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# ON SUBNEXUSES OF NEXUSES RELATED TO FUZZY SETS

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**Abstract**

The notion of intuitionistic fuzzy subnexus of a nexus is introduced. Some characteristic properties and connections are investigated. Finally, some equivalence relations, constructed by intuitionistic fuzzy subnexuses, are discussed.

## 1. Introduction

The space structure research center of university of Surrey was founded by Z. S. Makoswski as a part of civil engineering in 1963. The aim of the center is to carry out research into the design and analysis of space structures. Space structures include structural forms such as single and double layer girds, barrel vaults, shells and various forms of tension structures. The basic idea of a nexus has been further developed as a mathematical object for general use. Some researchers are working on nexuses and its applications in architecture. One of the most famous of these researchers is K. Williams. She became interested in mathematics and architecture while writing “Italian Pavements: Patterns in Space (Houston: Anchorage Press, 1997)” about the role of decorated pavements in the history of Italian architecture.

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In 1996, she founded the international conference series “Nexus: Architecture and Mathematics”. She also published many articles on the use of mathematical principles in architecture, some of them are mentioned in [12-15].

Atanassov [1, 2] introduced intuitionistic fuzzy sets which constitute a generalization of the notion of fuzzy sets. Fuzzy sets give the degree of membership of an element in a given set, while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership.

The aim of recent study has been to evolve a mathematical object that allows complex processes on groups of mathematical objects to be formulated with ease of elegant. This notion is very useful for study of space structures. In fact, this paper creates a link between nexuses and fuzzy sets. The notion of intuitionistic fuzzy subnexus of a nexus is introduced. Some characteristic properties and connections are investigated. Finally, some equivalence relations, constructed by intuitionistic fuzzy subnexuses, are discussed.

## 2. Preliminaries and Notations

**Definition 2.1** [10, 11]**.** (i) An *address* is a sequence of $\mathbb{N}^* = \mathbb{N} \bigcup \{0\}$ such that $a_k = 0$ implies that $a_i = 0,$ for all $i \geq k.$ The sequence of zero is called *empty* address and denoted by $(\ ).$ In other words, every nonempty address is of the form

$$(a_1, ..., a_n, 0, 0, \cdots),$$

where $n \in \mathbb{N}.$ This address will be denoted by $(a_1, a_2, ..., a_n).$

(ii) A *nexus N* is a set of addresses with the following properties:

(a) $(a_1, a_2, ..., a_n) \in N \Rightarrow (a_1, ..., a_{n-1}, t) \in N, \ \ \forall 0 \leq t \leq a_n,$

(b) $\{a_i\}_{i=1}^{\infty} \in N, a_i \in \mathbb{N} \Rightarrow \forall n \in \mathbb{N}, (a_1, a_2, ..., a_n) \in N.$

In what follows, *N* denotes a nexus unless otherwise specified.

**Definition 2.2** [10, 11]**.** Let $\omega \in N.$ Then the level of ω is said to be:

(i) *n*, if $\omega = (a_1, a_2, ..., a_n)$ for some $a_n \in \mathbb{N},$

(ii) $\infty,$ if ω is an infinite sequence of $\mathbb{N},$

(iii) 0, if $\omega = (\ ).$

The level of ω is denoted by $l(\omega).$

**Definition 2.3** [10, 11]**.** Let $\omega = \{a_i\}$ and $\nu = \{b_i\}$ be addresses, where $a_i, b_i \in \mathbb{N}$. Then $\omega \leq \nu$ if $l(\omega) = 0$ or one of the following cases satisfies:

(i) If $l(\omega) = 1$, i.e., $\omega = (a_1)$, for all $a_1 \in \mathbb{N}$, then $a_1 \leq b_1$.

(ii) If $1 < l(\omega) < \infty$, then $l(\omega) \leq l(\nu)$ and $a_{l(\omega)} \leq b_{l(\omega)}$ and for any $1 \leq i < l(\omega)$, $a_i = b_i$.

(iii) If $l(\omega) = \infty$, then $\omega = \nu$.

For example, in the nexus:

$$N = \{(\ ), (1), (2), (1, 1), (1, 2), (1, 3), (1, 3, 1), (1, 3, 2)\},$$

we have $(1) \leq (2)$, $(1, 2) \leq (1, 3, 1)$ and $(1, 3, 1) \leq (1, 3, 2)$.

