B}}(\ell)$) as direct sum of indecomposable $\mathbb{Z}_\ell[G]$-modules ($\mathbb{F}_\ell[G]$-modules).

1. Introduction

Let k be an algebraically closed field of characteristic $p \geq 0$, ℓ be a prime number, K/k be an algebraic function field of one variable with field of constants k, and L/K be a finite Galois ℓ-extension of function fields with Galois group $\mathrm{Gal}(L/K) = G$. The group G acts naturally on $\mathbb{J}_L(\ell)$, the ℓ-torsion of the Jacobian variety $\mathbb{J}_L$ associated to the function field L/k. By restriction, G acts on ${}_{\ell^m}\mathbb{J}_L$, the group of points of $\mathbb{J}_L$ of order dividing ℓ^m. Then the direct limit $\mathbb{J}_L(\ell) := \varinjlim_{m} {}_{\ell^m}\mathbb{J}_L = \bigcup_{m=1}^{\infty} {}_{\ell^m}\mathbb{J}_L$ has a natural $\mathbb{Z}_\ell[G]$-module structure, where $\mathbb{Z}_\ell$ denotes the ring of ℓ-adic integers and $\mathbb{Z}_\ell[G]$ denotes the group ring over $\mathbb{Z}_\ell$. It is well known that $\mathbb{J}_L(\ell)$ is naturally G-isomorphic to $\mathscr{C}_{0L}(\ell)$, the Sylow ℓ-subgroup of the group $\mathscr{C}_{0L}$ of divisor classes of degree zero of L.

In [2], it is proved that, as groups,

$$\mathscr{C}_{0L}(\ell) \cong \begin{cases} R^{\tau_L}, & \text{if } p = \ell, \\ R^{2g_L}, & \text{if } p \neq \ell, \end{cases}$$

where τ_L denotes the Hasse-Witt invariant of L and g_L denotes the genus of L, $R := \dfrac{\mathbb{Q}_\ell}{\mathbb{Z}_\ell}$ and $\mathbb{Q}_\ell$ denotes the field of ℓ-adic numbers.

The basic tool used successfully in the study of the Galois module structure of the usual Jacobian $\mathscr{C}_{0L}(\ell)$, that is, for finding the decomposition of $\mathscr{C}_{0L}(\ell)$ as direct sum of indecomposable $\mathbb{Z}_\ell[G]$-modules, in both wildly and tamely ramified cases, i.e., $\ell = p$ and $\ell \neq p$, respectively, has turned out to be the use of the generalized Jacobian variety $\mathscr{C}_{\mathfrak{N}}$, where the modulus $\mathfrak{N}$ in L is induced from a modulus $\mathfrak{M}$ in

K which contains in its support all prime divisors of K ramified in L, and the exact sequence of $\mathbb{Z}_\ell[G]$-modules

$$0 \to \mathfrak{R} \to \mathscr{C}_{0\mathfrak{N}}(\ell) \to \mathscr{C}_{0L}(\ell) \to 0, \tag{1}$$

where $\mathscr{C}_{0\mathfrak{N}}(\ell)$ denotes the Sylow ℓ-subgroup of $\mathscr{C}_{0\mathfrak{N}}$, the group of classes of divisors of degree 0 relatively prime to the modulus $\mathfrak{N}$ in the field L, and $\mathfrak{R}$ is the kernel of the natural map, which was characterized as $\mathbb{Z}_\ell[G]$-module by Villa-Salvador and Madan (see [12, Theorem 1, page 257]).

A difference between the cases $p = \ell$ and $p \neq \ell$, occurs in the Galois module structure of the generalized Jacobian $\mathscr{C}_{0\mathfrak{N}}(\ell)$. More specifically, in [11, Proposition 8], and in [13, Theorem 6], it is proved that, as $\mathbb{Z}_\ell[G]$-modules

$$\mathscr{C}_{0\mathfrak{N}}(\ell) \cong \begin{cases} R[G]^{\tau_K + t - 1,} & \text{if } p = \ell, \\ R[G]^{2g_K + t - 1 - d} \oplus S, & \text{if } p \neq \ell, \end{cases}$$

where t is the total number of prime divisors in K ramified in L, d denotes the minimum number of generators of G and S is an indecomposable $\mathbb{Z}_\ell[G]$-module such that, as groups, $S \cong R^{s_0}$ with $s_0 = |G|(d-1)+1$ and $|G|$ denotes the order of G. In [4], we obtain two explicit characterizations of ${}_\ell S$, the ℓ-part of the $\mathbb{Z}_\ell[G]$-module S.

