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SOME RESULTS CONCERNING COMPACTIFICATIONS, SUBMAXIMAL SPACES, SPECTRAL SPACES AND PRIMITIVE WORDS

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Abstract

It is shown that the Fan-Gottesman compactification of a regular space is a submaximal space. Considering the relation between spectral spaces and submaximal spaces, the relation among F-spectral, W-spectral, A-spectral and up-spectral spaces is investigated. Providing a new class of spectral spaces in connection with combinatorics on words, it is shown that Fan-Gottesman compactification of *PL-space* is a spectral space.

0. Introduction

The first section of this paper contains some information about compactification of topological spaces, especially, the Aleksandrov (one-point), the Wallman, the Fan-Gottesman compactifications. In 1929, Aleksandrov proved that all local compact Hausdorff spaces may be completed to a compact Hausdorff by the addition of one-point [2]. In 1938, Wallman introduced compactification of T_1 spaces having a normal base [17, 27] which is also called *Wallman compactification* [29]. In 1952,

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Fan and Gottesman constructed a compactification, also called *Fan-Gottesman compactification*, for a regular space with a normal base [17]. Their method is similar to Wallman compactification. In [15], the relation between Fan-Gottesman and Wallman compactifications is investigated and showed that Fan-Gottesman compactification of some interesting and specific spaces such as normal A_2 and T_4 is Wallman-type compactification. In [16], it is shown that Aleksandrov, Stone-Cech, Wallman and Fan-Gottesman compactifications of local compact Hausdorff space are homeomorphic to each other.

The second section of this paper contains some preliminaries about submaximal spaces. Different properties of submaximal spaces have been studied by several authors [1, 3, 4, 11, 25, 28]. For example, Arhangel'skiĭ and Collins characterized the submaximal spaces [3]. This paper contains maximal spaces, nodec space, irresolvable and maximal connected space. Authors investigated the relations among these spaces. Also, Bezhanishvili et al. [8] investigated modal logics of submaximal spaces. At the end of this section, we investigate a relation between the Fan-Gottesman compactification and submaximal space. In this part, we showed that Fan-Gottesman compactification of a topological space is a submaximal space.

The third section of this paper is about spectral space. In order that an order set $(X, \leq)$ be spectral, Kaplansky gave two conditions, called *first condition* and *second condition* of Kaplansky, for ring spectrum [23]. After, it is shown that there exists a partially ordered set which satisfies the first, the second condition of Kaplansky and the condition added by Lewis and Ohm [20]. Belaid and Echi defined up-spectral and down-spectral spaces [6]. In 2004, Belaid et al. [5] characterized topological spaces X such that one-point compactification of X is a spectral space. They called these spaces as an *A-spectral space*. Echi and Gargouri investigated the relation between the up-spectral and A-spectral spaces [13]. In 2006, Belaid gave some properties of H-spectral space, i.e., a topological space X such that its T_0-compactification is spectral. He introduced W-spectral space. Also, he gave necessary and sufficient condition on the T_1-space X in order to get its Wallman compactification spectral [7]. Although many results about spectral sets have been obtained by Dobbs, Hochster, Fontana, Levy, Ohm, and others [10, 18, 19, 20], a complete algebraic characterization of spectral sets still seems very far off. We get some results about F-spectral, W-spectral, A-spectral and up-spectral spaces.

The last section of this paper is about primitive words. In [14], Echi and Naimi showed that the one-point compactification of *PL-space* is a spectral space. Similarly, at the end of this part, it is proved that Fan-Gottesman compactification of *PL-space* is a spectral space, providing a new class of spectral spaces in connection with combinatorics on words.

1. Aleksandrov, Wallman and Fan-Gottesman Compactifications

A compactification of a topological space X is a compact Hausdorff space Y containing X as a subspace such that $cl_X X = Y$ [26]. It is known that there are a lot of compactification methods applying different topological spaces such as Aleksandrov (one-point), Wallman, Fan-Gottesman.

Now, we will give some information about these compactification methods.

Let X be a locally compact and Hausdorff space. Then take some object outside denoted by ∞ for convenience, and adjoin it to X, forming the set $Y = X \bigcup \{\infty\}$. Topologize Y by defining the collection of open sets in Y to be all sets of the following types:

(i) U, where U is an open subset of X.