**Definition 2.4** [10, 11]**.** A nonempty subset $S$ of $N$ is called a *subnexus* of $N$ provided that $S$ itself is a nexus. The set of all subnexuses of $N$ is denoted by $SUB(N)$.

Let $M$ and $N$ be two nexuses. Then a function $f : M \to N$ is called a *homomorphism* of nexuses if $\omega \leq \nu$ implies $f(\omega) \leq f(\nu)$, for all $\omega, \nu \in M$. If $f$ is onto, then we say $f$ is an *epimorphism*, and if $f$ is one-to-one, then we say $f$ is a *monomorphism*.

**Definition 2.5** [10]**.** Let $\mu : N \to [0, 1]$ be a fuzzy subset of $N$. Then $\mu$ is called a *fuzzy subnexus* of $N$, if $\omega \leq \nu$ implies $\mu(\nu) \leq \mu(\omega)$, for all $\nu, \omega \in N$. The set of all fuzzy subnexuses of $N$ is denoted by $FSUB(N)$.

**Definition 2.6** [3, 5]**.** For any mapping $f$ from $N$ to $S$, we can define in $N$ a new fuzzy set $\mu^f$ putting $\mu^f(x) = \mu(f(x))$, for all $x \in N$. Clearly, $\mu^f(x_1) = \mu^f(x_2)$, for all $x_1, x_2 \in f^{-1}(x)$.

For each fuzzy set $\mu$ in $N$ and any $\alpha \in [0, 1]$, we define two sets:

$$U(\mu, \alpha) = \{x \in N, \mu(x) \geq \alpha\}, \quad L(\mu, \alpha) = \{x \in N, \mu(x) \leq \alpha\},$$

which are called an *upper* and *lower level cut* of $\mu$ and can be used to the characterization of $\mu$. The *complement* of $\mu$, denoted by $\overline{\mu}$, is a fuzzy set of $N$ defined by $\overline{\mu}(x) = 1 - \mu(x)$ (see [3] and [5]).

An *intuitionistic fuzzy set* (IFS for short) of $N$ is defined as an object having the form:

$$S = (\mu_S, \lambda_S) = \{(x, \mu_S(x), \lambda_S(x)), x \in N\},$$

where the fuzzy sets $\mu_S$ and $\lambda_S$ denoted the *degree of membership* (namely, $\mu_S(x)$) and the *degree of non-membership* (namely, $\lambda_S(x)$) of each element $x \in N$ (see [1] and [2]).

For every two intuitionistic fuzzy sets $S = (\mu_S, \lambda_S)$ and $P = (\mu_P, \lambda_P)$ in $N$, we define:

$S \subseteq P$ if and only if $\mu_S(x) \leq \mu_P(x)$ and $\lambda_S(x) \geq \lambda_P(x)$, for all $x \in N$. Obviously $S = P$ means that $S \subseteq P$ and $P \subseteq S$.

## 3. Intuitionistic Fuzzy Subnexuses

**Definition 3.1.** An $IFS\, S = (\mu_S, \lambda_S)$ on a nexus $N$ is called an *intuitionistic fuzzy subnexus* of $N$ $(IFSUB(N)$ for short$)$, if $\nu \leq \omega$ implies $\mu_S(\omega) \leq \mu_S(\nu)$ and $\lambda_S(\omega) \geq \lambda_S(\nu)$, for all $\nu, \omega \in N$. It is not difficult to see that $\mu_S(x) \leq \mu_S((\ ))$ and $\lambda_S(x) \geq \lambda_S((\ ))$ for each $S \in IFS(N)$ and $x \in N$.

**Example 3.2.** Let

$$N = \{(\ ), (1), (2), (3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 1, 1), (3, 1, 2)\}.$$

Consider an $IFS\, S = (\mu_S, \lambda_S)$, where $\mu_S((\ )) = 0.6$, $\lambda_S((\ )) = 0.2$ and $\mu_S(x) = 0.2$, $\lambda_S(x) = 0.5$, for all $x \neq (\ )$. It is not difficult to verify that $S \in IFSUB(N)$.