In (1), the generalized Jacobian $\mathscr{C}_{0\mathfrak{N}}(\ell)$ is associated to the modulus $\mathfrak{N}$. What is the $\mathbb{Z}_\ell[G]$-module structure of $\mathscr{C}_{0\mathfrak{B}}(\ell)$, if the modulus $\mathfrak{B}$ in L not necessarily contains in its support all prime divisors of K ramified in L? In this direction, in [3, Theorem 4.12], we obtained explicitly the Galois module structure of the incomplete generalized Jacobian $\mathscr{C}_{0\mathfrak{B}}(\ell)$ in the tamely ramified cyclic case.

Our main goals in this paper are two. First, we obtain explicitly the Galois module structure of the incomplete generalized Jacobian $\mathscr{C}_{0\mathfrak{B}}(p)$ in the wildly ramified case. That is, for $p = \ell$ and L/K any finite Galois ℓ-extension, we obtain explicitly the decomposition of $\mathscr{C}_{0\mathfrak{B}}(p)$ as direct sum of indecomposable $\mathbb{Z}_\ell[G]$-modules. This is Theorem 3.1. The tools used to obtain the injective component and the non-injective sums of $\mathscr{C}_{0\mathfrak{B}}(p)$ are similar to those used in [5]. Second, in the tamely ramified case, if L/K is any finite Galois ℓ-extension and $\mathfrak{B}$ is a modulus in L induced from a modulus $\mathfrak{A}$ in K which contains in its support all except one of the prime divisors of K ramified in L, we obtain the Galois module structure of the incomplete generalized Jacobian $\mathscr{C}_{0\mathfrak{B}}(\ell)$, i.e., we obtain explicitly the decomposition of $\mathscr{C}_{0\mathfrak{B}}(\ell)$ as direct sum of indecomposable $\mathbb{Z}_\ell[G]$-modules. This is Theorem 4.4. Furthermore, we obtain the decomposition of ${}_\ell\mathscr{C}_{0\mathfrak{B}}$, the ℓ-part of $\mathscr{C}_{0\mathfrak{B}}(\ell)$, as direct sum of indecomposable $\mathbb{F}_\ell[G]$-modules, where $\mathbb{F}_\ell$ denotes the finite field with ℓ elements. This is Theorem 4.5. More precisely, in Section 4, we determine the non-injective component of $\mathscr{C}_{0\mathfrak{B}}(\ell)$ and of ${}_\ell\mathscr{C}_{0\mathfrak{B}}$. In Section 2, we collect several results that will be of use in the rest of the paper.

2. Notation and Preliminaries

In this section, we establish notations and auxiliary results which will be needed along the paper. Let ℓ be a prime number and L/K denote a finite Galois ℓ-extension of function fields of degree ℓ^n with Galois group $G = \mathrm{Gal}(L/K)$ and field of constants k, an algebraically closed field of characteristic $p \geq 0$. Let

$$\mathscr{P} = \{\wp_1, \wp_2, \ldots, \wp_s, \wp_{s+1}, \ldots, \wp_t\}$$

be the collection of the different prime divisors of K ramified in L. Let

$$\hat{\mathscr{P}} = \{\mathscr{Q}_j^{(i)} \mid i \in \{1, \ldots, t\},\ j \in \{1, \ldots, \ell^{n-n_i}\}\},$$

denote the set of prime divisors $\mathscr{Q}_j^{(i)}$ of L such that $\mathscr{Q}_j^{(i)}$ divides the prime divisor $\wp_i$, for $1 \leq j \leq \ell^{n-n_i}$, where ℓ^{n_i} denotes the ramification index of the prime divisor $\wp_i$. Let $\mathfrak{M}$ and $\mathfrak{A}$ be the moduli in K defined by

$$\mathfrak{M} = \prod_{i=1}^{t} \wp_i \quad \text{and} \quad \mathfrak{A} = \prod_{i=1}^{s} \wp_i.$$

Let $\mathfrak{N}$ and $\mathfrak{B}$ be the moduli in L induced by $\mathfrak{M}$ and $\mathfrak{A}$, respectively, i.e., $\mathfrak{N}$ and $\mathfrak{B}$ are the conorms of $\mathfrak{M}$ and $\mathfrak{A}$, respectively, given by

$$\mathfrak{N} = \prod_{\substack{\mathscr{Q} \mid \wp_i \\ 1 \leq i \leq t}} \mathscr{Q} \quad \text{and} \quad \mathfrak{B} = \prod_{\substack{\mathscr{Q} \mid \wp_i \\ 1 \leq i \leq s}} \mathscr{Q}.$$

We use the following notations:

$\mathbb{P}_L$ is the set of prime divisors of L.