(ii) $Y - C$, where C is a compact subset of X.

The space Y is called *Aleksandrov* (*one-point*) *compactification* of X. In order to avoid the confusion, it is denoted by αX [26, 30].

Let θ be a class of closed sets in X. If it satisfies the following three conditions, then θ is called *normal base*.

(1) θ is closed under finite intersection and unions.

(2) If x is not contained in the closed set A, then there is a set $B \in \theta$ such that $x \in B \subset X - A$.

(3) If $A_1 \bigcap A_2 = \varnothing$, for $\forall\ A_1, A_2 \in \theta$, then there exist sets $A_m, A_n \in \theta$ such that $A_1 \subset X - A_n$, $A_2 \subset X - A_m$, $A_n \bigcup A_m = X$.

Let X be a T_1 space having a normal base and θ be a normal base in X. Then it is considered K space whose element is denoted by letters as $a', b', \ldots$ consisting of finite number of F_i in X such that

$$F_1 \bigcap F_2 \bigcap F_3 \bigcap \cdots \bigcap F_n \neq \varnothing$$

and maximal with respect to above property. Let $\tau(F) = \{a' \in K : F \in a'\}$. Then it is defined topology of K with a family of sets $\delta = \{\tau(F) : F \in \theta\}$ a subbase of closed set. K is a compact space and compactification of X. This compactification is called *Wallman compactification* [17, 27, 29]. In order to avoid the confusion, it is denoted by γX.

Fan-Gottesman compactification of a regular space is introduced and studied by Ky Fan and Noel Gottesman. Let β be a class of open sets in X. Then β contains $\varnothing$ and satisfies the following three conditions:

(1) If $B_1, B_2 \in \beta$, then $B_1 \cap B_2 \in \beta$.

(2) If $B \in \beta$, then $X - cl_X B \in \beta$, where closure of B in X will be denoted by $cl_X B$.

(3) For every open set U in X and every $B \in \beta$ such that $cl_X B \subset U$, there exists a set $D \in \beta$ such that $cl_X B \subset D \subset cl_X D \subset U$.

They named β the normal base.

We consider a regular space having a normal base for an open set, i.e., which satisfies above three properties of normal base. A chain family on β is a nonempty family of sets of β such that

$$cl_X B_1 \cap cl_X B_2 \cap \cdots \cap cl_X B_n \neq \varnothing$$

for any finite number of sets B_i of the family. Every chain family on β is contained in at least one maximal chain family on β by Zorn's lemma. Maximal chain families on β will be denoted by letters as $a^*, b^*, \ldots$ and also the set of all maximal chain families on β will be denoted by $(X, \beta)^*$, whose topology is defined as follows. For each $B \in \beta$, let

$$\tau(B) = \{b^* \in (X, \beta)^* : B \in b^*\}.$$

Then the topology of $(X, \beta)^*$ is defined by taking

$$\beta^* = \{\tau(B) : B \in \beta\},$$

$(X, \beta)^*$ is a compact, Hausdorff space and a compactification of our regular spaces. Afterwards this compactification is called *Fan-Gottesman compactification* [17].

2. Submaximal Spaces

Let (X, τ) be a topological space. Then the regular open sets with respect to τ form a base of a topology τ^* on X, which is coarser than τ and semi-regular; τ^* is said to be the *semi-regular topology* associated with τ. The set $E(\tau_0)$ of topologies τ such that $\tau^* = \tau_0$ is inductive with respect to the relation "τ is coarser than τ'". If τ_0 is semi-regular topology on X, then a maximal element of $E(\tau_0)$ is called *submaximal topology* and the set X with such a topology is called *submaximal space*.

As a result of this definition, a topological space X is said to be *submaximal* if and only if every dense subset of X is open [9].

In [8], Bezhanishvili et al. characterized submaximal space as follows:

Proposition 2.1. *Let X be a topological space. Then the following statements are equivalent*:

(i) *X is submaximal.*

(ii) *$cl_X S - S$ is closed, for each $S \subseteq X$.*

(iii) *$cl_X S - S$ is closed and discrete, for each $S \subseteq X$.*

Proposition 2.1 gives the following results.

