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NUMERICAL SOLUTION OF VOLTERRA INTEGRAL EQUATIONS BY USING CAS WAVELETS

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Abstract

This paper presents a computational technique for Volterra integral equations of the second kind type. The method is based on CAS wavelets. We give a general procedure of forming product operational matrix of CAS wavelets. CAS wavelets approximation method is utilized to reduce the Volterra integral equations to algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

1. Introduction

Wavelets theory is relatively new and an emerging area in mathematical research. Recently, wavelets have been applied in different fields of science and engineering. Wavelets permit the accurate representation of a variety of function and operator. Orthogonal functions and polynomial series have received considerable attention in solving various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem. The approach [2] is based on converting the underlying differential equations into integral equations through

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integration, approximating various signals involved in the equation by truncated orthogonal series and using the operational matrix P of integration, to eliminate integral operations. The matrix P is given by

$$\int_0^t \Phi(t')dt' \approx P\Phi(t),$$

where $\Phi(t) = [\phi_0, \phi_1, ..., \phi_{n-1}]^T$ and the matrix P can uniquely be determined on the basis of particular orthogonal functions. The elements $\phi_0, \phi_1, ..., \phi_{n-1}$ are the basic functions, orthogonal on the certain interval $[a, b]$. Special attention has been given to the application of Legendre wavelets [1] and the linear Legendre wavelets [2].

In this paper, we introduce a new numerical method to solve Volterra integral equations of the form

$$y(t) = f(t) + \int_0^t K(s, t)y(s)ds, \quad 0 \leq s, t \leq 1, \tag{1}$$

where $f(t) \in L^2[0, 1]$, kernels $K(s, t) \in L^2([0, 1] \times [0, 1])$ and $y(t)$ is an unknown function. This method reduces integral equations to a set of algebraic equations by expanding $y(t)$ as CAS wavelets with unknown coefficients. In recent years, many different basic functions have been used to estimate the solution of (1), such as Haar wavelets [7] and Legendre wavelets.

The paper is organized as follows: In Section 2, it describes the CAS wavelets. The CAS wavelets product operational matrix will be derived in Section 3. In Section 4, it describes the Volterra integrations of the second kind and the proposed method is used to approximate the unknown function $y(t)$. Finally, in Section 5, illustrative examples are given.

2. Properties of CAS Wavelets

2.1. Wavelets and CAS wavelets

Wavelets constitute a family of functions constructed from the dilation and translation of a single function called the *mother wavelet*. When the dilation parameter a and the translation parameter b vary continuously, the following family of continuous wavelets are obtained [5],

$$\varphi_{a,b} = |a|^{-\frac{1}{2}} \varphi\left(\frac{t-b}{a}\right), \quad a, b \in R, \; a \neq 0.$$

If the parameters a and b are restricted to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 0$ and n, k are positive integers, then we have the following family of discrete wavelets

$$\varphi_{k,n} = |a_0|^{\frac{k}{2}} \varphi(a_0^k t - nb_0),$$

where $\varphi_{k,n}$ forms a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\varphi_{k,n}$ forms an orthonormal basis [5].

CAS wavelets $\varphi_{mn}(t) = \varphi(k, m, n, t)$ have four arguments, $n = 0, 1, ..., 2^k - 1$, k can be any non-negative integer, m is any integer and t is the normalized time. They are defined in the interval $[0, 1)$ as [3],

$$\varphi_{mn}(t) = \begin{cases} 2^{k/2} \mathrm{CAS}_m(2^k t - n), & \text{for } \dfrac{n}{2^k} \leq t \leq \dfrac{n+1}{2^{K+1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where

$$\mathrm{CAS}_m(t) = \cos(2m\pi t) + \sin(2m\pi t). \tag{3}$$

The dilation parameter is $a = 2^{-k}$ and translation parameter is $b = 2^{-k} n$. The set of CAS wavelets forms an orthonormal basis for $L^2(R)$.