**Example 3.3.** Let

$$N = \{(\ ), (1), (2), (1, 1), (1, 2), (1, 3), (1, 3, 1), (1, 3, 2)\}.$$

Consider an $IFS\, S = (\mu_S, \lambda_S)$, where $\mu_S((\ )) = \alpha_{1,}$

$\mu_S((1)) = \mu_S((2)) = \mu_S((3)) = \alpha_2$,

$\mu_S((1, 1)) = \mu_S((1, 2)) = \mu_S((1, 3)) = \alpha_3$,

$\mu_S((1, 3, 1)) = \mu_S((1, 3, 2)) = \alpha_4$, where $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$, and $\lambda_S((\ )) = \beta_1$,

$\lambda_S((1)) = \lambda_S((2)) = \lambda_S((3)) = \beta_2,$

$\lambda_S((1, 1)) = \lambda_S((1, 2)) = \lambda_S((1, 3)) = \beta_3,$

$\lambda_S((1, 3, 1)) = \lambda_S((1, 3, 2)) = \beta_4,$ where $\beta_1 < \beta_2 < \beta_3 < \beta_4.$

Then $S = (\mu_S, \lambda_S) \in IFSUB(N).$

**Proposition 3.4.** *A fuzzy set* $\mu_S$ *is a fuzzy subnexus of* $N$ *if and only if* $S = (\mu_S, \overline{\mu_S})$ *is an* $IFSUB(N)$.

**Proof.** Let $\mu_S$ be an $FSUB(N)$. Then $\omega \leq \nu$ implies $\mu_S(\nu) \leq \mu_S(\omega)$, for all $\omega, \nu \in N$. Thus $1 - \mu_S(\nu) \geq 1 - \mu_S(\omega)$, therefore $\overline{\mu_S}(\nu) \geq \overline{\mu_S}(\omega)$. So $S = (\mu_S, \overline{\mu_S})$ is an $IFSUB(N)$. The converse is clear. □

**Proposition 3.5.** *An IFS* $S = (\mu_S, \lambda_S)$ *is an* $IFSUB(N)$ *if and only if* $\mu_S$ *and* $\overline{\lambda_S}$ *are fuzzy subnexuses of N.*

**Proof.** Let $S = (\mu_S, \lambda_S) \in IFSUB(N)$ and $\omega, \nu \in N$ such that $\nu \leq \omega$. So

$$\mu_S(\omega) \leq \mu_S(\nu), \quad \lambda_S(\nu) \leq \lambda_S(\omega).$$

Therefore, $\mu_S$ is a fuzzy subnexus of $N$ and since $1 - \lambda_S(\omega) \leq 1 - \lambda_S(\nu)$, so $\overline{\lambda_S}(\omega) \leq \overline{\lambda_S}(\nu)$. Then $\overline{\lambda_S}$ is a fuzzy subnexus of $N$.

Conversely, let $\mu_S$ and $\overline{\lambda_S}$ be fuzzy subnexuses of $N$. Let $\omega, \nu \in N$ such that $\nu \leq \omega$. Thus $\mu_S(\omega) \leq \mu_S(\nu)$ and $\overline{\lambda_S}(\omega) \leq \overline{\lambda_S}(\nu)$. So $1 - \lambda_S(\omega) \leq 1 - \lambda_S(\nu)$, hence $\lambda_S(\omega) \geq \lambda_S(\nu)$, so $S = (\mu_S, \lambda_S) \in IFSUB\,(N)$. □

**Proposition 3.6.** *Let S be a nonempty subset of a nexus N. Then an* $IFS\,(\mu_S, \lambda_S)$ *is defined by*

$$\mu_S(x) = \begin{cases} \alpha_2, & \text{if } x \in S, \\ \alpha_1, & \text{if } x \notin S, \end{cases} \qquad \lambda_S(x) = \begin{cases} \beta_2, & \text{if } x \in S, \\ \beta_1, & \text{if } x \notin S, \end{cases}$$

*where* $0 \leq \alpha_1 < \alpha_2 \leq 1$, $0 \leq \beta_2 < \beta_1 \leq 1$ *and* $\alpha_i + \beta_i \leq 1$ *for* $i = 1, 2,$ *is an* $IFSUB(N)$ *if and only if S is a subnexus of N.*