Proposition 2.2. *Let (X, τ) be a compact submaximal space. Then $cl_X S - S$ is finite for every subset of X.*

It is clear that $cl_X S - S$ is compact and hence finite, because it is a closed with discrete relative topology.

Lemma 2.1. *Let (X, τ) be a topological space that has a finite number of accumulation points. If each accumulation point of X is closed, then (X, τ) is a submaximal space* [4].

Recall that an *Alexandroff topology* on a set X is a topology τ for which any intersection of open sets is again open. By the Alexandroff topology on X associated with an ordering $\leq$, we mean the topology on X that has a basis $\{(\downarrow, x) : x \in X\}$, where $(\downarrow, x) = \{y \in X : y \leq x\}$. If $x \in X$, then we denote by $(\uparrow, x)$ the subset $\{y \in X : x \leq y\}$ of X.

Let (X, τ) be a T_0 space and $\leq$ be the ordering induced by τ. Then a chain $x_0 \prec x_1 \prec \cdots \prec x_n$ of elements of X is said to be of *length n*, the supremum of lengths is called *Krull dimension* of (X, τ) which we write $\dim_K(X, \tau)$.

Bezhanishvili et al. [8] characterized submaximal Alexandroff space as follows.

Proposition 2.3. *Let* (X, τ) *be an Alexandroff* T_0 *space. Then the following are equivalent*:

(i) *X is submaximal space*.

(ii) $\dim_K(X, \tau) \leq 1$.

Now, we will give our theorem.

Theorem 2.1. *Let* $(X, \beta)^*$ *be the Fan-Gottesman compactification of X. Then the following statements are equivalent*:

(i) $(X, \beta)^*$ *is a submaximal space*.

(ii) $(X, \beta)^*$ *has a finite number of accumulation points*.

Proof. (ii) $\Rightarrow$ (i) Suppose that X^* has a finite number of accumulation points. Since X^* is Hausdorff, each one-point set is closed. By Lemma 2.1, it is a submaximal space.

(i) $\Rightarrow$ (ii) Suppose that X^* has infinitely many accumulation points. Since X^* is Hausdorff, we may consider a set D, which is an infinite discrete subset of set of accumulation points. Then, clearly, $X^* - D$ is dense in X^*. But $X^* - D$ is not open; indeed X^* is compact, D has an accumulation point that is necessarily in $X^* - D$, because D is discrete. This contradicts the fact that X^* is submaximal. □

3. Spectral Spaces

Let R be a commutative ring with identity. *Spectrum or prime spectrum* of R, denoted by $Spec(R)$, is the set of prime ideals of R. The topology on $Spec(R)$ defined by closed set $Z(I) = \{C \in Spec(R) : I \subseteq C\}$ for ideals I of R is called *Zariski topology* on $Spec(R)$.

By definition, the closure in the Zariski topology of the singleton set $\{P\}$ in $Spec(R)$ consists of all prime ideals of R containing P. In particular, a point P in $Spec(R)$ is closed in the Zariski topology if and only if the prime ideal P is not contained in any other prime ideals of R, i.e., if and only if P is a maximal ideal.

A topological space is called *spectral* if it is homeomorphic to the prime spectrum or a ring equipped with Zariski topology [12].

Recall that a closed subset C of a topological space X has a *generic point* if there is some $x \in C$ such that $cl_X\{x\} = C$. A topological space in which every nonempty *irreducible* closed subset has a generic point is called *sober*.

Hochster [18] characterized spectral spaces as follows:

A space X is spectral if and only if the following axioms hold:

(i) Every nonempty *irreducible* closed subset of X is the closure of a unique point (that is, sober). In other words, it has a generic point.

(ii) X is compact.

(iii) The compact open sets form a basis of X.

(iv) The family of compact open sets of X is closed under finite intersections.

Belaid and Echi introduced up-spectral and down-spectral spaces in [6].

X is said to be *up-spectral space* if it satisfies the axioms of a spectral space with the exception that X is not necessarily compact.

X is said to be *down-spectral space* if it satisfies the axioms of a spectral space with the exception that X does not necessarily have a generic point when it is irreducible.

