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HAUSDORFF DIMENSION OF THE RECURRENCE SET OF LÜROTH TRANSFORMATION

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Abstract

Let $x_0 \in [0, 1)$ be an irrational number with Lüroth series expansion $x_0 = [i_1, i_2, ...]$ and t_n be a nondecreasing sequence of natural numbers. Define the recurrence set of Lüroth transformation T as follows:

$$E(x_0) = \{x \in [0, 1) : T^n(x) \in I_{t_n}(x_0) \text{ for infinitely many } n\},$$

where $I_{t_n}(x_0)$ denotes t_nth order cylinder of x_0. In this paper, the Hausdorff dimension of the set $E(x_0)$ is determined.

1. Introduction

For $x \in (0, 1]$, notice that there is a unique $a_1(x) \in \mathbb{N}$ such that

$$\frac{1}{a_1(x)} < x \le \frac{1}{a_1(x) - 1}. \tag{1.1}$$

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Indeed, $a_1(x) = [1/x] + 1$, where $[\cdot]$ denotes the integer part. Define $T : (0, 1] \to (0, 1]$ by

$$T(x) := a_1(x)(a_1(x) - 1)\left(x - \frac{1}{a_1(x)}\right).$$

Now, we introduce the sequence $(a_k)_{k \in \mathbb{N}}$, where

$$a_k(x) = a_1(T^{k-1}(x)), \tag{1.2}$$

where T^k denotes the kth iterate of T $(T^0 = Id_{(0,1]})$. From (1.1) and (1.2), we notice that any $x \in (0, 1]$ can be developed uniquely into infinite series expansion of the form

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1 - 1)a_2} + \cdots + \frac{1}{a_1(a_1 - 1)a_2 \cdots a_{n-1}(a_{n-1} - 1)a_n} + \cdots. \tag{1.3}$$

This series expansion, called *Lüroth expansion*, was introduced in 1883 by Lüroth [8]. We may denote the Lüroth series expansion of $x \in (0, 1]$ by $x = [a_1(x), a_2(x), \ldots]$ for simplicity.

The behavior of the sequence $a_n(x)$ is of interest and the metric and ergodic properties of the sequence $\{a_n(x),\ n \geq 1\}$ and T have been investigated by a number of authors (see [1, 2, 3, 6, 7, 9]).

In [5], Fernández and Melián have considered the quantitative recurrence properties in continued fraction dynamical system. In this paper, we consider the analogous problem for the Lüroth series expansion. We determine the Hausdorff dimensions of $E(x_0)$.

Theorem 1.1. *Let* $\liminf_{n\to\infty} \frac{\log i_1(i_1 - 1)\cdots i_{t_n}(i_{t_n} - 1)}{n} = \log B$, *if* $1 < B < +\infty$. *Then*

$$\dim_H E(x_0) = s(B).$$

2. Preliminaries

In this section, we collect some known facts and establish some elementary properties of Lüroth series that will be used later [3, 6].

Lemma 2.1 [6]**.** *The series in* (1.2) *is a Lüroth expansion of some* $x \in (0, 1]$ *if and only if* $a_n \geq 2$, *for all* $n \geq 1$.

For any $a_1, a_2, \ldots, a_n \in \mathbb{N}$, we call $I_n(a_1, \ldots, a_n) = \{x \in [0, 1) : a_k(x) = a_k, 1 \leq k \leq n\}$ an nth order cylinder.

Lemma 2.2 [6]**.** *The length of* $I_n(a_1, \ldots, a_n)$ *is equal to*

$$|I_n(a_1, \ldots, a_n)| = \frac{1}{a_1(a_1 - 1)\cdots a_n(a_n - 1)}, \tag{2.1}$$

where $|\cdot|$ *denotes the Euclidean length.*

Lemma 2.3. *Let* $s(B)$ *be the unique solution of*

$$f_B(s) := \sum_{k=2}^{\infty} \left(\frac{1}{Bk(k-1)}\right)^s = 1. \tag{2.2}$$

Then $s(B)$ *is continuous with respect to* B. *Furthermore,* $\lim_{B \to 1} s(B) = 1$, $\lim_{B \to \infty} s(B) = 1/2$.