$\mathscr{D}_{0L}$ is the group of divisors of degree zero of L.

P_L is the group of principal divisors of L.

$\mathscr{C}_{0L} = \dfrac{\mathscr{D}_{0L}}{P_L}$ is the group of classes of divisors of degree zero of L.

$\mathscr{D}_{\mathfrak{B}}(\mathscr{D}_{\mathfrak{N}})$ is the group of divisors of L relatively prime to $\mathfrak{B}$, $(\mathfrak{N})$.

$\mathscr{D}_{0\mathfrak{B}}(\mathscr{D}_{0\mathfrak{N}})$ is the group of divisors of degree zero relatively prime to $\mathfrak{B}$, $(\mathfrak{N})$.

$P_{\mathfrak{B}}(P_{\mathfrak{N}})$ is the group of principal divisors (α) such that $\alpha \equiv 1 \bmod \mathfrak{B}$, $(\mathfrak{N})$.

$\mathscr{C}_{0\mathfrak{B}} = \dfrac{\mathscr{D}_{0\mathfrak{B}}}{P_{\mathfrak{B}}}$ is the group of classes of divisors of degree zero associated the modulus $\mathfrak{B}$.

$\mathscr{C}_{0\mathfrak{N}} = \dfrac{\mathscr{D}_{0\mathfrak{N}}}{P_{\mathfrak{N}}}$ is the group of classes of divisors of degree zero associated the modulus $\mathfrak{N}$.

The Sylow ℓ-subgroup $\mathscr{C}_{0\mathfrak{B}}(\ell)$ of the group of classes of divisors of degree zero relatively prime to $\mathfrak{B}$, will be called the *incomplete generalized Jacobian* of L. The Sylow ℓ-subgroup $\mathscr{C}_{0\mathfrak{N}}(\ell)$ of the group of classes of divisors of degree zero associated to $\mathfrak{N}$, will be called the *generalized Jacobian* of L. The Sylow ℓ-subgroup $\mathscr{C}_{0L}(\ell)$ of the group of classes of divisors of degree zero, will be called the *usual Jacobian* of L.

Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be moduli over L. We say that $\mathfrak{B}_2$ divides $\mathfrak{B}_1$, denoted by $\mathfrak{B}_2 \mid \mathfrak{B}_1$, if $v_{\mathscr{P}}(\mathfrak{B}_1) \geq v_{\mathscr{P}}(\mathfrak{B}_2)$ for all $\mathscr{P} \in \mathbb{P}_L$. The general result giving a relationship between two moduli of a field L is the following:

Lemma 2.1. *Let* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *be two moduli of L such that* $\mathfrak{B}_2$ *divides* $\mathfrak{B}_1$. *Then there exists a unique epimorphism* $\varphi : \mathscr{C}_{0\mathfrak{B}_1}(\ell) \to \mathscr{C}_{0\mathfrak{B}_2}(\ell)$ *such that* $H_{\mathfrak{B}_2 \mid \mathfrak{B}_1} = \ker(\varphi)$ *is a connected subgroup* (*in the Zariski topology*) *of* $\mathscr{C}_{0\mathfrak{B}_1}(\ell)$.

Proof. See [10, page 91, Proposition 6]. □

In [12, page 267, (29)], it was obtained the basic exact sequence of $\mathbb{Z}_\ell[G]$-modules

$$0 \to \frac{\bigoplus_{i=1}^{t} R[G/G_i]}{Re_t^*} \to \mathscr{C}_{0\mathfrak{N}}(\ell) \to \mathscr{C}_{0L}(\ell) \to 0, \tag{2}$$

where G_i denotes the decomposition group of the prime divisor $\wp_i$ of K, G/G_i denotes the set of left cosets of G_i in G, $R[G/G_i]$ is the $\mathbb{Z}_\ell[G]$-module $\left\{\sum_{\sigma \in G/G_i} a_\sigma \sigma \,|\, a_\sigma \in R\right\}$ on which G acts naturally and

$$Re_t^* := \left\{\left(\sum_{\sigma_1 \in G/G_i} x\sigma_i, \ldots, \sum_{\sigma_t \in G/G_t} x\sigma_t\right) \middle| \, x \in R\right\},$$

i.e., Re_t^* is isomorphic to $R := \frac{\mathbb{Q}_\ell}{\mathbb{Z}_\ell}$ and it is embedded diagonally in $\bigoplus_{i=1}^{t} R[G/G_i]$.

