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A CLASS OF NEW NONLINEAR DIFFERENCE INEQUALITY WITH TWO VARIABLES

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Abstract

The main objective of this paper is to establish a class of new nonlinear difference inequality with two variables, which provides explicit bounds on unknown functions. This inequality given here can be used as tools in the study of partial difference equations with the initial and boundary conditions.

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1. Introduction

Being important tools in the study of difference equations, some discrete versions of integral inequalities (e.g., in [1, 2, 4, 6, 8, 10, 12] and some references therein) have attracted great interests of many mathematicians. Some recent works can be found (e.g., in [3, 5, 7-9, 11, 13] and some references therein). Pachpatte [8] obtained an upper bound on the following inequality:

$$u^2(n) \le \left(c_1^2 + 2\sum_{s=0}^{n-1} f(s)u(s)\right)\left(c_2^2 + 2\sum_{s=0}^{n-1} h(s)u(s)\right).$$

However, the bound given on such inequality in [8] is not directly applicable in the study of certain difference equations. It is desirable to establish new inequalities of the above type, which can be used more effectively in the study of certain classes of difference equations.

In this paper, we establish a new difference inequality

$$\psi(u(m, n)) \le \left(c_1(m, n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_1(s, t)\varphi(u(s, t))\right) \times \left(c_2(m, n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s, t)\varphi(u(s, t))\right). \tag{1.1}$$

2. Main Result

Throughout this paper, m_0, m_1, n_0, n_1 are given natural numbers. $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$, $\mathbb{N}_+ := \{1, 2, 3, \ldots\}$, $I := [m_0, m_1] \cap \mathbb{N}_+$, $I_m := [m_0, m] \cap \mathbb{N}_+$, $J := [n_0, n_1] \cap \mathbb{N}_+$, $J_n := [n_0, n] \cap \mathbb{N}_+$, $\mathbb{R}_+ := [0, \infty)$. For functions $z(m, n)$, $m, n \in \mathbb{N}$, its first-order differences are defined by $\Delta_1 z(m, n) = z(m+1, n) - z(m, n)$. Obviously, the linear difference equation $\Delta u(m) = b(m)$ with the initial condition $u(m_0) = 0$ has the solution $\sum_{s=m_0}^{m-1} b(s)$. For convenience, in the sequel, we complementarily define that $\sum_{s=m_0}^{m_0-1} b(s) = 0$.

(H_1) $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a strictly increasing function with $\psi(0) = 0$ and $\psi(t) \to \infty$ as $t \to \infty$;

(H_2) $c_1, c_2 : I \times J \to (0, \infty)$ are nondecreasing in each variable;

(H_3) $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing with $\varphi(r) > 0$ for $r > 0$;

(H_4) $f_i \in C(I \times J, \mathbb{R}_+)$, $i = 1, 2$.

Theorem 1. *Suppose that* (H_1)-(H_4) *hold and* $u(m, n)$ *is a nonnegative and continuous function on* $I \times J$ *satisfying* (1.1). *Then we obtain*

$$u(m, n) \leq \psi^{-1}(\Phi^{-1}(\Omega^{-1}(A(m, n)))), \tag{2.1}$$

for all $(m, n) \in I_{M_1} \times J_{N_1}$, *where*

$$\Phi(r) = \int_1^r \frac{ds}{\varphi(\psi^{-1}(s))}, \quad r > 0, \quad \Phi(0) := \lim_{r \to 0^+} \Phi(r), \tag{2.2}$$

$$\Omega(r) = \int_1^r \frac{ds}{\varphi(\psi^{-1}(\Phi^{-1}(s)))}, \quad r > 0, \quad \Omega(0) := \lim_{r \to 0^+} \Omega(r), \tag{2.3}$$

$$\begin{aligned} A(m, n) = \Omega(B(m, n)) &+ \sum_{s=m_0}^{m-1} \left(\sum_{t=n_0}^{n-1} f_1(s, t) \sum_{\sigma=m_0}^{s-1} \sum_{t=n_0}^{n-1} f_2(\sigma, t) \right) \\ &+ \sum_{s=m_0}^{m-1} \left(\sum_{t=n_0}^{n-1} f_2(s, t) \sum_{\sigma=m_0}^{s} \sum_{t=n_0}^{n-1} f_1(\sigma, t) \right), \end{aligned} \tag{2.4}$$

$$\begin{aligned} B(m, n) = {} & \Phi(c_1(m, n) c_2(m, n)) \\ &+ c_2(m, n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f_1(s, t) + c_1(m, n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f_2(s, t), \end{aligned} \tag{2.5}$$