**Proof.** Let $(\mu_S, \lambda_S)$ be an $IFSUB(N)$. Let $x \in S$ and $y \leq x$. Since $(\mu_S, \lambda_S)$ is an $IFSUB(N)$, we have $\alpha_2 = \mu_S(x) \leq \mu_S(y)$ and $\beta_2 = \lambda_S(x) \geq \lambda_S(y)$. So by definition of $\mu_S$ and $\lambda_S$, $\mu_S(y) = \alpha_2$ and $\lambda_S(y) = \beta_2$. Thus $y \in S$. Therefore, $S$ is a subnexus of $N$.

Conversely, let $S$ be a subnexus of $N$ and $\nu \leq \omega$ for $\omega, \nu \in N$. If $\nu, \omega \in S$, then $\alpha_2 = \mu_S(\omega) \leq \mu_S(\nu) = \alpha_2$ and $\beta_2 = \lambda_S(\nu) \leq \lambda_S(\omega) = \beta_2$. In a similar way, we can verify other cases. Therefore, $(\mu_S, \lambda_S)$ is an $IFSUB(N)$. □

**Definition 3.7.** Let $S = (\mu_S, \lambda_S)$ be an $IFSUB(N)$ and $\alpha, \beta \in [0, 1]$ be such that $0 < \alpha + \beta \leq 1$. Then the set

$$N_S^{(\alpha,\beta)} = \{x \in N \mid \alpha \leq \mu_S(x), \lambda_S(x) \leq \beta\}$$

is called an $(\alpha, \beta)$-*level subset* of $S = (\mu_S, \lambda_S)$. The set of all $(\alpha, \beta) \in \mathrm{Im}(\mu_S) \times \mathrm{Im}(\lambda_S)$ such that $\alpha + \beta \leq 1$ is called the *image* of $S = (\mu_S, \lambda_S)$. Clearly $N_S^{(\alpha,\beta)} = U(\mu_S, \alpha) \cap L(\lambda_S, \beta)$, where $U(\mu_S, \alpha)$ and $L(\lambda_S, \beta)$ are upper and lower level subsets of $\mu_S$ and $\lambda_S$, respectively.

**Theorem 3.8.** *An IFS* $S = (\mu_S, \lambda_S)$ *is an* $IFSUB(N)$ *if and only if* $N_S^{(\alpha,\beta)}$ *is a subnexus of N for every* $(\alpha, \beta) \in \mathrm{Im}(\mu_S) \times \mathrm{Im}(\lambda_S)$ *such that* $\alpha + \beta \leq 1$, *i.e., if and only if all nonempty level subsets* $U(\mu_S, \alpha)$ *and* $L(\lambda_S, \beta)$ *are subnexuses.*

**Proof.** Let $S = (\mu_S, \lambda_S)$ be an $IFSUB(N)$. Let $\omega \in U(\mu_S, \alpha)$ and $\nu \leq \omega$. Since $S = (\mu_S, \lambda_S)$ is an $IFSUB(N)$, $\mu_S(\omega) \leq \mu_S(\nu)$ and $\lambda_S(\omega) \geq \lambda_S(\nu)$. On the other hand, $\omega \in U(\mu_S, \alpha)$, so $\alpha \leq \mu_S(\omega) \leq \mu_S(\mu)$, therefore $\nu \in U(\mu_S, \alpha)$. Hence $U(\mu_S, \alpha)$ is a subnexus of $N$. Similarly, $L(\lambda_S, \beta)$ is a subnexus of $N$.