Belaid at al. [5] characterized A-spectral spaces (that is; one-point compactification of X is spectral space) and get the following definition:

Definition 3.1. Let X be a topological space and U be a subset of X. Then

(1) U is called *intersection compact open* or *ICO*, if for each compact open subset O of X, $U \bigcap O$ is compact.

(2) U is called *intersection compact closed* or *ICC*, if for each compact closed subset O of X, $U \bigcap O$ is compact.

(3) U is called *intersection compact open closed* or *ICOC*, if it is ICO and ICC.

(4) Let P be a property. Then U is said to be *co-P* if $X - U$ satisfies P [5].

Belaid gave some properties of H-spectral spaces (that is; T_0 -compactification of X is spectral space) and defined W-spectral spaces (that is; Wallman compactification of X is spectral space) and characterized W-spectral spaces [7].

A *semispectral space* is a space in which the intersection of two compact open sets is compact [18]. If X is semispectral space, then the following properties hold:

(i) Any finite union of closed co-ICO subsets of X is co-ICO.

(ii) Any finite union of a closed, compact and co-ICO subset of X is a closed, compact and co-ICO subset.

(iii) The complement of compact open set of X is co-ICO.

(iv) The union of a co-ICO set with the complement of a compact open set of X is co-ICO.

Proposition 3.1. *Let X be a semispectral space with a basis of compact open sets and C be a nonempty subset of X. Then the following statements are equivalent*:

(i) *C is a closed compact co-ICOC subset of X*.

(ii) *C is a closed*, *compact and co-ICO subset of X*.

(iii) *C is closed in X and there exist two compact open sets U and V of X such that* $C = U \cap (X - V)$ [13].

Now, we will give our definition and theorem.

Definition 3.2. Let X be a T_3 space. If its Fan-Gottesman compactification is spectral, then it is called *F-spectral space*.

Theorem 3.1. *Let X be a* T_4 *space*. *If it is an F-spectral*, *then it is a W-spectral*.

Proof. If X is an F-spectral space, then its Fan-Gottesman compactification is spectral. It is known that Fan-Gottesman compactification of T_4 space is Wallman-type compactification [15]. Thus, Wallman compactification of X is spectral. That is; X is a W-spectral.

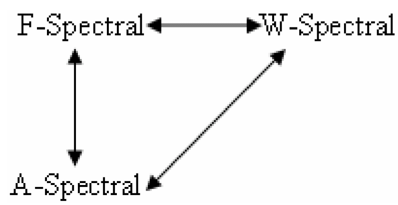
Theorem 3.2. *Let X be a local compact Hausdorff space*. *Then*

(i) *It is an A-spectral if and only if it is an F-spectral*.

(ii) *It is an A-spectral if and only if it is an H-spectral*.

(iii) *It is an F-spectral if and only if it is an H-spectral*.

Proof. If X is an A-spectral space, then its one-point compactification is spectral. It is known that Aleksandrov, Wallman and Fan-Gottesman compactifications of a local compact space are homeomorphic to each other [16]. Thus its Fan-Gottesman and Wallman compactifications are spectral, too. Hence X is an F-spectral space and also, X is a W-spectral space. That is; if X is a local compact Hausdorff space, then we get this diagram:



It is clear that F-spectral space is up-spectral. A natural question is whether an up-spectral space is necessarily F-spectral. We give answer of the above question.

Theorem 3.3. *Let X be a topological space*. *Then the following statements are equivalent*:

(1) *X is F-spectral*.

(2) *X satisfies the following properties*:

(i) *X is up-spectral*.

(ii) *For each compact closed subset C of X*, *there exists a compact closed and co-ICO subset D of X such that $C \subseteq D$*.

Proof. $(1) \Rightarrow (2)$ Obviously, a space X is up-spectral if and only if it satisfies the following conditions:

(a) X has a basis of compact open sets which are closed under finite intersections.

(b) X is sober.

Hence F-spectral property implies the up-spectral one. Let C be a compact closed subset of X. Then there exists a co-compact ICOC open subset O of X such that $O \subseteq X - C$ by definition of up-spectral space. Clearly, $D = X - O$ is a compact closed and co-ICO subset of X and $C \subseteq D$.

