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COEFFICIENT INEQUALITIES FOR INTEGRAL OPERATOR CONTAINING FOX-WRIGHT FUNCTION

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Abstract

In the present paper, we study the coefficient estimates for integral operator containing Fox-Wright function and concave univalent functions. The sharpness of these estimates is also investigated.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be the class of functions analytic in U and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = z + a_2 z^2 + \cdots$. For complex parameters $\alpha_1, \ldots, \alpha_q$ $\left(\frac{\alpha_j}{A_j} \neq 0, -1, -2, \ldots;\ j = 1, \ldots, q\right)$ and $\beta_1, \ldots, \beta_p$ $\left(\frac{\beta_j}{B_j} \neq 0, -1, -2, \ldots;\ j = 1, \ldots, p\right)$, the Fox-Wright generalization ${}_q\Psi_p[z]$ of the

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hypergeometric ${}_qF_p$ function, studied by the authors in [6]:

$$ {}_q\Psi_p[(\alpha_j,\, A_j)_{1,q};\, (\beta_j,\, B_j)_{1,p};\, z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q} \Gamma(\alpha_j + nA_j)}{\prod_{j=1}^{p} \Gamma(\beta_j + nB_j)} \frac{z^n}{n!}, $$

where $A_j > 0$, for all $j = 1, \ldots, q$, $B_j > 0$, for all $j = 1, \ldots, p$ and $1 + \sum_{j=1}^{p} B_j - \sum_{j=1}^{q} A_j \geq 0$. For special case, when $A_j = 1$, for all $j = 1, \ldots, q$, and $B_j = 1$, for all $j = 1, \ldots, p$ we have the following relationship:

$$ {}_qF_p(\alpha_1, \ldots, \alpha_q;\, \beta_1, \ldots, \beta_p;\, z) = \Omega_q\Psi_p[(\alpha_j,\, 1)_{1,q};\, (\beta_j,\, 1)_{1,p};\, z], $$

$$ q \leq p + 1;\; q,\, p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\; z \in U, \text{ where } \Omega := \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_p)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_q)}. $$

In [7], the authors introduced a function $(z_q\Psi_p[(\alpha_j,\, A_j)_{1,q};\, (\beta_j,\, B_j)_{1,p};\, z])^{-1}$ given by

$$ (z_q\Psi_p[(\alpha_j,\, A_j)_{1,q};\, (\beta_j,\, B_j)_{1,p};\, z]) * (z_q\Psi_p[(\alpha_j,\, A_j)_{1,q};\, (\beta_j,\, B_j)_{1,p};\, z])^{-1} $$

$$ = \frac{z}{(1-z)^{\lambda+1}} $$

and obtained the following integral operator:

$$ I_\lambda[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z) $$

$$ = (z_q\Psi_p[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z])^{-1} * f(z), \tag{1} $$

where $f \in \mathcal{A}$, $z \in U$, $\dfrac{\prod_{j=1}^{p} \Gamma(\beta_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j)} = 1$ and

$$ (z_q\Psi_p[(\alpha_j,\, A_j)_{1,q};\, (\beta_j,\, B_j)_{1,p};\, z])^{-1} $$

$$ = z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\beta_j + (n-1)B_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j + (n-1)A_j)} (\lambda + 1)_{n-1} z^n. $$

For some computation, we have

$$I_\lambda[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\beta_j + (n-1)B_j)}{\prod_{j=1}^{q}\Gamma(\alpha_j + (n-1)A_j)}(\lambda+1)_{n-1}a_n z^n, \tag{2}$$

where $(a)_n$ is the Pochhammer symbol. Note that operator (2) is a generalization of the one introduced by Noor [9].

In the theory of univalent functions the most important question is to find the coefficient estimates for

$$f(z) = z + \sum_{n=2}^{\infty} A_n(f)z^n \tag{3}$$

that are analytic and univalent in the unit disk $U := \{z : |z| < 1\}$. Let $Co(p)$ be the family of functions $f : U \to \overline{C}$, where $p \in (0, 1)$ that satisfy the following assumption:

Assumption (A):

(i) f is meromorphic in U and has a simple pole at the point p.

(ii) $f(0) = f'(0) - 1 = 0$.

