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ON A CUBICALLY ITERATIVE SCHEME FOR SOLVING NONLINEAR EQUATIONS

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Abstract

An accurate iterative scheme for solving nonlinear equations is developed by considering a two-step cycle. The proposed method includes one evaluation of the function and two evaluations of the first derivative per iteration. The efficiency of the presented method is illustrated by two numerical examples.

1. Introduction

Assume that the scalar function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a simple zero in the real domain D. Finding the roots of nonlinear equation $f(x) = 0$, has various applications in science and technology [3]. In this paper, we consider the iterative methods to approximate such zeros, wherein α is the simple root and the first derivative of the function in a neighborhood of the simple root α does not vanish. The aim of this paper is to provide an efficient method which has less computational burden in contrast with other iterative schemes whilst we use two evaluations of the first derivative of the function per iteration. Unlike the other two-step methods which use two evaluations of the function and one evaluation of the first derivative, we consider

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the case with same functional evaluations, i.e., three evaluations per iteration, but with one evaluation of the function and two evaluations of the first derivative per iteration. Our aim is achieved in Section 3, by providing an estimation for the unknown quantity. When taking account of, a practical utility of any method, the study of its efficiency is required. Hence the efficiency index of the nonlinear equation solver which is defined by (whereas p is the order of convergence and n is whole functional evaluation per iteration for the method) $p^{1/n}$, provides 1.441 as the efficiency index of the presented method in this paper. Many researchers proposed new algorithms of different orders of convergence up to now. The higher order methods such as [8, 11, 13] are, in fact, three-step iterative methods. In this paper, we consider the two-step iterative methods because of some advantages of these methods. The main challenge for the three-step methods for finding the simple roots of nonlinear equations is high computational burden. That is, for such algorithms, the CPU run time is much more than two-step iterative methods. For this reason, we take into consideration the two-step iterative methods. Before constructing our method, we review some well- known methods in this field of study in the next section and in Section 3, we propose our iteration scheme. In Section 4, we provide some examples and finally, our conclusion has been given in Section 5.

2. Background Literature

The familiar Newton's method $x_{n+1} = x_n - f(x_n)/f'(x_n)$ for calculating of the simple root α converges quadratically whereas x_0 is an initial approximation sufficiently close to α. This method has the efficiency index equals to 1.413. Motivated by this method, many methods of higher orders have been presented to this field. The well-known Potra-Ptak method, which is a third-order method, is defined by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n) + f(y_n)}{f'(x_n)}. \end{cases} \tag{1}$$

Another famous third-order method, Weerakoon and Fernando iterative method [10] is given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{2f(x_n)}{f'(x_n) + f'(y_n)}. \end{cases} \tag{2}$$

As we can see, this method consists of three functional evaluations per iteration just like method (1), and consequently has the efficiency index equals to 1.441. Sharma in [5] produced another kind of cubically convergent scheme by composition of Newton's method and Steffensen's method. This scheme, also called *Newton-Steffensen scheme*, may be expressed as

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{(f(x_n))^2}{f'(x_n)(f(x_n) - f(y_n))}. \end{cases} \tag{3}$$

Newly a third-order method has been investigated by Fang et al. in [1], and it can be written in the following form:

$$\begin{cases} y_n = x_n + \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(y_n) - f(x_n)}{f'(x_n)}. \end{cases} \tag{4}$$

To see more on this matter, see [4, 6, 7]. For obtaining a geometric view-point of Newton's method, see [9], and for iterative methods in operator form or in Banach spaces, kindly see [2, 12], respectively. Motivated and inspired by the ongoing research about the iterative schemes for solving nonlinear equations, in this paper, we are concerned with the iterative method improving the Newton's method, and then we present a new interesting method. By analysis of convergence, we establish that the local order of convergence of the proposed method is three, and by numerical examples, we demonstrate its performance in comparison with other methods of the same order.

