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ON THE NICHOLS ALGEBRA ASSOCIATED TO $(q_{ij}) = \begin{pmatrix} -\zeta^3 & -\zeta \\ -1 & -1 \end{pmatrix}$, OF TYPE B_2

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Abstract

We examine the defining relations of the Nichols algebra associated to $(q_{ij}) = \begin{pmatrix} -\zeta^3 & -\zeta \\ -1 & -1 \end{pmatrix}$, of type B_2, by using the method introduced by Nichols [1] (see also [3]).

1. Introduction

Nichols algebras are graded braided Hopf algebras with the base field in degree 0 and which are coradically graded and generated by its primitive elements ([4-7]). Let V be a vector space and $c : V \otimes V \to V \otimes V$ be a linear isomorphism. Then (V, c) is called a *braided vector space*, if c is a solution of the braid equation, that is, $(c \otimes id)(id \otimes c)(c \otimes id) = (id \otimes c)(c \otimes id)(id \otimes c)$. The pair (V, c) determines the Nichols algebras up to isomorphism. Let G be a group. Then a Yetter-Drinfeld

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module V over $\mathbb{K}G$ is a G-graded vector space $V = \bigoplus_{g \in G} V_g$, which is a G-module such that $g \cdot V_h \subset V_{ghg^{-1}}$ for all $g, h \in G$. The category ${}^G_G YD$ of $\mathbb{K}G$-Yetter-Drinfeld module is braided. For $V, W \in {}^G_G YD$, the braiding $c : V \otimes W \to W \otimes V$ is defined by $c(v \otimes w) = (g \cdot w) \otimes v$, $v \in V_g$, $w \in W$. Let V be a Yetter-Drinfeld module over G and let $T(V) = \bigoplus_{n \geq 0} T(V)(n)$ denote the tensor algebra of the vector space V. Let S be the set of all ideals and coideals I of $T(V)$ which are generated as ideals by $\mathbb{N}$-homogeneous elements of degree ≥ 2, and which are Yetter-Drinfeld submodules of $T(V)$. Let $I(V) = \sum_{I \in S} I$. Then $B(V) := T(V)/I(V)$ is called the *Nichols algebra* of $V \in {}^G_G YD$. In this paper, we examine the defining relations of the Nichols algebra $B(V)$ associated to $(q_{ij}) = \begin{pmatrix} -\zeta^3 & -\zeta \\ -1 & -1 \end{pmatrix}$, of type B_2.

2. Nichols Algebras of Cartan Type

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. Let G be an abelian group and V be a finite dimensional Yetter-Drinfeld module. Then the braiding is given by a nonzero scalar $q_{ij} \in \mathbb{K}$, $1 \leq i, j \leq \theta$, in the form $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, where $x_1, \ldots, x_\theta$, is a basis of V. If there is a basis such that $g \cdot x_i = \chi_i(g) x_i$ and $x_i \in V_{g_i}$, then V is called *diagonal type*. For the braiding, we have $c(x_i \otimes x_j) = \chi_j(g_i) x_j \otimes x_i$ for $1 \leq i, j \leq \theta$. Hence, we have $(q_{ij})_{1 \leq i, j \leq \theta} = (\chi_j(g_i))_{1 \leq i, j \leq \theta}$. Let $B(V)$ be the Nichols algebra of V. Then we can construct the Nichols algebra by $B(V) \cong T(V)/I$, where I denotes the sum of all ideals of $T(V)$ that are generated by homogeneous elements of degree ≥ 2 and that are coideals. If $B(V)$ is finite-dimensional, then the matrix (a_{ij}) defined by for all $1 \leq i \neq j \leq \theta$ by $a_{ii} := 2$ and $a_{ij} := -\min\{r \in \mathbb{N} \mid q_{ij} q_{ji} q_{ii}^r = 1 \text{ or } (r+1)_{q_{ii}} = 0\}$ is a generalized Cartan matrix fulfilling $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ or $ord\, q_{ii} = 1 - a_{ij}$. (a_{ij}) is called the *Cartan* *matrix* associated to $B(V)$. To examine the defining relations of $B(V)$, we use the technique introduced by Nichols [1] and the following proposition [3]. For all $1 \le i \le \theta$, let $\sigma_i : B(V) \to B(V)$ be the algebra automorphism given by the action of g_i. If $\sigma : B(V) \to B(V)$ is an algebra automorphism, then an (id, σ)-derivation $D : B(V) \to B(V)$ is a $\mathbb{K}$-linear map such that $D(xy) = D(x)\sigma(y) + xD(y)$, for all $x, y \in B(V)$.

Proposition 2.1 ([3])**.** (1) *For all* $1 \le i \le \theta$, *there exists a uniquely determined* (id, σ)*-derivation* $D_i : B(V) \to B(V)$ *with* $D_i(x_j) = \delta_{ij}$ (*Kronecker* δ) *for all j*.