$(2) \Rightarrow (1)$ Let C be a compact closed subset of X. Then there exists a compact closed and co-ICO subset D of X such that $C \subseteq D$. According to Proposition 3.1, $O = X - D$ is ICOC. Thus $X - D$ is an open co-compact ICOC subset of X satisfying $O \subseteq X - C$. Therefore, X is F-spectral. □

Example. The digital line, also known as the Khalimski line, is the set of integers $\mathbb{Z}$ equipped with the topology $\mathcal{L}$, generated by the subbase $G_{\mathcal{L}} = \{\{2n - 1, 2n, 2n + 1\} : n \in \mathbb{Z}\}$ [24]. It follows readily that an even point is closed and that an odd point is open. In terms of smallest neighborhood, we have $N(m) = \{m\}$ if m is odd and $N(n) = \{n \pm 1, n\}$ if n is even. The Khalimski line $(\mathbb{Z}, \mathcal{L})$ satisfies the following properties:

(i) $\mathcal{L}$ is an Alexandroff topology.

(ii) $(\mathbb{Z}, \mathcal{L})$ is a submaximal space.

(iii) $(\mathbb{Z}, \mathcal{L})$ is an F-spectral space.

Really, Property (i) is straightforward. (ii) This property follows clearly from proposition and the fact that the digital line is 1-dimensional. (iii) The digital line is 1-dimensional which implies that it is a sober space. The collection $\beta = \{(\downarrow, x) : x \in \mathbb{Z}\} \cup \{\varnothing\}$ is a basis of compact open sets of the digital line, which is closed under finite intersections. It is easily seen that compact subsets of digital line are exactly finite sets. Thus each subset of $\mathbb{Z}$ is ICO. Therefore, $(\mathbb{Z}, \mathcal{L})$ is an F-spectral by Theorem 3.3.

4. Primitive Word

By an alphabet, we mean a finite nonempty set A. The elements of A are called *letters* of A. A finite word over an alphabet A is a finite sequence of elements of A. The set of all finite words is denoted by $A^{\bullet}$. The sequence of zero letters is called the *empty word* and denoted by ε_A. We will denote by A^+ the set of all finite nonempty words. If $u = u_1 \cdots u_n$ is a finite sequence of n letters, then n is called the *length* of the word u and we denote it by $|u|$. Let us denote by A^n the set of all finite words over A of length n. The concatenation of two words $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_m$ of lengths, respectively, n and m, is the word $uv = u_1 \cdots u_n v_1 \cdots v_m$ of length $n + m$. The set $A^{\bullet}$ equipped with the concatenation operation is a monoid with ε_A as a unit element. A *power* of a word u is a word of the form u^k for some $k \in \mathbb{N}$. It is convenient to set $u^0 = \varepsilon_A$, for each word u. When $k \in \mathbb{N} \setminus \{1\}$, we say that u^k is a *proper power* of u. A word u is said to be a *prefix* of a word v if there exists a word t such that $ut = v$. A word u is said to be a *suffix* of a word v if there exists a word t such that $tu = v$. A word u is said to be a *factor* of a word v if there exist two words t and s such that $tus = v$. If $u = vt$, then we set $ut^{-1} = v$ or $v^{-1}u = t$. The prefix of length k of a word u will be denoted by $pref_k(u)$. A word is called *primitive* if it is nonempty and not a proper power of another word. The concept of primitive words plays a crucial role in combinatorial theory of words. Let u be a nonempty word. Then there exist a unique primitive word z and a unique integer $k \geq 1$ such that $u = z^k$. The word z is called the *primitive root* of u and is denoted by $z = p_A(u)$ [21, 22].

The operation $A \to cl_X A$ in a topological space X has the following properties:

(1) $E \subset cl_X E$.

(2) $cl_X(cl_X E) = cl_X E$.

(3) $cl_X(A \cup B) = cl_X A \cup cl_X B$.

(4) $cl_X \varnothing = \varnothing$.

Given a set X and a mapping $A \to cl_X A$ of $P(X)$ into $P(X)$ which satisfies above four conditions is called *Kuratowski closure operation* [30]. Moreover, if we suppose that E is closed in X, if $cl_X E = E$, then the result is a topology on X whose closure operation is just the operation $A \to cl_X A$ we began with.