3. The Method and the Analysis of Convergence

Let us consider the Newton's method in two steps as follows:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(y_n)}. \end{cases} \tag{5}$$

We wish to present a method in which we have two evaluations of the first derivative of the function and one evaluation of the function. Hence we approximate the function f on the real domain D, as if $w(t) = a_1 + a_2(t - x) + a_3(t - x)^2$, i.e., the approximation function, meets the function $f(x)$ as follows: $f(x_n) = w(x_n)$, $f'(x_n) = w'(x_n)$ and $f'(y_n) = w'(y_n)$. As we can see, we have three known equations by writing the conditions for $w(t)$ and three unknown parameters a_1, a_2 and a_3. By solving a system of three linear equations and finding three unknown parameters, we obtain $a_1 = f(x_n)$, $a_2 = f'(x_n)$ and $a_3 = (f'(y_n) - f'(x_n))/2(y_n - x_n)$. Hence the method is reduced to the following form which carries three functional evaluations per iteration only:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{w(y_n)}{f'(y_n)}. \end{cases} \tag{6}$$

Theorem 1. *Assume that the function* $f : D \subseteq \mathbb{R} \to \mathbb{R}$ *has a single root* $\alpha \in D$, *where D is an open interval. If* $f(x)$ *has first, second and third derivatives in the interval D, then the method defined by* (6) *converges cubically to* α *in a neighborhood of the simple zero and satisfies the following error equation*:

$$e_{n+1} = \left(\frac{c_3}{2}\right)e_n^3 + O(e_n^4), \tag{7}$$

where $e_n = x_n - \alpha$, $c_k = f^{(k)}(\alpha)/(k!\,f'(\alpha))$, $k = 2, 3$.

Proof. Using Taylor's expansion and take into consideration $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)] \tag{8}$$

and

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3)]. \tag{9}$$

Dividing (8) by (9) yields

$$\frac{f(x_n)}{f'(x_n)} = [e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)][1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3)]^{-1}$$

$$= e_n - c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + O(e_n^4). \tag{10}$$

In the same way, we have $u_n = c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + O[e_n^4]$, where $u_n = y_n - \alpha$ and by using this relation, we get that

$$w(y_n) - \alpha = c_2 f'(\alpha)e_n^2 + \left(-2c_2^2 f'(\alpha) + \frac{3c_2 f'(\alpha)}{2}\right)e_n^3 + O[e_n^4], \tag{11}$$

and finally by obtaining the Taylor expansion about the simple root in the $(n+1)$th iterate gives us $e_{n+1} = x_{n+1} - \alpha = \left(\frac{c_3}{2}\right)e_n^3 + O(e_n^4)$, which shows that the order of convergence for our proposed method, is three.

4. Computational Aspects

We provide some numerical test results for different iterative methods in Tables 1 and 2. The following methods are compared: the Newton method (NM), Potra-Ptak method (PP), the method of Weerakoon and Fernando (WF), the method of Sharma (SH), Fang et al. third-order method (FA) and our presented iteration scheme (PM). All computations were done using Mathematica Ver. 6.1 with 16 digit floating point arithmetic $(\text{Digits} := 16)$. We accept an approximate solution rather than the exact root, depending on the precision (ε) of the computer. We use the following stopping criterion for computer programs: $|f_k(x_{n+1})| < \varepsilon.$ The test functions are as follows:

$f_1(x) = x\exp(x^2) - \sin^2(x) + 3\cos(x) + 5,\quad \alpha = -1.207647827130919,\quad x_0 = -1.19.$

$f_2(x) = \exp(-x^2 + x + 2) - 1,\quad \alpha = 2,\quad x_0 = 1.6.$

Table 1. The comparison of various methods for Example 1

Methods	$\vert x_1 - \alpha \vert$	$\vert x_2 - \alpha \vert$
NM	4.7(4)	3.3(7)
PP	2.5(5)	7.7(14)
WF	1.8(5)	1.9(14)
SH	1.2(5)	4.4(15)
FA	2.5(6)	0
PM	5.3(6)	0

Table 2. The comparison of various methods for Example 2

Methods	$\vert x_1 - \alpha \vert$	$\vert x_2 - \alpha \vert$
NM	1.0(1)	1.1(2)
PP	4.8(2)	2.5(4)
WF	3.4(2)	6.1(5)
SH	3.4(2)	5.0(5)
FA	5.8(2)	3.1(4)
PM	1.1(2)	4.1(7)

Tables 1 and 2, show the effectiveness of our proposed method. Our second derivative-free method performs better or equal in the examples considered in this paper.

5. Conclusion

In this paper, we have presented a new third-order method for solving single variable nonlinear equations. We observed from numerical examples that the proposed method has at least equal performance as compared with the other methods of the same order. Also, the practical utility of our method is good, the above-mentioned third-order method requires one function and two first derivative evaluations per iteration to improve the order of convergence and consequently, the efficiency of our method, measured by the efficiency index, is 1.441.

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