(2) $\bigcap_{i=1}^{\theta} \ker(D_i) = \mathbb{K}1.$

Let $B(V)$ be a Nichols algebra with Cartan matrix $(a_{ij}) = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ of type B_2. From the results of Helbig [2], let $q_{11} = -\zeta^3$, $q_{12}q_{21} = \zeta$, $q_{22}^{-1} = -1$, $\begin{pmatrix} -\zeta^3 & \zeta & -1 \\ \circ & \text{———} & \circ \end{pmatrix}$. Then $B(V) = T(V)/([x_1x_1x_2x_1x_2], x_1^4, x_2^2)$ with basis

$$\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}[x_1x_1x_1x_2]^{r_{1112}}x_1^{r_1} \mid 0 \le r_1 < 4,$$

$$0 \le r_{12}, r_{112} < 3, 0 \le r_{1112}, r_2 < 2\}$$

and $\dim_{\mathbb{K}} B(V) = 144.$ Using this, we obtain the following:

Proposition 2.2. *Let* $(q_{ij}) = \begin{pmatrix} -\zeta^3 & -\zeta \\ -1 & -1 \end{pmatrix}$, (*type* B_2) (*where* ζ *is a primitive root of unity of order* 12). *Then the Nichols algebra* $B(V)$ *is described as follows*:

Generators: x_1, x_2.

Relations: $x_1^4 = 0$, $x_2^2 = 0$,

$$x_1^2x_2x_1x_2 + (1 + \zeta + \zeta^2)x_1x_2x_1x_2x_1 + \zeta^2x_2x_1x_2x_1^2 + (\zeta + \zeta^2)x_2x_1^2x_2x_1$$

$$+ (1 + \zeta)x_1x_2x_1^2x_2 + \zeta x_2x_1^3x_2 = 0.$$

Its basis is given as follows:

$$\{1,\ x_1,\ x_2,\ x_1^2,\ x_1x_2,\ x_2x_1,\ x_1^2x_2,\ x_2x_1^2,\ x_1x_2x_1,\ x_2x_1x_2,\ x_1^3x_2,\ x_1^2x_2x_1,\ x_1x_2x_1^2,$$

$$(x_1x_2)^2,\ (x_2x_1)^2,\ x_2x_1^2x_2,\ x_2x_1^3,\ (x_1x_2)^2x_1,\ (x_2x_1)^2x_1,\ x_2x_1^2x_2x_1,\ x_1x_2x_1^2x_2,$$

$$x_2x_1^3x_2,\ x_1^2x_2x_1^2,\ x_1x_2x_1^3,\ (x_2x_1)^2x_2,\ x_1^3x_2x_1,\ (x_1x_2)^3,\ x_1(x_2x_1)^3x_1,\ x_1x_2x_1^2x_2x_1,$$

$$(x_1^2x_2)^2,\ (x_2x_1^2)^2,\ x_1x_2x_1^3x_2,\ x_2x_1^3x_2x_1,\ x_1^3x_2x_1^2,\ x_1^2x_2x_1^3,$$

$$(x_2x_1)^2x_1x_2,\ (x_2x_1)^2x_1^2,\ (x_1x_2)^3x_1,\ (x_1^2x_2)^2x_1,\ (x_2x_1)^2x_1x_2x_1,\ x_2(x_1^2x_2)^2,$$

$$(x_2x_1^2)^2x_1,\ x_2x_1^3x_2x_1^2,\ x_1^2x_2x_1^3x_2,\ x_1(x_2x_1^2)^2,\ (x_1x_2)^2x_1^3,\ x_1x_2x_1^3x_2x_1,$$

$$(x_2x_1)^2x_1^2x_2,\ x_1^3x_2x_1^3,\ (x_1x_2)^2x_1^2x_2,\ (x_1x_2)^3x_1^2,\ (x_1^2x_2)^2x_1^2,\ (x_2x_1^2)^2x_2x_1,$$

$$(x_1x_2)^2x_1^2x_2x_1,\ x_1x_2(x_1^2x_2)^2,\ x_1x_2x_1^2x_2x_1^3,\ x_1x_2x_1^3x_2x_1^2,$$

$$(x_2x_1^2)^2x_1x_2,\ x_1^2x_2x_1^3x_2x_1,\ (x_2x_1^3)^2,\ x_2x_1(x_2x_1^2)^2,\ x_2x_1(x_1^2x_2)^2,$$

$$x_1(x_1^2x_2)^2x_1,\ (x_1x_2)^2x_1^3x_2,\ (x_1x_2)^4,\ (x_1x_2)^4x_1,\ x_1^2x_2x_1^3x_2x_1^2,$$

$$x_1(x_2x_1^3)^2,\ (x_2x_1^2)^3,\ x_1(x_2x_1^2)^2x_2x_1,\ (x_2x_1)^3x_1x_2x_1,\ x_1x_2x_1(x_2x_1^2)^2,$$

$$x_1x_2x_1^2x_2x_1^3x_2,\ x_1x_2x_1^3x_2x_1^2x_2,\ x_1^2x_2x_1^3x_2x_1^2,\ (x_2x_1)^3x_1^2x_2,$$

$$(x_1^2x_2)^3,\ (x_2x_1)^2x_1x_2x_1^3,\ x_2x_1^3x_2x_1^2x_2x_1,\ (x_1^3x_2)^2x_1^2,\ (x_2x_1^3)^2x_1,$$