If A is an alphabet, then the map $_A : P(X) \to P(X)$ defined by $\overline{L}_A = L \cup p_A(L)$ is a Kuratowski closure defining an Alexandroff topology on A^*.

The topology defined previously will be called the *topology of primitive languages* on A^* (*PL-topology*, for short) and we denote it by $PL(A)$. In other words, the *PL-topology* is the topology which has primitive languages as closed sets.

Each topological space X which is homeomorphic to A^+ equipped with the topology of primitive languages for some alphabet A is called a *PL-space* [21, 22].

Following proposition is obtained from Proposition 2.3.

Proposition 4.1. *Let A be an alphabet. We equip* A^* *by its PL-topology. Then the following properties hold*:

(1) *If* $u \in A^+$ *is a primitive word, then u is a closed point and the smallest open set containing u is*

$$V_A(u) = \{u^n : n \in \mathbb{N}^+\}.$$

(2) *If* $u \in A^+$ *is not a primitive word, then u is an open point and*

$$cl_X\{u\} = \{u, p_A(u)\}.$$

(3) $\dim_K(A^*) = 1$.

(4) A^* *is a submaximal space*.

Now, we will give the main theorem in this part.

Theorem 4.1. *Let* (X, τ) *be a PL-space and* $\leq$ *be the ordering defined on X by* $u \leq v$ *if and only if* $v \in cl_X\{u\}$. *Then the following properties hold*:

(1) $(X, \leq)$ *is a spectral set*.

(2) (X, τ) *is an up-spectral space*.

(3) (X, τ) *is spectral if and only if X has a unique closed point*.

(4) (X, τ) *is an F-spectral space*.

Proof. We may suppose without loss of generality that $X = A^+$, for some finite alphabet A.

(1) Let $\mathrm{Prim}(A)$ be the set of all primitive words over A. For $u \in \mathrm{Prim}(A)$, we let X_u to be the set $\{u^n : n \in \mathbb{N}^+\}$. Then

$$X = \bigcup_{u \in \mathrm{Prim}(A)} X_n$$

is an ordered disjoint union, where the ordering on X_u is defined by $u^n \leq u^m$ if and only if $n = m$ or $m = 1$.

By Theorem 4.1 in [21], in order to prove that $(X, \leq)$ is spectral, it is enough to show that $(X_u, \leq)$ is spectral, for each $u \in \mathrm{Prim}(A)$. But this is an immediate consequence of the characterization of L(eft)-spectral space done in [10]. Therefore $(X, \leq)$ is a spectral set.

(2) (i) (X, τ) is a sober space. This follows immediately from Proposition 4.1.

(ii) (X, τ) has a basis of compact open sets which is closed under finite intersections. It is easily seen that $B = \{V_A(u) : u \in A^*\} \cup \{\emptyset\}$ is a basis of compact open sets; and for each pair of words $u, v \in A^*$, the two sets $V_A(u)$ and $V_A(v)$ are comparable or do not meet.

(3) According to (2), X is spectral if and only if it is compact.

If A is not a singleton, then there are infinitely many primitive words. Hence

$$X = \bigcup_{u \in \mathrm{Prim}(A)} V_A(u)$$

is an open covering of X which has no finite subcover. Thus X is not compact.

Now, if A is a singleton, then the unique closed point of A^* is the word a. In this case, $X = V_A(a)$ and thus X is a compact.

To prove that X is an F-spectral, it is enough to show that (X, τ) satisfies the condition (ii) of Theorem 3.3 because X is an up-spectral.

Let C be a compact closed set of X. Then there exist finitely many primitive words $p_1, p_2, \ldots, p_n$ over A such that

$$C \subseteq V_A(p_1) \cup V_A(p_2) \cup \cdots \cup V_A(p_n)$$

set

$$O = \bigcup_{u \in \mathrm{Prim}(A) \setminus \{p_1, p_2, \ldots, p_n\}} V_A(u),$$

then clearly O is both open and closed subset of X. It is easy to check that each closed set of topological space is ICOC. Thus O is co-compact ICOC open subset of X such that $O \subseteq X \setminus C$. Therefore, (X, τ) is an F-spectral space.

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