2.2. Function approximation

A function $f(t)$ defined over $[0, 1)$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m \in Z} c_{nm} \varphi_{nm}(t), \tag{4}$$

where

$$c_{nm} = (f(t), \varphi_{nm}(t)), \tag{5}$$

in which $(\cdot,\cdot)$ denotes the inner product. If the infinite series in equation (3) are truncated, then equation (3) can be written as

$$f(t) \simeq \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{nm}\varphi_{nm}(t) = C^T \psi(t), \tag{6}$$

where C and $\psi(t)$ are $2^k(2M+1)\times 1$ matrices given by

$$C = [c_{0(-M)}, \ldots, c_{0(M)}, c_{1(-M)}, \ldots, c_{1(M)}, \ldots, c_{2^k(-M)}, \ldots, c_{2^k(M)}]^T, \tag{7}$$

$$\psi(t) = [\varphi_{0(-M)}, \ldots, \varphi_{0(M)}, \varphi_{1(-M)}, \ldots, \varphi_{1(M)}, \ldots, \varphi_{2^k(-M)}, \ldots, \varphi_{2^k(M)}]^T. \tag{8}$$

3. CAS Wavelets Operational Matrix

3.1. CAS wavelets operational matrix of integration

The operational matrix of integration P has been derived in [4]. First for the 6×6 matrix P and $M=1$ and $k=1.$ The six basis functions are given by

$$\left.\begin{aligned} \varphi_{0(-1)}(t) &= \sqrt{2}(\cos(4m\pi t) - \sin(4m\pi t)) \\ \varphi_{00}(t) &= \sqrt{2} \\ \varphi_{01}(t) &= \sqrt{2}(\cos(4m\pi t) + \sin(4m\pi t)) \end{aligned}\right\}\ 0 \le t < \frac{1}{2}, \tag{9}$$

$$\left.\begin{aligned} \varphi_{1(-1)}(t) &= \sqrt{2}(\cos(4m\pi t) - \sin(4m\pi t)) \\ \varphi_{10}(t) &= \sqrt{2} \\ \varphi_{11}(t) &= \sqrt{2}(\cos(4m\pi t) + \sin(4m\pi t)) \end{aligned}\right\}\ \frac{1}{2} \le t < 1. \tag{10}$$

By integrating (7) and (8) from 0 to t and using (4), it can be obtained

$$\int_0^t \varphi_{0(-1)}(t')dt' = \begin{cases} \dfrac{\sqrt{2}}{4\pi}(\cos(4m\pi t) + \sin(4m\pi t) - 1), & 0 \le t < \dfrac{1}{2} \\ 0, & \dfrac{1}{2} \le t < 1 \end{cases}$$

$$\simeq \frac{1}{4\pi}(-\varphi_{00}(t) + \varphi_{01}(t)) = \left[0, -\frac{1}{4\pi}, \frac{1}{4\pi}, 0, 0, 0\right]\psi_6,$$

$$\int_0^t \varphi_{00}(t')dt' = \begin{cases} \sqrt{2}t, & 0 \le t < \dfrac{1}{2} \\ \dfrac{\sqrt{2}}{2}, & \dfrac{1}{2} \le t < 1 \end{cases}$$

$$\simeq \frac{1}{4\pi}\varphi_{0(-1)}(t) + \frac{1}{4\pi}\varphi_{00}(t) - \frac{1}{4\pi}\varphi_{01}(t) + \frac{1}{2}\varphi_{10}(t)$$

$$= \left[\frac{1}{4\pi}, \frac{1}{4}, -\frac{1}{4\pi}, 0, \frac{1}{2}, 0\right]\psi_6(t).$$

Similarly we have

$$\int_0^t \varphi_{01}(t')\,dt' \simeq \frac{1}{4\pi}(\varphi_{0(-1)}(t) + \varphi_{00}(t)) = \left[\frac{1}{4\pi}, \frac{1}{4\pi}, 0, 0, 0, 0\right]\psi_6(t),$$

$$\int_0^t \varphi_{1(-1)}(t')\,dt' \simeq \frac{1}{4\pi}(-\varphi_{10}(t) + \varphi_{11}(t)) = \left[0, 0, 0, 0, -\frac{1}{4\pi}, \frac{1}{4\pi}\right]\psi_6(t),$$

$$\int_0^t \varphi_{10}(t')\,dt' \simeq \frac{1}{4\pi}\varphi_{1(-1)}(t) + \frac{1}{4}\varphi_{10}(t) - \frac{1}{4\pi}\varphi_{11}(t) = \left[0, 0, 0, \frac{1}{4\pi}, \frac{1}{4}, -\frac{1}{4\pi}\right]\psi_6(t),$$

$$\int_0^t \varphi_{11}(t')\,dt' \simeq \frac{1}{4\pi}(\varphi_{1(-1)}(t) + \varphi_{10}(t)) = \left[0, 0, 0, 0, \frac{1}{4\pi}, \frac{1}{4\pi}\right]\psi_6(t).$$

