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ON *Hom* (-, -) AS *B*-ALGEBRAS

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Abstract

In this paper, we give an example to show that $Hom(\text{-}, \text{-})$ may not, in general, be a *B*-algebra. Moreover, we find conditions under which $Hom(\text{-}, \text{-})$ is a *B*-algebra. Also, we introduce the notion of an orthogonal subset and discuss some related properties.

1. Introduction

Iseki and Tanaka introduced two classes of abstract algebras: *BCK*-algebras and *BCI*-algebras [4, 5]. It is known that the class of *BCK*-algebras is a proper subclass of the class of *BCI*-algebras. In [2, 3] Hu and Li introduced a wide class of abstract algebras: *BCH*-algebras. They have shown that the class of *BCI*-algebras is a proper subclass of the class of *BCH*-algebras. In [9] the authors introduced the notion of *d*-algebras, which is another useful generalization of *BCK*-algebras, and then they investigated several relations between *d*-algebras and *BCK*-algebras as well as some other interesting relations between *d*-algebras and oriented digraphs. Jun et al. [7] introduced a new notion, called *BH-algebras*, which is a generalization of *BCH*, *BCI*, *BCK*-algebras. They also defined the notions of ideals in *BH*-algebras. Recently Neggers and Kim [10] introduced the notion of *B*-algebra, and then Cho and Kim [1] studied some of its properties. In [6] Jun and Meng investigated some properties of $Hom(X, Y)$ the set of all homomorphisms of a *BCI*-algebra *X* into an arbitrary *BCI*-

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algebra Y. In this paper, we investigate some properties of $Hom(X, Y)$ as B-algebras. We show that $Hom(X, Y)$ may not, in general, be a B-algebra for an arbitrary B-algebra, and we prove that if X is a B-algebra and Y is an associative B-algebra, then $Hom(X, Y)$ is an associative B-algebra. Also, we prove that if X is a B-algebra and Y is a 0-commutative B-algebra, then $Hom(X, Y)$ is a 0-commutative B-algebra. Also, we introduce the notion of orthogonal subsets and investigate some related properties.

2. Preliminaries

Definition 2.1 [10]**.** A B-algebra is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = x * (z * (0 * y))$,

for all $x, y, z \in X$.

Proposition 2.2 [10]**.** *If* $(X, *, 0)$ *is a B-algebra, then*

(1) $(x * y) * (0 * y) = x$,

(2) $x * (y * z) = (x * (0 * z)) * y$,

(3) $x * y = 0$ *implies* $x = y$,

(4) $0 * (0 * x) = x$,

for all $x, y, z \in X$.

Theorem 2.3 [10]**.** $(X, *, 0)$ *is a B-algebra if and only if it satisfies the following axioms*:

(5) $x * x = 0$,

(6) $0 * (0 * x) = x$,

(7) $(x * z) * (y * z) = x * y$,

(8) $0 * (x * y) = y * x$,

for all $x, y, z \in X$.

Definition 2.4 [8]. A *B*-algebra $(X, *, 0)$ is said to be 0-*commutative* if

$$x * (0 * y) = y * (0 * x),$$

for all $x, y \in X$.

Proposition 2.5 [8]. *If* $(X, *, 0)$ *is a* 0-*commutative B-algebra, then*

(9) $(0 * x) * (0 * y) = y * x,$

(10) $(z * y) * (z * x) = x * y,$

(11) $(x * y) * z = (x * z) * y,$

(12) $(x * (x * y)) * y = 0,$

(13) $(x * z) * (y * t) = (t * z) * (y * x),$

for all $x, y, z, t \in X$.

A *B*-algebra *X* is said to be *associative* if $(x * y) * z = x * (y * z),$ for all *x*, *y*, $z \in X$. A nonempty subset *S* of *X* is called a *subalgebra* of *X* if $x * y \in S,$ for all $x, y \in S$.

Definition 2.6 [10]. A nonempty subset *N* of a *B*-algebra *X* is said to be *normal subalgebra* of *X* if

$$(x * a) * (y * b) \in N,$$

for any $x * y, a * b \in N$.

A mapping $f : x \to y$ between *B*-algebras *X* and *Y* is called a *homomorphism* if $f(x * y) = f(x) * f(y),$ for all $x, y \in X$. Define the trivial homomorphism 0 as $0(x) = 0,$ for all $x \in X$. Denote by $Hom(X, Y)$ the set of all homomorphisms of a *B*-algebra *X* into a *B*-algebra *Y* (see [11]).