On the other hand, we have that the moduli $\mathfrak{B}$ and $\mathfrak{N}$ in L satisfy that $\mathfrak{B} \,|\, \mathfrak{N}$. In [3, page 764, (21)], it was obtained the exact sequence of $\mathbb{Z}_\ell[G]$-modules

$$0 \to \bigoplus_{i=s+1}^{t} R[G/G_i] \to \mathscr{C}_{0\mathfrak{N}}(\ell) \to \mathscr{C}_{0\mathfrak{B}}(\ell) \to 0, \tag{3}$$

where t is the number of prime divisors of K ramified in L and s is the number of prime divisors in the support of the modulus $\mathfrak{A}$ which is associated to the modulus $\mathfrak{B}$ of L.

Remark 2.2. The exact sequence of $\mathbb{Z}_\ell[G]$-modules (3), holds in general, that is, it is true either for $p = \ell$ or $p \neq \ell$.

Let M be a $\mathbb{Z}_\ell[G]$-module and let $0 \to M \to Y \to P \to 0$ be any exact sequence of G-modules, with Y an injective $\mathbb{Z}_\ell[G]$-module. We write $P = P^{(1)} \oplus P^{(0)}$, where $P^{(1)}$ is an injective $\mathbb{Z}_\ell[G]$-module and $P^{(0)}$ has no $\mathbb{Z}_\ell[G]$-injective components. Then $\Omega^{\#}(M) := P^{(0)}$ is the *dual of Heller's loop operator of M*. The $\mathbb{Z}_\ell[G]$-module $\Omega^{\#}(M)$ is unique up to isomorphism. Note that $\Omega^{\#}$ is well defined since the Krull-Schmidt-Azumaya Theorem (see [1, (6.12), page 128]) holds for $\mathbb{Z}_\ell[G]$-modules.

Proposition 2.3. *Let G be a finite ℓ-group and let H be a subgroup of G. Then*

(i) $R[G/H]$ *and* $\dfrac{R[G]}{R[G/H]}$ *are indecomposable* $\mathbb{Z}_\ell[G]$*-modules.*

(ii) $\Omega^{\#}(R[G/H]) \cong \dfrac{R[G]}{R[G/H]}$ *as* $\mathbb{Z}_\ell[G]$*-modules.*

(iii) *If* M_1 *and* M_2 *are* $\mathbb{Z}_\ell[G]$*-modules, then* $\Omega^{\#}(M_1 \oplus M_2) \cong \Omega^{\#}(M_1) \oplus \Omega^{\#}(M_2)$.

(iv) *If M is an injective* $\mathbb{Z}_\ell[G]$*-module, then* $\Omega^{\#}(M) \cong \{0\}$.

Proof. (i), (ii) and (iii) were proved in [4, Proposition 2.8, page 108]. For (iv), since M is injective, we have the exact sequence

$$0 \to M \xrightarrow{\text{id}} M \to 0 \to 0$$

with id the identity function. Therefore, $\Omega^{\#}(M) \cong \{0\}$. □

Let M be a $\mathbb{Z}_\ell[G]$-module such that the *Pontryagin's dual* $\mathfrak{X}(M) := \mathrm{Hom}_{\mathbb{Z}_\ell}(M, R)$ is finitely generated, G is a finite ℓ-group and M is a $\mathbb{Z}_\ell$-injective module. Then, as groups, $M \cong R^s$ with $s < \infty$. If ${}_\ell M$ denotes the set of elements of M whose order divides ℓ, then ${}_\ell M$ is a finitely generated $\mathbb{F}_\ell[G]$-module and ${}_\ell M$ will be called the *ℓ-part of M*. With this terminology, we have

Theorem 2.4 (Rzedowski-Villa-Madan)**.** *Let M and G be given as above. If* ${}_\ell M \cong \mathbb{F}_\ell[G]^n \oplus U$, *where* $\mathbb{F}_\ell[G]$ *is not a component of U and* $M \cong R[G]^m \oplus V$, *where* $R[G]$ *is not a component of V, then* $n = m$.