Conversely, let $U(\mu_S, \alpha)$ and $L(\lambda_S, \beta)$ be subnexuses of $N$. Let $\nu \leq \omega$ for $\nu, \omega \in N$, $\mu_S(\omega) = \alpha$, $\lambda_S(\omega) = \beta$. Then $\omega \in U(\mu_S, \alpha)$. Since $U(\mu_S, \alpha)$ is a subnexus of $N$, $\nu \in U(\mu_S, \alpha)$. Thus $\mu_S(\nu) \geq \alpha = \mu_S(\omega)$. Similarly, $\lambda_S(\nu) \leq \beta = \lambda_S(\omega)$. Therefore, $S = (\mu_S, \lambda_S)$ is an $IFSUB(N)$. □

**Theorem 3.9.** *Let* $S = (\mu_S, \lambda_S)$ *be an* $IFSUB(N)$ *and* $x \in N$. *Then* $\mu_S(x) = \alpha$, $\lambda_S(x) = \beta$ *if and only if* $x \in U(\mu_S, \alpha), x \notin U(\mu_S, \gamma)$ *and* $x \in L(\lambda_S, \beta), x \notin L(\lambda_S, \sigma)$, *for all* $\gamma > \alpha$ *and* $\sigma < \beta$.

**Proof.** Let $\mu_S(x) = \alpha$, $\lambda_S(x) = \beta$. Then $x \in U(\mu_S, \alpha)$. If there exists $\gamma > \alpha$ such that $x \in U(\mu_S, \gamma)$, then $\mu_S(x) \geq \gamma > \alpha$, so $\mu_S(x) > \alpha$ which is a contradiction with $\mu_S(x) = \alpha$. Therefore, $x \notin U(\mu_S, \gamma)$, for all $\gamma > \alpha$. Similarly, $x \notin L(\lambda_S, \sigma)$, for all $\sigma < \beta$.

Conversely, let $x \in U(\mu_S, \alpha)$, $x \notin U(\mu_S, \gamma)$, for all $\gamma > \alpha$. Since $x \in U(\mu_S, \alpha)$, $\mu_S(x) \geq \alpha$. If $\mu_S(x) > \alpha$, then there exists $\gamma > \alpha$ such that $\mu_S(x) \geq \gamma$ and so $x \in U(\mu_S, \gamma)$ for $\gamma > \alpha$, which is a contradiction with hypothesis. Thus $\mu_S(x) = \alpha$. Similarly, $\lambda_S(x) = \beta$. □

## 4. Characteristic Intuitionistic Fuzzy Subnexuses

**Definition 4.1.** A subnexus $S$ of $N$ is said to be *characteristic* if $f(S) = S$, for all $f \in Aut(N)$, where $Aut(N)$ is the set of all automorphisms of $N$.

**Definition 4.2.** An $IFS\ S = (\mu_S, \lambda_S)$ of $N$ is called an *intuitionistic fuzzy characteristic* if $\mu_S^f(x) = \mu_S(x)$ and $\lambda_S^f(x) = \lambda_S(x)$, for all $x \in N$ and $f \in Aut(N)$.

**Theorem 4.3.** *$S \in IFSUB(N)$ is characteristic if and only if each nonempty level subset is a characteristic subnexus of $N$.*

**Proof.** An $IFS\ S = (\mu_S, \lambda_S)$ is an $IFSUB(N)$ if and only if all its nonempty level subsets are subnexuses, (Theorem 3.8). So we will prove only that $S$ is a characteristic if and only if all its nonempty level subsets are characteristic. If $S = (\mu_S, \lambda_S)$ is characteristic, $\alpha \in \mathrm{Im}(\mu_S)$, $f \in Aut(N)$ and $x \in U(\mu_S, \alpha)$, then $\mu_S^f(x) = \mu_S(f(x)) = \mu_S(x) \geq \alpha$ which means that $f(x) \in U(\mu_S, \alpha)$. Thus $f(U(\mu_S, \alpha)) \subseteq U(\mu_S, \alpha)$. Since for each $x \in U(\mu_S, \alpha)$, there exists $y \in N$ such that $f(y) = x$, we have

$$\mu_S(y) = \mu_S^f(y) = \mu_S(f(y)) = \mu_S(x) \geq \alpha.$$

Therefore, $y \in U(\mu_S, \alpha)$, thus $x = f(y) \in f(U(\mu_S, \alpha))$, so $U(\mu_S, \alpha) \subseteq f(U(\mu_S, \alpha))$. Hence $U(\mu_S, \alpha) = f(U(\mu_S, \alpha))$. Similarly, $L(\lambda_S, \beta) = f(L(\lambda_S, \beta))$. This proves that $U(\mu_S, \alpha)$ and $L(\lambda_S, \beta)$ are characteristic.