(iii) f maps U conformally onto a set whose complement with respect to $\overline{C}$ is convex.

The family $Co(p)$ has been investigated recently in [1-4, 10]. In [8], Livingston introduced a necessary and sufficient condition for a function f to be in $Co(p)$:

$$\Re\left\{-(1+p^2) + 2pz - \frac{(z-p)(1-pz)f''(z)}{f'(z)}\right\} > 0, \quad \forall z \in U.$$

In [4], Avkhadiev and Wirths proved that for each $f \in Co(p)$ with the expansion in (3):

$$\left| A_n(f) - \frac{1-p^{2n+2}}{p^{n-1}(1-p^4)} \right| \le \left| \frac{p^2(1-p^{2n-2})}{p^{n-1}(1-p^4)} \right|$$

is valid. Equality is attained if and only if

$$f(z) = \frac{z - \frac{p}{1+p^2}(1+e^{i\theta})z^2}{\left(1-\frac{z}{p}\right)(1-zp)}. \tag{4}$$

In [5], Bhowmik and Pommerenke obtained certain coefficient estimates for functions

$$f(z) = \sum_{n=-1}^{\infty} a_n(f)(z-p)^n, \quad z \in \Delta, \tag{5}$$

where $\Delta := \{z \in C : |z-p| < 1-p\}$ and $p \in (0, 1)$,

$$\left| a_{n-2} - \frac{(1-p^2 a_{n-1})}{p} \right| \le \frac{p}{(1-p^4)(1-p)^{n-1}} \left[1 - \left(\frac{1-p^4}{p^4} \right) \left| a_{-1} + \frac{p^2}{1-p^4} \right|^2 \right], \; n \ge 3.$$

In this paper, we determine some estimates the real part of $A_n(f)$ for $n \ge 2$ and a_n for $n = 0, 1$ and $n \ge 2.$ For this purpose, we need the following result:

Theorem 1.1 [10]. *For each* $f \in Co(\mathrm{p}),$ *there exists a function* ω *holomorphic in U such that* $\omega(U) \subset \overline{U}$ *and*

$$f(z) = \frac{z - \frac{p}{1+p^2}(1+\omega(z))z^2}{\left(1-\frac{z}{p}\right)(1-zp)}, \quad z \in U. \tag{6}$$

2. Coefficient Estimates

In this section, we introduce some coefficient estimates for the integral operator (1). Denoted by

$$H_{n-1} := \frac{\prod_{j=1}^{p} \Gamma(\beta_j + (n-1)B_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j + (n-1)A_j)} (\lambda+1)_{n-1}.$$

Theorem 2.1. *Let* $p \in (0, 1)$ *and* $I_\lambda[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z) \in Co(\mathrm{p})$ *have the Laurent expansion*

$$I_\lambda[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z) = \sum_{n=-1}^{\infty} \mathbf{a}_n(f)(z-p)^n. \tag{7}$$

Then

$$|\mathbf{a}_{-1}| \le \frac{p^2}{1+p^2}. \tag{8}$$

Proof. Let $I_\lambda[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z) \in Co(\mathrm{p})$. Then by Theorem 1.1, there exist a function $\omega(z)$ holomorphic in U and $\omega(U) \subset \overline{U}$ satisfying (6). Assume that

$$\omega(z) = \sum_{n=0}^{\infty} c_n(z-p)^n, \quad z \in \Delta \tag{9}$$

with $|c_0| < 1$. By using these two expansions (7) and (9),

$$(1+p^2)\left(z - \frac{1}{p}\right)\sum_{n=-1}^{\infty} \mathbf{a}_n(z-p)^n = \left(1 + \frac{p}{z-p}\right)\left[1 + p^2 - pz\sum_{n=0}^{\infty} c_n(z-p)^n\right]. \tag{10}$$

Comparing the coefficient of $z(z-p)^{-1}$ on both sides of (10), yields

$$\mathbf{a}_{-1} = \frac{-p^2}{1+p^2}c_0, \tag{11}$$

we pose the assertion (7).

