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NON-PERTURBATIVE ANALYTICAL APPROXIMATE SOLUTIONS OF FALKNER-SKAN EQUATION

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Abstract

We use Adomian's decomposition method to obtain accurate non-perturbative approximate solutions to the Falkner-Skan equation. The latter describes some 2D laminar boundary-layers.

1. Introduction

When a body moves in a fluid with moderately large Reynolds number, the effect of viscosity is pronounced in a thin layer adjacent to its surface. In this small region, frictional forces overtake inertial ones whereas the inverse occurs in the main flow. A century ago, Prandtl [15] made these observations precise in his boundary-layer theory. He found equations that accurately approximate Navier-Stokes ones in boundary-layers. For a two-dimensional incompressible steady flow, Prandtl's boundary-layers equations read

$$uu_x + vu_y = U(x)\frac{dU(x)}{dx} + \nu u_{yy}, \tag{1}$$

$$u_x + v_y = 0, \tag{2}$$

where u and v are respectively the x- and y-components of the velocity

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field in the boundary-layer, $U(x)$ is the mainstream velocity, ν is the kinematic viscosity of the fluid. The system formed by the equations (1)-(3) must be solved subject to the following boundary conditions:

$$u(x, 0) = 0, \quad v(x, 0) = 0, \quad u(x, \infty) = U(x). \tag{3}$$

The first two equations in (3) are the no-slip conditions. The third condition in (3) enforces the continuity of the velocity field at the interface between the boundary-layer and the mainstream.

We assume that $U(x) = U_0 x^m$, $m \neq -1$. Note that if $0 \leq m \leq 1$, such a mainstream flow occurs on the surface of a wedge of angle $2m\pi/(1 + m)$. Now, we introduce a stream function $\Psi(x, y)$ such that $u = \partial\Psi/\partial y$, $v = -\partial\Psi/\partial x$. Then, we look for $\Psi(x, y)$ in the form

$$\Psi(x, y) = \left(\frac{2\nu U_0 x^{m+1}}{1 + m}\right)^{1/2} f(\eta), \tag{4}$$

where

$$\eta = y\left(\frac{(1 + m)U_0 x^{m-1}}{2\nu}\right)^{1/2}. \tag{5}$$

The system (1)-(2) reduces to the single equation

$$f''' + ff'' + \beta(1 - f'^2) = 0, \tag{6}$$

where $\beta = 2m/(1 + m)$ and the prime stands for differentiation with respect to η. The boundary conditions (3) become

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1. \tag{7}$$

Equation (6) was first derived by Falkner and Skan [8]. The particular case $\beta = 0$ is the Blasius equation. The parameter β is sometimes called *Falkner-Skan pressure gradient*.

The theoretical investigation of the boundary-value problem (6)-(7) was pioneered by Weyl [19]. In order to give a brief review of the theoretical results available, we need to define the different type of solutions of the Falkner-Skan equation considered in the literature (see for instance [13]). A *classical solution* of (6)-(7) is one that is sufficiently smooth and satisfies $f'(t) > 0$ for $t > 0$. A *reverse flow solution* of (6)-(7) is one for which $f'(\tau) < 0$ for some $\tau > 0$. An *overshoot solution* of (6)-(7) is a reverse flow solution for which $|f'(\tau)| > 1$ for some $\tau > 0$. Weyl [19] proved the existence of classical solutions of (6)-(7) for $\beta \geq 0$. So far global uniqueness of classical solutions has been established for $0 \leq \beta \leq 1$ [12]. There is a number $\beta^* \approx -0.198838$ such that for $\beta^* < \beta < 0$ there are infinitely many solutions bounded by two extremal solutions. The upper solution is a classical solution and the lower solution is of the reverse flow type. The latter was discovered through numerical experiments by Stewartson [16] and theoretically investigated by Hastings [10]. When $\beta < \beta^*$, all possible solutions are of overshoot type [10].

Since the general exact solution of (6) does not exist, the natural way of tackling the problem (6)-(7) is to use approximate methods such as numerical and perturbation techniques. Amongst numerical methods employed on (6)-(7), we may conservatively cite shooting and invariant imbedding methods [6, 9, 13], finite difference methods [3, 4] and finite element methods [5].