Conversely, if all levels of $S = (\mu_S, \lambda_S)$ are subnexuses of $N$, then for $x \in N$, $f \in Aut(N)$ and $\mu_S(x) = \alpha$, $\lambda_S(x) = \beta$, by Theorem 3.9, we have $x \in U(\mu_S, \alpha)$, $x \notin U(\mu_S, \gamma)$ and $x \in L(\lambda_S, \beta)$, $x \notin L(\lambda_S, \sigma)$, for all $\gamma > \alpha$, $\sigma < \beta$. Thus $f(x) \in f(U(\mu_S, \alpha)) = U(\mu_S, \alpha)$ and $f(x) \in f(L(\lambda_S, \beta)) = L(\lambda_S, \beta)$, i.e., $\mu_S(f(x)) \geq \alpha$ and $\lambda_S(f(x)) \leq \beta$. For $\mu_S(f(x)) = \gamma > \alpha$, $\lambda_S(f(x)) = \sigma < \beta$, we have $f(x) \in U(\mu_S, \gamma) = f(U(\mu_S, \gamma))$, $f(x) \in L(\lambda_S, \sigma) = f((\lambda_S, \sigma))$ which implies that $x \in U(\mu_S, \gamma)$, $x \in L(\lambda_S, \sigma)$. This is a contradiction. Thus $\mu_S(f(x)) = \mu_S(x)$ and $\lambda_S(f(x)) = \lambda_S(x)$. So, $S = (\mu_S, \lambda_S)$ is characteristic. □

**Proposition 4.4.** *Let* $f : N \to N'$ *be a homomorphism of nexuses. If* $S = (\mu_S, \lambda_S)$ *is an* $IFSUB(N')$, *then* $S^f = (\mu_S^f, \lambda_S^f)$ *is an* $IFSUB(N)$.

**Proof.** Let $x, y \in N$. Since $f$ is a homomorphism, $x \leq y$ implies that $f(x) \leq f(y)$. Since $S = (\mu_S, \lambda_S)$ is an $IFSUB(N')$,

$$\mu_S^f(x) = \mu_S(f(x)) \geq \mu_S(f(y)) = \mu_S^f(y),$$

$$\lambda_S^f(x) = \lambda_S(f(x)) \leq \lambda_S(f(y)) = \lambda_S^f(y).$$

Therefore, $S^f = (\mu_S^f, \lambda_S^f)$ is an $IFSUB(N)$. □

**Proposition 4.5.** *Let* $f : N \to N'$ *be an epimorphism of nexuses. If* $S^f = (\mu_S^f, \lambda_S^f)$ *is an* $IFSUB(N)$, *then* $S = (\mu_S, \lambda_S)$ *is an* $IFSUB(N')$.

**Proof.** Since $f$ is a surjective mapping, for $x, y \in N'$, there are $x_1, y_1 \in N$ such that $x = f(x_1)$, $y = f(y_1)$. If $x \leq y$, then $f(x_1) \leq f(y_1)$. Since $f$ is a homomorphism, $x_1 \leq y_1$. Otherwise, if $x_1 > y_1$, then $f(x_1) > f(y_1)$ which is a contradiction. Thus since $S^f = (\mu_S^f, \lambda_S^f)$ is an $IFSUB(N)$,

$$\mu_S(x) = \mu_S(f(x_1)) = \mu_S^f(x_1) \geq \mu_S^f(y_1) = \mu_S(f(y_1)) = \mu_S(y),$$

$$\lambda_S(x) = \lambda_S(f(x_1)) = \lambda_S^f(x_1) \leq \lambda_S^f(y_1) = \lambda_S(f(y_1)) = \lambda_S(y).$$

Therefore, $S = (\mu_S, \lambda_S)$ is an $IFSUB(N')$. □

**Theorem 4.6.** *Let* $f : N \to N'$ *be an epimorphism of nexuses. Then* $S^f = (\mu_S^f, \lambda_S^f)$ *is an* $IFSUB(N)$ *if and only if* $S = (\mu_S, \lambda_S)$ *is an* $IFSUB(N')$.