Thus

$$\int_0^t \psi_{6\times 1}(t')\,dt' = P_{6\times 6}\psi_6(t), \tag{11}$$

where

$$\psi_6(t) = [\varphi_{0(-1)}, \varphi_{00}, \varphi_{01}, \varphi_{1(-1)}, \varphi_{10}, \varphi_{11}]^T$$

and

$$P_{6\times 6} = \frac{1}{4}\begin{bmatrix} 0 & -\frac{1}{\pi} & \frac{1}{\pi} & 0 & 0 & 0 \\ \frac{1}{\pi} & 1 & -\frac{1}{\pi} & 0 & 2 & 0 \\ \frac{1}{\pi} & \frac{1}{\pi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\pi} & \frac{1}{\pi} \\ 0 & 0 & 0 & \frac{1}{\pi} & 1 & -\frac{1}{\pi} \\ 0 & 0 & 0 & 0 & \frac{1}{\pi} & \frac{1}{\pi} \end{bmatrix}.$$

In (8), the subscript of $P_{6\times 6}$ and ψ_6 denotes the dimensions. In (8), the matrix $P_{6\times 6}$ can be written as

$$P_{6\times 6} = \begin{bmatrix} L_{3\times 3} & F_{3\times 3} \\ 0_{3\times 3} & L_{3\times 3} \end{bmatrix},$$

where

$$L_{3\times 3} = \begin{bmatrix} 0 & -\frac{1}{\pi} & \frac{1}{\pi} \\ \frac{1}{\pi} & 1 & -\frac{1}{\pi} \\ \frac{1}{\pi} & \frac{1}{\pi} & 0 \end{bmatrix}, \quad F_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In general, we have

$$\int_0^t \psi(t')\,dt' = P\psi(t), \tag{12}$$

where $\psi(t)$ is given in equation (6) and P is a $2^k(2M+1)\times 2^k(2M+1)$ matrix given by

$$P = \frac{1}{2^{k+1}} \begin{bmatrix} L & F & F & \cdots & F \\ 0 & L & F & \cdots & F \\ 0 & 0 & \ddots & \ddots & F \\ & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & L & F \\ 0 & 0 & 0 & \cdots & L \end{bmatrix},$$

where F and L are $(2M+1)\times(2M+1)$ matrix given by

$$F = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\frac{1}{M\pi} & 0 & \cdots & 0 & \frac{1}{M\pi} \\ 0 & 0 & \cdots & 0 & -\frac{1}{(M-1)\pi} & 0 & \cdots & \frac{1}{(M-1)\pi} & 0 \\ & \cdots & \cdots & & \cdots & & & & \\ 0 & 0 & \cdots & 0 & -\frac{1}{\pi} & \frac{1}{\pi} & \cdots & 0 & 0 \\ \frac{1}{\pi} & \frac{1}{\pi} & \cdots & \frac{1}{\pi} & 1 & \frac{1}{\pi} & \cdots & \frac{1}{\pi} & \frac{1}{\pi} \\ 0 & 0 & \cdots & \frac{1}{\pi} & \frac{1}{\pi} & 0 & \cdots & 0 & 0 \\ & \cdots & \cdots & & \cdots & & & & \\ 0 & \frac{1}{(M-1)\pi} & \cdots & 0 & \frac{1}{(M-1)\pi} & 0 & \cdots & 0 & 0 \\ \frac{1}{M\pi} & 0 & \cdots & 0 & \frac{1}{M\pi} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

3.2. The product operational matrix of CAS wavelets

Let

$$\psi(t)\psi^T(t) \simeq \tilde{C}\psi(t), \tag{13}$$

where $\tilde{C}$ is a $2^k(2M+1) \times 2^k(2M+1)$ product operational matrix. The 6×6 product operational matrix was given in [3]. In this paper, a general procedure of forming product operational matrix is proposed. In general, by using the vector C in equation (7), the $2^k(2M+1) \times 2^k(2M+1)$ matrix $\tilde{C}$ is

$$\tilde{C} = 2^{k/2} \begin{bmatrix} \tilde{C}_1 & & & \\ & \tilde{C}_2 & & \\ & & \ddots & \\ & & & \tilde{C}_{2^k} \end{bmatrix},$$