$$x_1x_2(x_1^3x_2)^2,\ x_2x_1^2x_2x_1^3x_2x_1^2,\ x_1x_2x_1^2x_2x_1^3x_2x_1,\ x_1^2(x_2x_1^3)^2,\ x_2x_1(x_2x_1^3)^2,$$

$$x_2x_1x_2x_1^2x_2x_1^3x_2,\ x_1x_2x_1(x_2x_1^3)^2,\ (x_1x_2)^2x_1^2x_2x_1^3,\ (x_2x_1)^2(x_2x_1^2)^2,$$

$$(x_1x_2)^2(x_1^2x_2)^2,\ (x_2x_1)^3x_1^2x_2x_1,\ x_2x_1x_2(x_1^2x_2)^2x_1,\ (x_2x_1^2)^3x_2,$$

$$(x_2x_1^3)^2x_2x_1^2,\ x_1(x_2x_1^3)^2x_2x_1,\ x_1x_2x_1^2x_2x_1^3x_2x_1^2,\ x_1^2(x_2x_1^3)^2x_2,$$

$$x_2x_1^2(x_2x_1^3)^2,\ x_2x_1(x_2x_1^3)^2x_2,\ x_1x_2x_1(x_2x_1^3)^2,\ (x_2x_1)^3x_1x_2x_1^3,$$

$$(x_1x_2)^2x_1^2x_2x_1^3x_2,\ x_2x_1x_2x_1^2x_2x_1^3x_2x_1,\ x_2x_1(x_2x_1^2)^3,\ (x_1x_2)^2(x_1^2x_2)^2x_1,$$

$(x_2x_1)^2(x_2x_1^2)^2x_2,\ x_1x_2(x_1^3x_2)^2x_1^2,\ x_1^2x_2(x_1^3x_2)^2x_1,\ x_2x_1^2x_2(x_1^3x_2)^2,$

$x_1x_2x_1^2(x_2x_1^3)^2,\ (x_2x_1)^2(x_2x_1^3)^2,\ (x_2x_1)^3x_1x_2x_1^3x_2,\ x_2x_1x_2(x_1^3x_2)^2x_1,$

$x_2x_1x_2x_1^2x_2x_1^3x_2x_1^2,\ (x_2x_1)^2x_1^2x_2x_1^3x_2x_1,\ x_1x_2x_1(x_2x_1^2)^3,\ (x_2x_1)^2(x_2x_1^2)^2x_2x_1,$

$x_1^2(x_2x_1^3)^2x_2x_1^2,\ x_2x_1^2(x_2x_1^3)^2x_2x_1,\ x_1x_2x_1^2(x_2x_1^3)^2x_2,\ (x_2x_1)^2(x_2x_1^3)^2x_2,$

$x_1x_2x_1(x_2x_1^3)^2x_2x_1,\ (x_1x_2)^2x_1^2x_2x_1^3x_2x_1^2,\ (x_2x_1)^2(x_2x_1^2)^3,$

$(x_1x_2)^2(x_1^2x_2)^2x_1^3,\ x_2x_1x_2(x_1^3x_2)^2x_1^2,\ x_2x_1^2x_2(x_1^3x_2)^2x_1^2,\ x_1x_2x_1^2x_2(x_1^3x_2)^2x_1,$

$x_2x_1x_2x_1^2(x_2x_1^3)^2x_2,\ (x_1x_2)^2(x_1^3x_2)^2x_1^2,\ (x_1x_2)^2x_1^2(x_2x_1^3)^2,\ (x_2x_1)^3x_1x_2x_1^3x_2x_1^2,$

$(x_2x_1)^2(x_2x_1^2)^2x_2x_1^3,\ x_1x_2x_1^2x_2(x_1^3x_2)^2x_1^2,\ x_2x_1x_2x_1^2x_2(x_1^3x_2)^2x_1,$

$(x_1x_2)^2x_1^2x_2(x_1^3x_2)^2,\ (x_2x_1)^3x_1(x_2x_1^3)^2,\ x_2x_1x_2x_1^2x_2(x_1^3x_2)^2x_1^2,$

$(x_1x_2)^2x_1^2x_2(x_1^3x_2)^2x_1,\ (x_2x_1)^3x_1x_2(x_1^3x_2)^2,\ (x_1x_2)^2x_1^2x_2(x_1^3x_2)^2x_1^2,$

$(x_2x_1)^3x_1x_2(x_1^3x_2)^2x_1,\ (x_2x_1)^3x_1x_2(x_1^3x_2)^2x_1^2\}.$

Hence, the Hilbert polynomial of $B(V)$ *is*

$$P(t) = 1 + 2t + 3t^2 + 4t^3 + 7t^4 + 9t^5 + 11t^6 + 13t^7 + 15t^8 + 14t^9 + 15t^{10} + 13t^{11}$$
$$+ 11t^{12} + 9t^{13} + 7t^{14} + 4t^{15} + 3t^{16} + 2t^{17} + t^{18}.$$

Proof. They are similarly shown as in [8-12]. □

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