Theorem 2.2. *Let* $p \in (0, (\sqrt{5}-1)/2)$ *and* $I_\lambda[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z) \in Co(p)$ *have the Laurent expansion* (7). *Then*

$$|\mathbf{a}_0| \le \frac{p}{1-p^4}. \tag{12}$$

Proof. By using [5, Theorem 1.1] and Theorem 2.1.

For $n \geq 3$, we have the following result:

Theorem 2.3. *Let* $p \in (0, 1)$ *and* $I_\lambda[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z) \in Co(p)$ *have the Laurent expansion* (7). *Then*

$$\left| \mathbf{a}_{n-2} - \frac{(1-p^2\mathbf{a}_{n-1})}{p} \right|$$

$$\leq \frac{p}{(1-p^4)(1-p)^{n-1}} \left| 1 - \left(\frac{1-p^4}{p^4}\right)^2 \left(\frac{p^2}{1+p^2} + \frac{p^2}{1-p^4}\right)^2 \right|. \tag{13}$$

Proof. By using [5, Theorem 1.2] and Theorem 2.1.

Theorem 2.4. *Let* $p \in (0, (\sqrt{5}-1)/2)$ *and* $I_\lambda[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z)$ $\in Co(p)$ *have the Laurent expansion* (7). *Then for* $n \geq 3$,

$$| \mathbf{A}_n(I_\lambda[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z)) |$$

$$\leq \frac{p^2}{(1+p^2)(1-p)} + \frac{p}{1-p^4}$$

$$+ (1-p)^n \left[\frac{p}{(1-p^4)(1-p)^{n-1}} \left| 1 - \left(\frac{1-p^4}{p^4}\right)^2 \left(\frac{p^2}{1+p^2} + \frac{p^2}{1-p^4}\right)^2 \right| \right], \tag{14}$$

where

$$\mathbf{A}_n(I_\lambda[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}]f(z)) := H_{n-1}a_n.$$

Proof. Equating the right sides of (2) and (7), assuming $|z| < 1$ and $|z - p| < 1 - p$. Then applying Theorems 2.1, 2.2 and 2.3.

Corollary 2.1. *Let the assumptions of Theorem* 2.4 *hold*. *Then* $n \geq 3$,

$$|a_n(f)| \leq \left[\frac{p^2}{(1+p^2)(1-p)} + \frac{p}{1-p^4} + (1-p)^n \right.$$

$$\left. \times \left[\frac{p}{(1-p^4)(1-p)^{n-1}} \left| 1 - \left(\frac{1-p^4}{p^4}\right)^2 \left(\frac{p^2}{1+p^2} + \frac{p^2}{1-p^4}\right)^2 \right| \right] \right] \Big/ H_{n-1}. \tag{15}$$

Next we consider the class $\mathcal{I}$ of all functions F of the form (2) and belongs to the family $Co(p)$, where $p \in (0, 1)$.

Let F and G be analytic in the unit disk U. Then the function F is *subordinate* to G, written $F \prec G$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. In more general case, given two functions $F(z)$ and $G(z)$, which are analytic in U, the function $F(z)$ is said to be *subordination* to $G(z)$ in U if there exists a function $\rho(z)$, analytic in U with $\rho(0) = 0$ and $|\rho(z)| < 1$ such that $F(z) = G(\rho(z))$.

Let φ be analytic in U of the form (2) and subordinate to the function $f \in Co(p)$. Then we have the following result:

Theorem 2.5. *Let* $p \in (0, 1)$. *If* $F \in \mathcal{I}$ *and* $\varphi \prec F$, *where* φ *has the form* (2), *then*

$$|a_n(\varphi)| < \frac{(1+p)^2 + p\left[1 + \dfrac{4p}{1+p^2}\right]}{(1+p)^2 H_{n-1}}, \quad \forall n \geq 2. \tag{16}$$

Proof. Since $\varphi \prec F$, there exists a function $\rho(z)$ satisfies $\rho(0) = 0$, $|\rho(z)| < 1$, holomorphic in U and $\varphi(z) = F(\rho(z))$. Also, since $F \in Co(p)$, by Theorem 1.1, F satisfies (6). Now replacing z by $\rho(z)$ in (6), putting $\varphi(z) = F(\rho(z))$ and using the fact that $|\omega(\rho(z))| < e$ with $|z| = 1$, yield the assertion (16).

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