where $\tilde{C}_i$, $i = 1, \ldots, 2^k - 1$ are $(2M+1) \times (2M+1)$ matrices given by

$$\tilde{C}_i = \begin{bmatrix} \varphi_{i0} & & & \cdots & \varphi_{i(-M)} & \cdots & 0 \\ \frac{1}{2}\varphi_{i(-1)} + \frac{1}{2}\varphi_{i1} & & & \cdots & \varphi_{i(-M+1)} & \cdots & \frac{1}{2}\varphi_{i(-1)} - \frac{1}{2}\varphi_{i1} \\ \cdots & \cdots & \cdots & & & & \\ \frac{1}{2}\varphi_{i(-M+1)} - \frac{1}{2}\varphi_{i(M-1)} & & & \cdots & \varphi_{i(-1)} & \cdots & -\frac{1}{2}\varphi_{i(-M+1)} + \frac{1}{2}\varphi_{i(M-1)} \\ \varphi_{i(-M+2)} & & & \cdots & \varphi_{i0} & \cdots & \varphi_{i(M-2)} \end{bmatrix}.$$

4. Solution of the Volterra Integral Equations

Consider the Volterra integral equations given in equation (1). We first approximate $y(t)$ as

$$y(t) = C^T \psi(t), \quad f(t) = X^T \psi(t), \quad K(s, t) = \psi(t)^T K \psi(s),$$

where C and $\psi(t)$ are defined similarly to (7) and (8). K is $2^k(2M+1) \times 2^k(2M+1)$ matrix where the elements of K calculate as follows:

$$\int_0^1 \int_0^1 \psi_{ni}(t)\psi_{lj}(s) K(t, s) dtds, \quad n, l = 0, \ldots, 2^k - 1, i, j = -M, \ldots, M.$$

Then

$$C^T \psi(t) = X^T \psi(t) + \int_0^t \psi(t)^T K \psi(s) \psi(s)^T C ds. \tag{14}$$

Thus with equation (12) and equation (13), we have

$$C^T \psi(t) = X^T \psi(t) + \psi(t)^T K \tilde{C} P \psi(t). \tag{15}$$

Equation (15) is a linear system in terms of C by using collocating method.

5. Illustrative Examples

Example 1.1. Consider the Volterra integral equation

$$y(t) = (4\pi^2 - 1)\cos 2\pi t + 1 + \int_0^t (s - t) y(s) ds. \tag{16}$$

The exact solution of this problem is $y(t) = 4\pi^2 \cos(2\pi t)$.

Equation (16) can be solved using the method with $k = 0$ and $M = 1$ in Section 4, we have

$$X = \left[\frac{1}{2}(4\pi^2 - 1), -1, \frac{1}{2}(4\pi^2 - 1)\right]^T$$

and

$$K = \begin{bmatrix} 0 & \frac{1}{2\pi} & 0 \\ -\frac{1}{2\pi} & 0 & \frac{1}{2\pi} \\ 0 & \frac{1}{2\pi} & 0 \end{bmatrix},$$

and then $C = [2\pi^2, 0, 2\pi^2]^T$. Therefore

$$y(t) = C^T \psi(t) = 4\pi^2 \cos(2\pi t).$$

Example 1.2. Consider the Volterra integral equation

$$y(t) = \cos t - \int_0^t (t - s)\cos(t - s)\, y(s)\, ds. \tag{17}$$

The exact solution of this problem is $y(t) = \frac{1}{3}(2\cos(\sqrt{3}t) + 1)$.

Table 1. Comparison of CAS wavelets method and the method in [6]

	$k = 2, \ M = 1$	$k = 3, \ M = 2$
X	$\lvert y - \tilde{y} \rvert$	$\lvert y - \tilde{y} \rvert$
0.1	2.43117832e-02	1.11008211e-03
0.2	1.59182044e-02	4.68651323e-04
0.3	5.55352592e-03	5.08730892e-04
0.4	5.93105645e-02	7.55356316e-03
0.5	6.63233072e-02	2.31888592e-02
0.6	4.03928772e-02	1.09551714e-02
0.7	5.41201624e-03	1.04133232e-03
0.8	3.43154135e-02	6.94512700e-04
0.9	1.32045209e-02	1.00043250e-03

We solve equation (17) using the method with $k = 2, \ M = 1$ and $k = 3, \ M = 2$ in Section 4, respectively. Table 1 shows the numerical result of the example.

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