Proof. See [7, Lemma 3, page 81]. □

For any G-module A, the i-th Tate cohomology group $\hat{H}^i(G, A)$ with $i \in \mathbb{Z}$ is denoted by $H^i(G, A)$. The trivial group is denoted by $\{0\}$, whether its structure is additive or multiplicative. Finally, we denote by C_m the cyclic group with m elements.

3. Wildly Ramified Case

In this section, we assume that L/K is any finite Galois ℓ-extension of function fields with field of constants k, an algebraically closed field of characteristic $p = \ell$. Our main goals in this section are to obtain the Galois module structure of the incomplete generalized Jacobian $\mathscr{C}_{0\mathfrak{B}}(p)$ and of ${}_p\mathscr{C}_{0\mathfrak{B}}$, the elements of order dividing p of $\mathscr{C}_{0\mathfrak{B}}(p)$, i.e., we obtain explicitly the decomposition of $\mathscr{C}_{0\mathfrak{B}}(p)$ and of ${}_p\mathscr{C}_{0\mathfrak{B}}$ as direct sum of indecomposable $\mathbb{Z}_p[G]$-modules and $\mathbb{F}_p[G]$-modules, respectively, where $\mathfrak{B}$ is a modulus in L induced by the modulus $\mathfrak{A}$ in K which not necessarily contains in its support all prime divisors of K ramified in L.

Theorem 3.1. *Let L/K be an arbitrary finite Galois ℓ-extension of function fields with field of constants k of characteristic $p = \ell$. Then the $\mathbb{Z}_p[G]$-module structure of $\mathscr{C}_{0\mathfrak{B}}(p)$ is given by*

$$\mathscr{C}_{0\mathfrak{B}}(p) \cong R[G]^{\tau_K + s - 1} \oplus \left(\bigoplus_{i=s+1}^{t} \frac{R[G]}{R[G/G_i]} \right),$$

where τ_K denotes the Hasse-Witt invariant of K, s is the number of prime divisors contained in the support of the modulus $\mathfrak{A}$ of K and t is the number of prime divisors ramified in L/K.

Proof. In [12, Theorem 9, page 267], it was proved:

$$\mathscr{C}_{0\mathfrak{N}}(p) \cong R[G]^{\tau_K + t - 1}.$$

From the exact sequence (3), we have the exact sequence of $\mathbb{Z}_p[G]$-modules

$$0 \to \bigoplus_{i=s+1}^{t} R[G/G_i] \to R[G]^{\tau_K + t - 1} \to \mathscr{C}_{0\mathfrak{B}}(p) \to 0. \tag{4}$$

By the Krull-Schmidt-Azumaya Theorem, we have:

$\mathscr{C}_{0\mathfrak{B}}(p) \cong R[G]^x \oplus W,$ where W does not have $R[G]$ as a component.

Now, we must find the value of x and decompose W as direct sum of indecomposable $\mathbb{Z}_p[G]$-modules.

Using the dual of Heller's loop operator in (4) and Proposition 2.3, we obtain

$$W \cong \Omega^{\#}\left(\bigoplus_{i=s+1}^{t} R\left[G/G_i\right]\right) \cong \bigoplus_{i=s+1}^{t} \Omega^{\#}(R[G/G_i]) \cong \bigoplus_{i=s+1}^{t}\left(\frac{R[G]}{R[G/G_i]}\right).$$

On the other hand, to compute x, we use the technique used to obtain the injective component of $\mathscr{C}_{0L}(p)$ (case $p = \ell$). We have, the exact sequences of $\mathbb{Z}_p[G]$-modules (4) and

$$0 \to \bigoplus_{i=s+1}^{t} R[G/G_i] \to R[G]^c \to \Omega^{\#}\left(\bigoplus_{i=s+1}^{t} R\left[G/G_i\right]\right) \to 0,$$

where c is the minimum natural number such that there exists a $\mathbb{Z}_p[G]$-monomorphism

$$\phi : \left(\bigoplus_{i=s+1}^{t} R\left[G/G_i\right]\right) \to R[G]^c.$$

Using that $R[G]^c$ and $R[G]^{\tau_k + t - 1}$ are injective $\mathbb{Z}_p[G]$-modules and Schanuel's Lemma for injective modules, we have

$$R[G]^c \oplus R[G]^x \oplus \left(\bigoplus_{i=s+1}^{t} \frac{R[G]}{R\left[G/G_i\right]}\right) \cong \Omega^{\#}\left(\bigoplus_{i=s+1}^{t} R\left[G/G_i\right]\right) \oplus R[G]^{\tau_k + t - 1}.$$