3. *Hom*(-, -) as *B*-algebras

Let $Hom(X, Y)$ be the set of all homomorphisms of a *B*-algebra *X* into a *B*-algebra *Y*. In the following example, we show that $(Hom(X, Y), *, 0)$ may not be a *B*-algebra in general, where $*$ is defined as follows:

$$(f * g)(x) = f(x) * g(x), \quad \forall f, g \in Hom(X, Y), \forall x \in X,$$

and 0 is a trivial homomorphism from a *B*-algebra *X* into a *B*-algebra *Y*.

Example 3.1. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a B-algebra with Cayley table (Table 1) as follows:

Table 1

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Define a map $f : X \to X$ by $f(x) = 0,$ for all $x \in X,$ and a map $g : X \to X$ by $g(x) = x,$ for all $x \in X.$ Then it is easily checked that $f, g \in Hom(X, Y),$ but $f * g \notin Hom(X, Y)$ for

$$(f * g)(3 * 1) = (f * g)(4) = f(4) * g(4) = 4$$

and

$$(f * g)(3) * (f * g)(1) = (f(3) * g(3)) * (f(1) * g(1)) = 3 * 2 = 5,$$

therefore,

$$(f * g)(3 * 1) \neq (f * g)(3) * (f * g)(1).$$

Hence, $Hom(X, Y)$ is not a B-algebra.

Theorem 3.2. *If X is a B-algebra and Y is an associative B-algebra, then $Hom(X, Y)$ is an associative B-algebra.*

Proof. Let $f, g \in Hom(X, Y)$ and $x \in X.$ Then

$$\begin{aligned}(f * g)(x * y) &= f(x * y) * g(x * y) \\ &= (f(x) * f(y)) * (g(x) * g(y)) \\ &= f(x) * ((f(y) * g(x)) * g(y)) \\ &= (f(x) * (0 * g(y))) * (f(y) * g(x)) \quad \text{by (2)}\end{aligned}$$

$$= ((f(x) * 0) * g(y)) * (f(y) * g(x))$$

$$= (f(x) * g(y)) * (f(y) * g(x)) \quad \text{by (II)}$$

$$= (f(x) * (g(y) * f(y))) * g(x)$$

$$= f(x) * (g(x) * (0 * (g(y) * f(y)))) \quad \text{by (III)}$$

$$= f(x) * (g(x) * (f(y) * g(y))) \quad \text{by (8)}$$

$$= (f(x) * g(x)) * (f(y) * g(y))$$

$$= (f * g)(x) * (f * g)(y).$$

Then $f * g \in Hom(X, Y)$, for all $f, g \in Hom(X, Y)$. Since Y is a B-algebra, it is easy to prove that the axioms in Definition 2.1 are satisfied for all $f, g, h \in Hom(X, Y)$, and so $Hom(X, Y)$ is a B-algebra. Now let $f, g, h \in Hom(X, Y)$ and $x \in X$. Then

$$((f * g) * h)(x) = (f(x) * g(x)) * h(x) = f(x) * (g(x) * h(x)) = (f * (g * h))(x),$$

because Y is an associative B-algebra, and the proof is completed.

Theorem 3.3. *If X is a B-algebra and Y is a* 0*-commutative B-algebra, then $Hom(X, Y)$ is a* 0*-commutative B-algebra.*

Proof. Let $f, g \in Hom(X, Y)$ and $x \in X$. Then

$$(f * g)(x * y) = f(x * y) * g(x * y)$$

$$= (f(x) * f(y)) * (g(x) * g(y))$$

$$= (g(y) * f(y)) * (g(x) * f(x)) \quad \text{by (13)}$$

$$= (0 * (f(y) * g(y))) * (0 * (f(x) * g(x))) \quad \text{by (8)}$$

$$= (f(x) * g(x)) * (f(y) * g(y)) \quad \text{by (9)}$$

$$= (f * g)(x) * (f * g)(y).$$

Therefore, $f * g \in Hom(X, Y)$, for all $f, g \in Hom(X, Y)$. Since Y is a B-algebra, it is easy to prove that the axioms in Definition 2.1 are satisfied for all $f, g, h \in Hom(X, Y)$, and so $Hom(X, Y)$ is a B-algebra. Let $f, g \in Hom(X, Y)$ and $x \in X$. Then

$$((f * 0) * g)(x) = (f(x) * 0) * g(x) = g(x) * (0 * f(x)) = ((g * 0) * f)(x),$$

because Y is a 0-commutative B-algebra, and the proof is completed.