Recently Wang [17] used an approximate analytical method to solve (6)-(7) in the case $\beta = 0$. Precisely, he showed that the Adomian decomposition method (ADM) [2] fails when it is applied to Blasius equation. Indeed the Adomian approximant for the Blasius equation does not satisfy the boundary condition at infinity even when it is replaced by its diagonal Padé approximation as suggested by Wazwaz [18] in another context. In order to overcome this difficulty he introduced a change of variable that reduces the order of Blasius equation by one and transforms the boundary condition at infinity to one at a finite point. He then applied the ADM to the transformed problem and obtained an accurate approximate analytical solution.

In contrast to the Blasius equation, we aim at showing that the classical ADM together with Wazwaz suggestion [18] provide accurate solutions to the Falkner-Skan equation when $\beta \neq 0.$ In more details, the outline of this paper is the following. Section 2 introduces the rudiments of the ADM. Section 3 deals with the application of the ADM to the Falkner-Skan equation. It is shown that when $\beta \neq 0,$ the use of diagonal Padé approximants enables the evaluation of the boundary condition at infinity. Thus we are able to calculate $f''(0)$ by solving an algebraic equation depending on β.

2. Overview of Adomian's Decomposition Method

In this section we provide some rudiments of ADM. For a comprehensive exposition, the reader is referred to the classical book of Adomian [2].

Consider the abstract functional equation

$$u = v + G(u), \tag{8}$$

where the unknown u belongs to a Banach space B, v is a given element of B and G is a nonlinear mapping from X into itself. We assume that G is sufficiently Fréchet differentiable at v. This last hypothesis will become clear as we proceed.

The fundamental idea of the ADM is to seek a solution to (8) in the form

$$u = \sum_{i=0}^{\infty} u_i \tag{9}$$

and to decompose the nonlinear operator G as

$$G(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \tag{10}$$

where

$$A_n = \frac{1}{n!} \frac{d^n G(u_\lambda)}{d\lambda^n}\bigg|_{\lambda=0}, \quad n = 0, 1, 2, \ldots, \tag{11}$$

$$u_\lambda = \sum_{i=0}^{\infty} \lambda^i u_i. \tag{12}$$

Note that if $X \subset R,$ then, for $n \geq 1,$ A_n is a polynomial in $u_1, u_2, \ldots, u_n$

called *Adomian polynomial*. In general, the form of A_n depends on G.

If we substitute (9) and (10) into (8), then we obtain

$$\sum_{i=0}^{\infty} u_i = v + \sum_{i=0}^{\infty} A_i(u_0, u_1, ..., u_i). \tag{13}$$

We may satisfy (13) by formally setting

$$u_0 = v, \tag{14}$$

$$u_n = A_{n-1}(u_0, u_1, ..., u_{n-1}), \quad n = 1, 2, \ldots . \tag{15}$$

The relations (14)-(15) are used to recursively calculate the terms of the series (9). In practice, only a finite number of terms are needed to obtain a good approximation of a solution to (8). It is worthwhile noting that the ADM provides an approximate solution to a nonlinear problem without recourse to linearization or perturbation. This is one of its most attractive feature.

Cherruault [7], then Abbaoui [1] were the first to prove the convergence of the ADM. Recently Himoun et al. [11] extended the previous results on the convergence of the ADM. They established the following theorem.

Theorem. *If there is* $M > 0$ *and* $L > 0$ *such that*

$$ML \leq e^{-1} \text{ and } \| G^{(n)}(v) \| \leq ML^n \text{ for } n \geq 0, \tag{16}$$

then the series defined by (9), (14) *and* (15) *converges to a solution of the nonlinear functional equation* (8).

3. Application of the Classical ADM to the Falkner-Skan Equation

In this section we apply the ADM to the problem (6)-(7).