From the Krull-Schmidt-Azumaya Theorem, we obtain

$$R[G]^c \oplus R[G]^x \cong R[G]^{\tau_k + t - 1},$$

i.e., $x = \tau_K + t - 1 - c$. Now, to determine c, we have

$$c = \dim_{\mathbb{F}_p}\left({}_p\left(\bigoplus_{i=s+1}^{t} R[G/G_i]\right)^G\right) = \dim_{\mathbb{F}_p}\left(\bigoplus_{i=s+1}^{t} \mathbb{F}_p[G/G_i]^G\right)$$

$$= \dim_{\mathbb{F}_p}\left(\bigoplus_{i=s+1}^{t} \mathbb{F}_p\right) = t - s.$$

Finally, $x = (\tau_K + t - 1) - (t - s) = \tau_K + s - 1.$ □

Corollary 3.2. *We keep the notation as above. Let* L/K *be any finite Galois* ℓ*-extension. Then the* $\mathbb{F}_p[G]$*-module structure of* ${}_p\mathscr{C}_{0\mathfrak{B}}$ *is given by*

$${}_p\mathscr{C}_{0\mathfrak{B}} \cong \mathbb{F}_p[G]^{\tau_K + s - 1} \oplus \left(\bigoplus_{i=s+1}^{t} \frac{\mathbb{F}_p[G]}{\mathbb{F}_p[G/G_i]}\right).$$

Proof. The result follows from Theorem 3.1, since

$${}_p(R[G]) \cong \mathbb{F}_p[G] \quad \text{and} \quad {}_p\left(\frac{R[G]}{R[G/G_i]}\right) \cong \frac{\mathbb{F}_p[G]}{\mathbb{F}_p[G/G_i]}.$$ □

4. Tamely Ramified Case

In this section, we assume that L/K is an arbitrary finite Galois ℓ-extension of function fields with field of constants k, an algebraically closed field of characteristic $p \neq \ell$. Our main goals in this section are to obtain the Galois module structure of the incomplete generalized Jacobian $\mathscr{C}_{0\mathfrak{B}}(\ell)$ and of ${}_\ell\mathscr{C}_{0\mathfrak{B}}$, where $\mathfrak{B}$ is a modulus in L induced by the modulus $\mathfrak{A}$ in K which contains in its support *all except one* of the prime divisors of K ramified in L.

Lemma 4.1. *Let* L/K *be a finite Galois* ℓ*-extension and* $G = \mathrm{Gal}(L/K)$*. Then*

(i) $H^1(G, \mathscr{C}_{0\mathfrak{B}}) \cong C_{\ell^{n-n_{i_o}}}$.

(ii) $H^0(G, \mathscr{C}_{0\mathfrak{B}}) \cong \bigoplus_{\substack{i=s+1\\ i\neq i_0}}^{t} (C_{\ell^{n_i}})$.

(iii) $H^1(G, {}_{\ell}\mathscr{C}_{0\mathfrak{B}}) \cong C_{\ell}^{t-s}$,

where n_{i_0} *denotes the maximum ramification index of the prime divisors not contained in the support of* $\mathfrak{A}$, s *is the number of prime divisors in the support of the modulus* $\mathfrak{A}$ *of K and t is the total of the prime divisors ramified in* L/K.

Proof. (i) and (ii) follow from Theorem 2.15 and Propositions 2.4 and 2.5 of [3]. From (17), [3, page 759], we obtain

$$H^1(G, {}_{\ell}\mathscr{C}_{0\mathfrak{B}}) \cong C_{\ell}^{\alpha_0(\mathscr{C}_{0\mathfrak{B}})+\alpha_1(\mathscr{C}_{0\mathfrak{B}})},$$

where

$$\alpha_i(\mathscr{C}_{0\mathfrak{B}}) = \dim_{\mathbb{F}_{\ell}} \frac{H^i(G, \mathscr{C}_{0\mathfrak{B}})}{\ell H^i(G, \mathscr{C}_{0\mathfrak{B}})} = \dim_{\mathbb{F}_{\ell}} {}_{\ell}H^i(G, \mathscr{C}_{0\mathfrak{B}}).$$

Since $\alpha_0(\mathscr{C}_{0\mathfrak{B}}) = 1$ and $\alpha_1(\mathscr{C}_{0\mathfrak{B}}) = t - s - 1$, (iii) follows. □

Lemma 4.2. *With the notation as above, if* $\mathfrak{B}$ *is a modulus in L induced by the modulus* $\mathfrak{A}$ *in K which contains in its support all except one of the prime divisors of K ramified in L, then*

(i) $H^1(G, \mathscr{C}_{0\mathfrak{B}}) \cong C_{\ell^{n-n_{i_0}}}$.