ψ^{-1}, Φ^{-1} *and* Ω^{-1} *denote the inverse functions of* ψ, Φ *and* Ω, *respectively, and* $M_1 \in I$, $N_1 \in J$ *satisfy*

$$A(M_1, N_1) \in Dom(\Omega^{-1}), \quad \Omega^{-1}(A(M_1, N_1)) \in Dom(\Phi^{-1}). \tag{2.6}$$

Proof. From assumption (H_2) and the inequality (1.1), we have

$$\psi(u(m, n)) \le \left(c_1(M, n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_1(s, t)\varphi(u(s, t))\right)$$

$$\times\left(c_2(M, n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s, t)\varphi(u(s, t))\right), \tag{2.7}$$

for all $(m, n) \in I_M \times J$, where $m_0 \le M \le M_1$ is chosen arbitrarily, M_1 is defined by (2.6). Define a function $\eta(m, n)$ by the right hand side of (1.1), i.e.,

$$\eta(m, n) = \left(c_1(M, n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_1(s, t)\varphi(u(s, t))\right)$$

$$\times\left(c_2(M, n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s, t)\varphi(u(s, t))\right). \tag{2.8}$$

Clearly, $\eta(m_0, n) = c_1(M, n)c_2(M, n) > 0$, $\eta(m, n)$ is a positive and nondecreasing function in each variable. From the above equality (2.8) and by making use of the formula

$$\Delta_1[a(m, n)b(m, n)] = \Delta_1 a(m, n)b(m, n) + a(m+1, n)\Delta_1 b(m, n)$$

and using the fact that

$$u(m, n) \le \psi^{-1}(\eta(m, n)),$$

we obtain

$$\Delta_1\eta(m, n)$$

$$= \left(\sum_{t=n_0}^{n-1} f_1(m, t)\varphi(u(m, t))\right)\left(c_2(M, n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s, t)\varphi(u(s, t))\right)$$

$$+ \left(\sum_{t=n_0}^{n-1} f_2(m, t)\varphi(u(m, t))\right)\left(c_1(M, n) + \sum_{s=m_0}^{m}\sum_{t=n_0}^{n-1} f_1(s, t)\varphi(u(s, t))\right)$$

$$\leq \varphi(\psi^{-1}(\eta(m, n)))\left[\left(\sum_{t=n_0}^{n-1} f_1(m, t)\right)\left(c_2(M, n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s, t)\varphi(\psi^{-1}(\eta(s, t)))\right)\right.$$

$$\left. + \left(\sum_{t=n_0}^{n-1} f_2(m, t)\right)\left(c_1(M, n) + \sum_{s=m_0}^{m}\sum_{t=n_0}^{n-1} f_1(s, t)\varphi(\psi^{-1}(\eta(s, t)))\right)\right], \tag{2.9}$$

for all $(m, n) \in I_M \times J$. From (2.9), we get

$$\frac{\Delta_1\eta(m, n)}{\varphi(\psi^{-1}(\eta(m, n)))}$$

$$\leq \left(\sum_{t=n_0}^{n-1} f_1(m, t)\right)\left(c_2(M, n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s, t)\varphi(\psi^{-1}(\eta(s, t)))\right)$$

$$+ \left(\sum_{t=n_0}^{n-1} f_2(m, t)\right)\left(c_1(M, n) + \sum_{s=m_0}^{m}\sum_{t=n_0}^{n-1} f_1(s, t)\varphi(\psi^{-1}(\eta(s, t)))\right). \tag{2.10}$$

On the other hand, for arbitrarily given $(m, n), (m+1, n) \in I_M \times J$, by the Mean Value Theorem for integrals, there exists ξ in the open interval $(\eta(m, n), \eta(m+1, n))$ such that

$$\Phi(\eta(m+1, n)) - \Phi(\eta(m, n))$$

$$= \int_{\eta(m,n)}^{\eta(m+1,n)} \frac{ds}{\varphi(\psi^{-1}(s))} = \frac{\Delta_1\eta(m, n)}{\varphi(\psi^{-1}(\xi))} \leq \frac{\Delta_1\eta(m, n)}{\varphi(\psi^{-1}(\eta(m, n)))}, \tag{2.11}$$