Definition 3.4. Let M and Θ be subsets of X and $Hom(X, Y)$, respectively. We define orthogonal subsets $M^{\perp}$ and $\Theta^{\perp}$ of M and Θ, respectively, by

$$M^{\perp} = \{f \in Hom(X, Y) \mid f(x) = 0, \text{ for all } x \in M\}$$

and

$$\Theta^{\perp} = \{x \in X \mid f(x) = 0, \text{ for all } f \in Hom(X, Y)\}.$$

Theorem 3.5. *Let X be a B-algebra, Y be an associative B-algebra, $M \subseteq X$ and $\Theta \subseteq Hom(X, Y)$. Then $M^{\perp}$ and $\Theta^{\perp}$ are normal subalgebras of $Hom(X, Y)$ and X, respectively.*

Proof. Let $f * g, h * k \in M^{\perp}$. Then $(f * g)(x) = 0$, for all $x \in M$ and $(h * k)(x) = 0$, for all $x \in M$, by Theorem 3.2, we have that $Hom(X, Y)$ is an associative B-algebra. Thus

$$\begin{aligned} ((f * h) * (g * k))(x) &= (((f * h) * g) * k)(x) \\ &= ((f * (g * (0 * h))) * k)(x) \quad \text{by (III)} \\ &= ((f * ((g * 0) * h)) * k)(x) \\ &= ((f * (g * h)) * k)(x) \quad \text{by (II)} \\ &= ((f * g) * (h * k))(x) \\ &= (f * g)(x) * (h * k)(x) = 0, \end{aligned}$$

for all $x \in M$. Thus, $(f * h) * (g * k) \in M^{\perp}$, and so $M^{\perp}$ is normal subalgebra of $Hom(X, Y)$.

Now let $x * y,\ a * b \in \Theta^{\perp}$, hence $f(x * y) = 0$ and $f(a * b) = 0$, for all $f \in Hom(X, Y)$. Since Y is an associative *B*-algebra, in similar way we can prove that $f((x * a) * (y * b)) = 0$, for all $f \in Hom(X, Y)$, and then $(x * a) * (y * b) \in \Theta^{\perp}$, for all $f \in Hom(X, Y)$. Therefore, $\Theta^{\perp}$ is normal subalgebra of *X*.

Theorem 3.6. *Let X be a B-algebra, Y be a* 0*-commutative B-algebra,* $M \subseteq X$ *and* $\Theta \subseteq Hom(X, Y)$. *Then* $M^{\perp}$ *and* $\Theta^{\perp}$ *are normal subalgebras of* $Hom(X, Y)$ *and X, respectively.*

Proof. Let $f * g,\ h * k \in M^{\perp}$. Then $(f * g)(x) = 0$, for all $x \in M$ and $(h * k)(x) = 0$, for all $x \in M$, from Theorem 3.3 we know that $Hom(X, Y)$ is a 0-commutative *B*-algebra. Hence

$$\begin{aligned}((f * h) * (g * k))(x) &= ((k * h) * (g * f))(x) \quad \text{by (13)}\\ &= ((0 * (h * k)) * (0 * (f * g)))(x) \quad \text{by (8)}\\ &= (0(x) * (h * k)(x)) * (0(x) * (f * g)(x)) = 0,\end{aligned}$$

for all $x \in M$. Thus, $(f * h) * (g * k) \in M^{\perp}$ and so $M^{\perp}$ is normal subalgebra of $Hom(X, Y)$.

Now, let $x * y,\ a * b \in \Theta^{\perp}$. Then $f(x * y) = 0$ and $f(a * b) = 0$, for all $f \in Hom(X, Y)$. Since Y is a 0-commutative *B*-algebra, in similar way we can prove that $f((x * a) * (y * b)) = 0$, for all $f \in Hom(X, Y)$, and then $(x * a) * (y * b) \in \Theta^{\perp}$, for all $f \in Hom(X, Y)$. Therefore, $\Theta^{\perp}$ is normal subalgebra of *X*.

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