**Proof.** The proof is obtained by Propositions 4.4 and 4.5. □

## 5. Equivalence Relations on $IFSUB(N)$

For any $t \in [0, 1]$ define on $IFSUB(N)$ two binary relations $\mathcal{U}^t$ and $\mathcal{L}^t$ as follows:

$$(K, M) \in \mathcal{U}^t \Leftrightarrow U(\mu_K; t) = U(\mu_M; t)$$

and

$$(K, M) \in \mathcal{L}^t \Leftrightarrow L(\lambda_K; t) = L(\lambda_M; t),$$

respectively, where $M = (\mu_M, \lambda_M)$, $K = (\mu_K, \lambda_K)$. Then clearly $\mathcal{U}^t$ and $\mathcal{L}^t$ are equivalence relations on $IFSUB(N)$.

For any $M = (\mu_M, \lambda_M) \in IFSUB(N)$, let $[M]_{\mathcal{U}^t}$ (resp., $[M]_{\mathcal{L}^t}$) be the equivalence class of $M$ with respect to $\mathcal{U}^t$ (resp., $\mathcal{L}^t$) and $IFSUB(N)/\mathcal{U}^t$ (resp., $IFSUB(N)/\mathcal{L}^t$) the set of all equivalence classes of $\mathcal{U}^t$ (resp., $\mathcal{L}^t$), so

$$IFSUB(N)/\mathcal{U}^t = \{[M]_{\mathcal{U}^t} \mid M = (\mu_M, \lambda_M) \in IFSUB(N)\}$$

$$(\text{resp., } IFSUB(N)/\mathcal{L}^t = \{[M]_{\mathcal{L}^t} \mid M = (\mu_M, \lambda_M) \in IFSUB(N)\}).$$

Now let $SUB(N)$ denote the family of all subnexus of nexus $N$ and let $t \in [0, 1]$. Define two maps $f_t$ and $g_t$ from $IFSUB(N)$ to $SUB(N)$ by

$$f_t(M) = U(\mu_M; t), \quad g_t(M) = L(\lambda_M; t),$$

for all $M = (\mu_M, \lambda_M) \in IFSUB(N)$. Then $f_t$ and $g_t$ are clearly well-defined.

**Theorem 5.1.** *For any* $0 < t < 1$, *the maps* $f_t$ *and* $g_t$ *are surjective from* $IFSUB(N)$ *to* $SUB(N) \cup \{\varnothing\}$.

**Proof.** Let $0 < t < 1$. Note that $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IFSUB(N)$, where $\mathbf{0}$ and $\mathbf{1}$ are fuzzy sets of $N$ defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$, for all $x \in N$. Obviously $f_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) = \varnothing = L(\mathbf{1}; t) = g_t(\mathbf{0}_\sim)$. Let $\varnothing \neq K \in SUB(N)$. For $K_\sim = (\chi_K, \overline{\chi_K}) \in IFSUB(N)$, we have $f_t(K_\sim) = U(\chi_K; t) = K$ and $g_t(K_\sim) = L(\overline{\chi_K}; t) = K$. Hence $f_t$ and $g_t$ are surjective. □

**Theorem 5.2.** *For any* $0 < t < 1$ *there are bijective maps from the quotient sets* $IFSUB(N)/\mathcal{U}^t$ *and* $IFSUB(N)/\mathcal{L}^t$ *to* $SUB(N) \cup \{\varnothing\}$.

**Proof.** For any $0 < t < 1$, let $f_t^*$ (resp., $g_t^*$) be a map from $IFSUB(N)/\mathcal{U}^t$ (resp., $IFSUB(N)/\mathcal{L}^t$) to $SUB(N) \cup \{\varnothing\}$ defined by $f_t^*([M]_{\mathcal{U}^t}) = f_t(M)$ (resp., $g_t^*([M]_{\mathcal{L}^t}) = g_t(M)$), for all $M = (\mu_M, \lambda_M) \in IFSUB(N)$. If $U(\mu_M; t) = U(\mu_K; t)$ and $L(\lambda_M; t) = L(\lambda_K; t)$ for $M = (\mu_M, \lambda_M)$, $K = (\mu_K, \lambda_K)$ of $IFSUB(N)$, then $(M, K) \in \mathcal{U}^t$ and $(M, K) \in \mathcal{L}^t$. Thus $[M]_{\mathcal{U}^t} = [K]_{\mathcal{U}^t}$ and $[M]_{\mathcal{L}^t} = [K]_{\mathcal{L}^t}$. This proves that the maps $f_t^*$ and $g_t^*$ are injective.