where Φ is defined by (2.2). From (2.10) and (2.11), we have

$$\Phi(\eta(m+1, n)) \leq \Phi(\eta(m, n))$$

$$+ \left(\sum_{t=n_0}^{n-1} f_1(m, t)\right)\left(c_2(M, n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s, t)\varphi(\psi^{-1}(\eta(s, t)))\right)$$

$$+ \left(\sum_{t=n_0}^{n-1} f_2(m, t)\right)\left(c_1(M, n) + \sum_{s=m_0}^{m}\sum_{t=n_0}^{n-1} f_1(s, t)\varphi(\psi^{-1}(\eta(s, t)))\right), \tag{2.12}$$

for $(m, n), (m+1, n) \in I_M \times J$. Keep n fixed, let $s = m$ in (2.12) and then, taking the sum on both sides of (2.12) over $s = m_0, m_0+1, m_0+2, \ldots, m-1$, we get

$$\begin{aligned}
\Phi(\eta(m, n)) &\le \Phi(\eta(m_0, n)) \\
&\quad + c_2(M, n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_1(s, t) + c_1(M, n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s, t) \\
&\quad + \sum_{s=m_0}^{m-1}\left(\sum_{t=n_0}^{n-1} f_1(s, t)\sum_{\sigma=m_0}^{s-1}\sum_{t=n_0}^{n-1} f_2(\sigma, t)\varphi(\psi^{-1}(\eta(\sigma, t)))\right) \\
&\quad + \sum_{s=m_0}^{m-1}\left(\sum_{t=n_0}^{n-1} f_2(s, t)\sum_{\sigma=m_0}^{s}\sum_{t=n_0}^{n-1} f_1(\sigma, t)\varphi(\psi^{-1}(\eta(\sigma, t)))\right) \\
&\le \Phi(c_1(M, n)c_2(M, n)) + c_2(M, n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{n-1} f_1(s, t) \\
&\quad + c_1(M, n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{n-1} f_2(s, t) \\
&\quad + \sum_{s=m_0}^{m-1}\left(\sum_{t=n_0}^{n-1} f_1(s, t)\sum_{\sigma=m_0}^{s-1}\sum_{t=n_0}^{n-1} f_2(\sigma, t)\varphi(\psi^{-1}(\eta(\sigma, t)))\right) \\
&\quad + \sum_{s=m_0}^{m-1}\left(\sum_{t=n_0}^{n-1} f_2(s, t)\sum_{\sigma=m_0}^{s}\sum_{t=n_0}^{n-1} f_1(\sigma, t)\varphi(\psi^{-1}(\eta(\sigma, t)))\right), \qquad (2.13)
\end{aligned}$$

for $(m, n) \in I_M \times J$. Let

$$\begin{aligned}
D(M, n) = \Phi(c_1(M, n)c_2(M, n)) &+ c_2(M, n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{n-1} f_1(s, t) \\
&+ c_1(M, n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{n-1} f_2(s, t). \qquad (2.14)
\end{aligned}$$

Define the function $\Theta(m, n)$ by the right hand side of (2.13). Clearly, $\Theta(m, n)$ is a positive and nondecreasing function in each variable, $\Theta(m_0, n) = D(M, n) > 0$.

Using $\eta(m, n) \leq \Phi^{-1}(\Theta(m, n))$, by the definition $\Theta(m, n)$, we obtain

$$\begin{aligned}
\Delta_1\Theta(m, n) &= \sum_{t=n_0}^{n-1} f_1(m, t) \sum_{\sigma=m_0}^{m-1} \sum_{t=n_0}^{n-1} f_2(\sigma, t)\varphi(\psi^{-1}(\eta(\sigma, t))) \\
&\quad + \sum_{t=n_0}^{n-1} f_2(m, t) \sum_{\sigma=m_0}^{m} \sum_{t=n_0}^{n-1} f_1(\sigma, t)\varphi(\psi^{-1}(\eta(\sigma, t))) \\
&\leq \varphi(\psi^{-1}(\Phi^{-1}(\Theta(m, n)))) \left(\sum_{t=n_0}^{n-1} f_1(m, t) \sum_{\sigma=m_0}^{m-1} \sum_{t=n_0}^{n-1} f_2(\sigma, t) \right. \\
&\quad \left. + \sum_{t=n_0}^{n-1} f_2(m, t) \sum_{\sigma=m_0}^{m} \sum_{t=n_0}^{n-1} f_1(\sigma, t) \right), \qquad (2.15)
\end{aligned}$$