We may rewrite (6) as

$$f(\eta) = \alpha \frac{\eta^2}{2} - L_{\eta\eta\eta}^{-1}[ff'' + \beta(1 - f'^2)], \tag{17}$$

where $\alpha = f''(0)$ and

$$L_{\eta\eta\eta}^{-1} = \int_0^{\eta}\int_0^{\eta}\int_0^{\eta}(\cdot)d\eta d\eta d\eta. \tag{18}$$

Note that the boundary conditions $f(0) = 0$ and $f'(0) = 0$ are readily satisfied. The idea is to solve (17) using the ADM and impose the boundary condition at infinity in order to find α.

According to the ADM, we look for a solution to (17) in the form

$$f = \sum_{i=0}^{\infty} f_i, \tag{19}$$

and we decompose the nonlinear operator

$$G(f) = ff'' + \beta(1 - f'^2) \tag{20}$$

as

$$G(f) = \sum_{i=1}^{\infty} A_i(f_0, f_1, \ldots, f_i), \tag{21}$$

where the A_i's are Adomian polynomials. The first four Adomian polynomials calculated using formula (11) are

$$A_0 = f_0 f_0'' + \beta(1 - f_0'^2), \tag{22}$$

$$A_1 = f_0'' f_1 + f_0 f_1'' - 2\beta f_0' f_1', \tag{23}$$

$$A_2 = f_0'' f_2 + f_1'' f_1 + f_2'' f_0 - \beta(2 f_0' f_2' + f_1'^2), \tag{24}$$

$$A_3 = f_0'' f_3 + f_1'' f_2 + f_2'' f_1 + f_3'' f_0 - \beta(f_0' f_3' + 2 f_1' f_2'). \tag{25}$$

The series terms are computed from the relations

$$f_0 = \alpha \frac{\eta^2}{2}, \tag{26}$$

$$f_{n+1} = -L_{\eta\eta\eta}^{-1}[A_n], \quad n \geq 0. \tag{27}$$

After some calculations, we find that

$$f_1 = -\frac{\beta}{3!}\eta^3 + \frac{(2\beta - 1)\alpha^2}{5!}\eta^5, \tag{28}$$

$$f_2 = \frac{2(2 - 3\beta)\alpha\beta}{6!}\eta^6 + \frac{(20\beta^2 - 32\beta + 11)\alpha^3}{8!}\eta^8, \tag{29}$$

$$f_3 = \frac{2(3\beta-2)\beta^2}{7!}\eta^7 - \frac{(132\beta^2-226\beta+90)\alpha^2\beta}{9!}\eta^9 + \frac{(600\beta^3-1596\beta^2+1398\beta-375)\alpha^4}{11!}\eta^{11}. \tag{30}$$

We may compute more terms to achieve better accuracy. A solution to the Falkner-Skan equation is approximated by

$$f(\eta) \approx \alpha\frac{\eta^2}{2} - \frac{\beta}{3!}\eta^3 + \frac{(2\beta-1)\alpha^2}{5!}\eta^5 + \frac{2(2-3\beta)\alpha\beta}{6!}\eta^6 + \frac{2(3\beta-2)\beta^2}{7!}\eta^7 + \frac{(20\beta^2-32\beta+11)\alpha^3}{8!}\eta^8 - \frac{(132\beta^2-226\beta+90)\alpha^2\beta}{9!}\eta^9 + \frac{(600\beta^3-1596\beta^2+1398\beta-375)\alpha^4}{11!}\eta^{11} + \cdots. \tag{31}$$

We deduce from (31) that

$$f'(\eta) \approx \alpha\eta - \frac{\beta}{2!}\eta^2 + \frac{(2\beta-1)\alpha^2}{4!}\eta^4 + \frac{2(2-3\beta)\alpha\beta}{5!}\eta^5 + \frac{2(3\beta-2)\beta^2}{6!}\eta^6 + \frac{(20\beta^2-32\beta+11)\alpha^3}{7!}\eta^7 - \frac{(132\beta^2-226\beta+90)\alpha^2\beta}{8!}\eta^8 + \frac{(600\beta^3-1596\beta^2+1398\beta-375)\alpha^4}{10!}\eta^{10} + \cdots. \tag{32}$$

It can be readily seen that we cannot apply the boundary condition at infinity to the truncated series (32). In order to bypass this difficulty, we use Padé approximation as suggested by Wazwaz [18]. Indeed, it is a well-known fact that the Padé approximant of a function offers a better approximation over a wide range of values than its truncated Taylor series. Below we provide the definition of a Padé approximant.