Now let $\varnothing \neq P \in SUB(N)$. For $P_\sim = (\chi_P, \overline{\chi_P}) \in IFSUB(N)$, we have

$$f_t^*([P_\sim]_{\mathcal{U}^t}) = f_t(P_\sim) = U(\chi_P; t) = P, \quad g_t^*([P_\sim]_{\mathcal{L}^t}) = g_t(P_\sim) = U(\overline{\chi_P}; t) = P.$$

Finally, for $\mathbf{0}_\sim$, we have

$$f_t^*([\mathbf{0}_\sim]_{\mathcal{U}^t}) = f_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) = \varnothing, \quad g_t^*([\mathbf{0}_\sim]_{\mathcal{L}^t}) = g_t(\mathbf{0}_\sim) = L(\mathbf{1}; t) = \varnothing.$$

This shows that $f_t^*$ and $g_t^*$ are surjective. □

For any $0 < t \leq 0.5$, we define another relation $\mathcal{R}^t$ on $IFSUB(N)$ as follows:

$$(M, K) \in \mathcal{R}^t \Leftrightarrow N_M^{(t,t)} = N_K^{(t,t)}.$$

Then the relation $\mathcal{R}^t$ also is an equivalence relation on $IFSUB(N)$.

**Theorem 5.3.** *For any* $0 < t \leq 0.5$ *the map* $\varphi_t : IFSUB(N) \to SUB(N) \cup \{\varnothing\}$ *defined by* $\varphi_t(M) = N_M^{(t,t)}$ *is surjective.*

**Proof.** Let $0 < t \leq 0.5$. Then $\varphi_t(\mathbf{0}_\sim) = N_M^{(t,t)} = U(\mathbf{0};\, t) \cap L(\mathbf{1},\, t) = \varnothing$. For any $K \in IFSUB(N)$, there exists $K_\sim = (\chi_K, \overline{\chi_K}) \in IFSUB(N)$ such that $\varphi_t(K_\sim) = N_M^{(t,t)} = U(\chi_K;\, t) \cap L(\overline{\chi_K};\, t) = K$. Therefore, $\varphi_t$ is surjective. □

**Theorem 5.4.** *For any* $0 < t \leq 0.5$ *there is a bijective map from the quotient set* $IFSUB(N)/\mathcal{R}^t$ *to* $SUB(N) \cup \{\varnothing\}$.

**Proof.** Let $0 < t \leq 0.5$ and let $\varphi_t^* : IFSUB(N)/\mathcal{R}^t \to SUB(N)$ be a map defined by $\varphi_t^*([M]_{\mathcal{R}^t}) = \varphi_t(M)$, for all $[M]_{\mathcal{R}^t} \in IFSUB(N)/\mathcal{R}^t$. If $\varphi_t^*([M]_{\mathcal{R}^t}) = \varphi_t^*([K]_{\mathcal{R}^t})$ for any $[M]_{\mathcal{R}^t}, [K]_{\mathcal{R}^t} \in IFSUB(N)/\mathcal{R}^t$, then $N_M^{(t,t)} = N_K^{(t,t)}$, i.e., $(M, K) \in \mathcal{R}^t$. It follows that $[M]_{\mathcal{R}^t} = [K]_{\mathcal{R}^t}$ so that $\varphi_t^*$ is injective. Moreover, $\varphi_t^*([\mathbf{0}_\sim]_{\mathcal{R}^t}) = \varphi_t(\mathbf{0}_\sim) = N_{\mathbf{0}_\sim}^{(t,t)} = \varnothing$. For any $F \in SUB(N)$ we have $F_\sim = (\chi_F, \overline{\chi_F}) \in IFSUB(N)$ and $\varphi_t^*([F]_{\mathcal{R}^t}) = \varphi_t(F_\sim) = N_M^{(t,t)} = U(\chi_F;\, t) \cap L(\overline{\chi_F};\, t) = F$. This proves that $\varphi_t^*$ is surjective. □

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