(ii) $H^0(G, \mathscr{C}_{0\mathfrak{B}}) \cong \{0\}$.

(iii) $H^1(G, {}_{\ell}\mathscr{C}_{0\mathfrak{B}}) \cong C_{\ell}$.

Proof. The result follows from Lemma 4.1, taking $s = t - 1$. □

Proposition 4.3. *Let* L/K *be an arbitrary finite Galois* ℓ*-extension. Then the integral representation of the incomplete generalized Jacobian* $\mathscr{C}_{0\mathfrak{B}}(\ell)$ *is of the form*

$$\mathscr{C}_{0\mathfrak{B}}(\ell) \cong R[G]^{2g_K+s-1-e} \oplus M,$$

where M has no $R[G]$ *components, e is the minimum number of generators of* $G/G'U$, *and U is the group generated by the inertia groups of the prime divisors different from those in the support of* $\mathfrak{A}$, G' *denotes the commutator subgroup of G*, *and as groups*,

$$M \cong R^m, \quad m = |G|\left(e + t - s - 1 - \sum_{i=s+1}^{t} \frac{1}{|G_i|}\right) + 1. \tag{5}$$

Proof. From the Krull-Schmidt-Azumaya Theorem, we have

$$\mathscr{C}_{0\mathfrak{B}}(\ell) \cong R[G]^{\alpha} \oplus M, \text{ where } M \text{ does not have } R[G] \text{ as a component.}$$

In [3, page 760], Theorem 3.5, it was obtained the exponent of the injective summand of $\mathscr{C}_{0\mathfrak{B}}(\ell)$, i.e., $\alpha = 2g_K + s - 1 - e$. Now, we will determine the rank of M. On the one hand, we have that, as groups, $\mathscr{C}_{0\mathfrak{B}}(\ell) \cong R^{\lambda_{\mathfrak{B}}}$, where

$$\lambda_{\mathfrak{B}} = |G|(2g_K - e + s - 1) + m. \tag{6}$$

On the other hand, using the exact sequence (3), we obtain that, as groups,

$$\mathscr{C}_{0\mathfrak{N}}(\ell) \cong R^{\lambda_{\mathfrak{N}}} \quad \text{with} \quad \lambda_{\mathfrak{N}} = 2g_L + |\mathfrak{N}| - 1$$

and

$$\bigoplus_{i=s+1}^{t} R[G/G_i] \cong R^{\sum_{i=s+1}^{t} |G/G_i|}.$$

Therefore,

$$\lambda_{\mathfrak{B}} = \lambda_{\mathfrak{N}} - \sum_{i=s+1}^{t} |G/G_i|.$$

Using, the computation for $\lambda_{\mathfrak{N}}$ in [13, page 47], we have

$$\lambda_{\mathfrak{B}} = [|G|(2g_K - 2 + t) + 1] - \sum_{i=s+1}^{t} |G/G_i|. \tag{7}$$

Now, from (6) and (7), it follows

$$\begin{aligned} |G|(2g_K - e + s - 1) + m &= |G|(2g_K - 2 + t) + 1 - \sum_{i=s+1}^{t} [G/G_i] \\ m &= |G|(e + t - s - 1) + 1 - |G| \sum_{i=s+1}^{t} \frac{1}{|G_i|} \\ m &= |G|\left(e + t - s - 1 - \sum_{i=s+1}^{t} \frac{1}{|G_i|}\right) + 1. \end{aligned} \qquad \square$$

Theorem 4.4. *Let* L/K *be any finite Galois* ℓ*-extension of function fields of one variable with field of constants k, an algebraically closed field of characteristic* $p \neq \ell$*. If* $\mathfrak{B}$ *is a modulus in L induced by a modulus* $\mathfrak{A}$ *in K which contains in its support* all except one *of the prime divisors of K ramified in L, then the* $\mathbb{Z}_\ell[G]$*-module structure of* $\mathscr{C}_{0\mathfrak{B}}(\ell)$ *is given by*

$$\mathscr{C}_{0\mathfrak{B}}(\ell) \cong R[G]^{2g_K+t-2-e_1} \oplus M_1,$$

where g_K *denotes the genus of K, t is the number of prime divisors ramified in* L/K*,* e_1 *is the minimum number of generators of group* $G/G'G_1$*,* G_1 *is the decomposition group of the prime divisor removed from* $\mathfrak{M}$ *and* M_1 *is an indecomposable* $\mathbb{Z}_\ell[G]$*-module such that, as groups* $M_1 \cong R^a$ *with* $a = |G|\left(e_1 - \frac{1}{|G_1|}\right) + 1.$