Definition. Let F be a function that admits a Taylor series expansion T_{F,x_0} about $x = x_0$. The Padé approximant of order $[M, N]$ of F is the irreducible rational function $[M, N]_F = P_M/Q_N$ defined by

$$T_{F,x_0}(x)Q_N(x) - P_M(x) = O(x^{M+N+1}), \tag{33}$$

where P_M and Q_N are polynomials of degree M and N, respectively.

Remark. It is customary to normalize Padé approximants by setting $Q_N(0) = 1.$ However in our symbolic calculations below we do not adopt this normalization as it complicates manipulations.

Since the limit at infinity of f' is finite, we employ diagonal Padé approximants, i.e., Padé approximants with $M = N$ to approximate it. It can be verified that diagonal Padé approximants of f' exist provided $\beta \neq 0.$ Using the package for symbolic computations MuPad [14], we obtain

$$[5,\, 5]_{f'} = \frac{a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 + a_5\eta^5}{b_0 + b_1\eta + b_2\eta^2 + b_3\eta^3 + b_4\eta^4 + b_5\eta^5}, \tag{34}$$

where the coefficients a_i, b_i are dependent on β and are not presented because they are quite long.

Provided $\beta \neq 0,$ the boundary condition at infinity applied to (34) is equivalent to $a_5/b_5 = 1$, i.e.,

$$\lambda_1(\beta)\alpha^{14} + \lambda_2(\beta)\alpha^{12} + \lambda_3(\beta)\alpha^{10} + \lambda_4(\beta)\alpha^8 + \lambda_5(\beta)\alpha^6 + \lambda_6(\beta)\alpha^4 + \lambda_7(\beta)\alpha^2 + \lambda_8(\beta) = 0 \tag{35}$$

with the coefficients $\lambda_i(\beta)$ omitted because of their length.

Given β, we need to solve the algebraic equation (35) in order to determine α. To the best of our knowledge an explicit relation linking α to β has not appeared in the literature. Once α is calculated from (35), f' is accurately approximated by (34) and of course f is obtained from the latter through a single integration.

In order to demonstrate the accuracy of our approximate analytical results, we consider the case $\beta = 1.$ In this case equation (35) reads

$$18822375\alpha^{14} - 618750\alpha^{12} - 12098520\alpha^{10} - 583632\alpha^8 - 13225968\alpha^6 - 2422560\alpha^4 - 73100160\alpha^2 - 3073280 = 0. \tag{36}$$

Using MuPad, we find that the only positive solution of (36) is

$$\alpha_{\text{analytic}} \approx 1.196529 \tag{37}$$

whereas the solution found by Asaithambi [5] using finite element method is

$$\alpha_{\text{numer}} \approx 1.232588. \tag{38}$$

Thus the relative error with respect to the numerical solution is about 2.9% in the case $\beta = 1$. As we mentioned earlier, the error can be reduced by truncating the series solution farther. However, note that since exact solutions are not known, the comparison of our solution to the numerical solution is quite rough. Indeed most numerical solutions rely on certain ad hoc assumptions. For instance the boundary condition at infinity is applied at some sufficiently large number. This last assumption affects the accuracy of the solution especially if the solution is of the overshoot type.

The shear-stress on the boundary is given by

$$\tau_{yx} = \rho\left[\frac{\nu U_0^3 x^{2(2\beta-1)/(2-\beta)}}{2-\beta}\right]^{1/2}\alpha, \tag{39}$$

where ρ is the fluid density, ν is its kinematic viscosity and U_0 is a constant related to the mainstream velocity. Hence the shear stress on the wall depends linearly on α. This emphasizes the importance of (35) in design problems where one wants to control the frictional force on the wall.

4. Conclusion

We have shown that we can use Adomian's decomposition method to obtain accurate non-perturbative analytical solution of the Falkner-Skan equation. The method is simple and is free from ad hoc assumptions that characterize many available numerical schemes for the Falkner-Skan equation.

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