for all $(m, n) \in I_M \times J_{N_1}$, N_1 is defined by (2.6). From (2.15), we have

$$\begin{aligned}
&\frac{\Delta_1\Theta(m, n)}{\varphi(\psi^{-1}(\Phi^{-1}(\Theta(m, n))))} \\
&\leq \sum_{t=n_0}^{n-1} f_1(m, t) \sum_{\sigma=m_0}^{m-1} \sum_{t=n_0}^{n-1} f_2(\sigma, t) + \sum_{t=n_0}^{n-1} f_2(m, t) \sum_{\sigma=m_0}^{m} \sum_{t=n_0}^{n-1} f_1(\sigma, t). \qquad (2.16)
\end{aligned}$$

Similar to (2.10) to (2.13), we obtain

$$\begin{aligned}
\Omega(\Theta(m, n)) &\leq \Omega(\Theta(m_0, n)) \\
&\quad + \sum_{s=m_0}^{m-1} \left(\sum_{t=n_0}^{n-1} f_1(s, t) \sum_{\sigma=m_0}^{s-1} \sum_{t=n_0}^{n-1} f_2(\sigma, t) \right) \\
&\quad + \sum_{s=m_0}^{m-1} \left(\sum_{t=n_0}^{n-1} f_2(s, t) \sum_{\sigma=m_0}^{s} \sum_{t=n_0}^{n-1} f_1(\sigma, t) \right) \\
&= \Omega(D(M, n)) + \sum_{s=m_0}^{m-1} \left(\sum_{t=n_0}^{n-1} f_1(s, t) \sum_{\sigma=m_0}^{s-1} \sum_{t=n_0}^{n-1} f_2(\sigma, t) \right) \\
&\quad + \sum_{s=m_0}^{m-1} \left(\sum_{t=n_0}^{n-1} f_2(s, t) \sum_{\sigma=m_0}^{s} \sum_{t=n_0}^{n-1} f_1(\sigma, t) \right). \qquad (2.17)
\end{aligned}$$

Using the fact $u(m, n) \leq \psi^{-1}(\eta(m, n))$ and $\eta(m, n) \leq \Phi^{-1}(\Theta(m, n))$, from (2.17), we obtain

$$u(m, n) \leq \psi^{-1}(\eta(m, n)) \leq \psi^{-1}(\Phi^{-1}(\Theta(m, n)))$$

$$\leq \psi^{-1}\left(\Phi^{-1}\left(\Omega^{-1}\left(\Omega(D(M, n)) + \sum_{s=m_0}^{m-1}\left(\sum_{t=n_0}^{n-1} f_1(s, t) \sum_{\sigma=m_0}^{s-1} \sum_{t=n_0}^{n-1} f_2(\sigma, t)\right)\right.\right.\right.$$

$$\left.\left.\left. + \sum_{s=m_0}^{m-1}\left(\sum_{t=n_0}^{n-1} f_2(s, t) \sum_{\sigma=m_0}^{s} \sum_{t=n_0}^{n-1} f_1(\sigma, t)\right)\right)\right)\right), \quad \forall (m, n) \in I_M \times J_{N_1}. \tag{2.18}$$

Let $m = M$, from (2.18), we observe that

$$u(M, n) \leq \psi^{-1}\left(\Phi^{-1}\left(\Omega^{-1}\left(\Omega\left(\Phi(c_1(M, n)c_2(M, n)) + c_2(M, n)\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{n-1} f_1(s, t)\right.\right.\right.\right.$$

$$\left. + c_1(M, n)\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{n-1} f_2(s, t)\right) + \sum_{s=m_0}^{M-1}\left(\sum_{t=n_0}^{n-1} f_1(s, t) \sum_{\sigma=m_0}^{s-1} \sum_{t=n_0}^{n-1} f_2(\sigma, t)\right)$$

$$\left.\left.\left. + \sum_{s=m_0}^{M-1}\left(\sum_{t=n_0}^{n-1} f_2(s, t) \sum_{\sigma=m_0}^{s} \sum_{t=n_0}^{n-1} f_1(\sigma, t)\right)\right)\right)\right), \tag{2.19}$$

for all $(m, n) \in I_M \times J_{N_1}$. Since $M \in I_{M_1}$ is arbitrary, from (2.19), we get the required estimation (2.1).

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