Proof. Taking $s = t - 1$ in Proposition 4.3, we find the injective component of $\mathscr{C}_{0\mathfrak{B}}(\ell)$, where $G_1 = U$ denotes the decomposition group of the ramified prime divisor $\wp_1$ which is not in the support of the modulus $\mathfrak{A}$ of *K*. The rank of M_1 follows from (5). Now, suppose that $M_1 \cong A \oplus B$ for some $\mathbb{Z}_\ell[G]$-modules *A* and *B* such that $R[G]$ is not a component neither of *A* nor of *B*. Since $R[G]$ is cohomologically trivial, we have

$$H^i(G, \mathscr{C}_{0\mathfrak{B}}) \cong H^i(G, M_1) \cong H^i(G, A) \oplus H^i(G, B).$$

Using Lemma 4.2, we obtain

$$\{0\} \cong H^0(G, \mathscr{C}_{0\mathfrak{B}}) \cong H^0(G, A) \oplus H^0(G, B), \tag{8}$$

$$C_{\ell^{n-n_{i_0}}} \cong H^1(G, \mathscr{C}_{0\mathfrak{B}}) \cong H^1(G, A) \oplus H^1(G, B). \tag{9}$$

Therefore, from (8), we obtain that $H^0(G, A) \cong \{0\}$ and $H^0(G, B) \cong \{0\}$. From (9), it follows that $H^1(G, A) \cong \{0\}$ or $H^1(G, B) \cong \{0\}$. Therefore, *A* or *B* is cohomologically trivial and $\mathbb{Z}_\ell$-divisible. Hence, *A* or *B* must be $\mathbb{Z}_\ell[G]$-injective, which is absurd. It follows that M_1 is an indecomposable $\mathbb{Z}_\ell[G]$-module. □

Theorem 4.5. *With the notation as above, if* $\mathfrak{B}$ *is a modulus in L induced by a modulus* $\mathfrak{A}$ *in K which contains in its support* all except one *of the prime divisors of K ramified in L, then the* $\mathbb{F}_\ell[G]$*-module structure of* ${}_\ell\mathscr{C}_{0\mathfrak{B}}$ *is given by*

$$ {}_\ell\mathscr{C}_{0\mathfrak{B}} \cong \mathbb{F}_\ell[G]^{2g_K+t-2-e_1} \oplus M_0, $$

where M_0 *is an indecomposable* $\mathbb{F}_\ell[G]$*-module, and as groups,*

$$ M_0 \cong \mathbb{F}_\ell^b \quad with \quad b = |G|\left(e_1 - \frac{1}{|G_1|}\right) + 1. $$

Proof. From Theorems 4.4 and 2.4, it suffices to prove that M_0 is an indecomposable $\mathbb{F}_\ell[G]$-module. Suppose that $M_0 \cong A \oplus B,$ for some not trivial $\mathbb{F}_\ell[G]$-modules A and B, i.e., M_0 is not indecomposable. Since $\mathbb{F}_\ell[G]$ is cohomology trivial, we have $H^i(G, {}_\ell\mathscr{C}_{0\mathfrak{B}}) \cong H^i(G, M_0)$. In particular, using (iii) of Lemma 4.2, we obtain

$$ H^1(G, M_0) \cong H^1(G, {}_\ell\mathscr{C}_{0\mathfrak{B}}) \cong C_\ell. $$

Then

$$ C_\ell \cong H^1(G, M_0) \cong H^1(G, A) \oplus H^1(G, B), $$

hence, $H^1(G, A) = \{0\}$ or $H^1(G, B) = \{0\}$. From [9, Theorem 5, page 142], we obtain $A \cong \mathbb{F}_\ell[G]^{\alpha_1}$ or $B \cong \mathbb{F}_\ell[G]^{\alpha_2}$, which is absurd. Therefore, M_0 is an indecomposable $\mathbb{F}_\ell[G]$-module. □

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