Proof. (i) Fix $B > 0$. For any $\varepsilon > 0$, when $1 < B < B' < B + \varepsilon$,

$$\sum_{k=2}^{\infty} \left(\frac{1}{Bk(k-1)}\right)^{s(B')+\varepsilon} \leq \frac{1}{B^{\varepsilon}} \sum_{k=2}^{\infty} \left(\frac{1}{Bk(k-1)}\right)^{s(B')}$$

$$= \frac{1}{B^{\varepsilon}} \left(\frac{B'}{B}\right)^{s(B')} \leq \frac{B'}{B^{1+\varepsilon}} < 1.$$

Notice that $s(\cdot)$ is monotonic decreasing with respect to B. We have $s(B') < s(B) < s(B') + \varepsilon$.

(ii) We prove that $\lim_{B \to 1} s(B) = 1$ only, the second limits can be proved by the similar method. It is easy to see that $s(B) \leq 1$ for any $B > 1$. On the other hand, fix $\varepsilon > 0$, take $B_0 = 2^{\varepsilon}$, for all $1 < B < B_0$, we have $s(B) > 1 - \varepsilon$, since

$$\sum_{k=2}^{\infty} \left(\frac{1}{Bk(k-1)}\right)^{1-\varepsilon} \geq 2^{\varepsilon} \sum_{k=2}^{\infty} \frac{1}{Bk(k-1)} > 1. \qquad \square$$

3. Proof of Main Results

Before proving Theorem 1.1, we state the mass distribution principle at first (see [4]), which will be applied to obtain the lower bound of $\dim_H E(x_0)$.

Lemma 3.1 [4]**.** *Let E be a Borel set in* $[0, 1]$ *and* μ *be a measure with* $\mu(E) > 0$. *If for any* $x \in E$,

$$\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq s,$$

where $B(x, r)$ *denotes the open ball with center x and radius r, then* $\dim_H E \geq s$.

Proof of Theorem 1.1. First, we give an upper bound of $\dim_H E(x_0)$.

Notice that

$$E(x_0) = \{x \in [0, 1) : T^n(x) \in I_{t_n}(x_0) \text{ for infinitely many } n\}$$

$$= \limsup_{n \to 0} \{x \in [0, 1) : T^n(x) \in I_{t_n}(x_0)\}$$

$$= \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} \{x \in [0, 1) : T^n(x) \in I_{t_n}(x_0)\}$$

$$= \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} \bigcup_{a_1, \ldots, a_n} \{x \in [0, 1) : a_k(x) = a_k \in \mathbb{N},$$

$$1 \leq k \leq n;\ a_{n+j}(x) = i_j,\ 1 \leq j \leq n\}.$$

For any $\varepsilon > 0$ and $\tau > 0$, when n is large enough, we have $i_1(i_1 - 1) \cdots i_{t_n}(i_{t_n} - 1) > (B - \varepsilon)^n$. So

$$\mathbf{H}^{s(B-\varepsilon)+\tau}(E(x_0))$$

$$\leq \liminf_{N \to \infty} \sum_{n \geq N} \sum_{a_1, \ldots, a_n} | I_{n+t_n}(a_1, \ldots, a_n, i_1, i_2, \ldots, i_{t_n}) |^{s(B-\varepsilon)+\tau}$$

$$\leq \liminf_{N \to \infty} \sum_{n \geq N} \sum_{a_1, \ldots, a_n} \left(\frac{1}{(B - \varepsilon)^n a_1(a_1 - 1) \cdots a_n(a_n - 1)} \right)^{s(B-\varepsilon)+\tau}$$

$$\le \liminf_{N\to\infty} \sum_{n\ge N} \frac{1}{(2^n)^\tau} \sum_{a_1, \ldots, a_n} \left(\frac{1}{(B-\varepsilon)^n a_1(a_1-1)\cdots a_n(a_n-1)}\right)^{s(B-\varepsilon)}$$

$$= \liminf_{N\to\infty} \sum_{n\ge N} \frac{1}{(2^n)^\tau} = 0.$$

So $\dim_H E(x_0) \le s(B-\varepsilon) + \tau$. Since $\tau > 0$ and $\varepsilon > 0$ are arbitrary, from Lemma 2.3, we have $\dim_H E(x_0) \le s(B)$.

Now, we give a lower bound of $\dim_H E(x_0)$.

In this paper, we shall use Γ to denote the set $\{n_k, n_k+1, n_k+2, \ldots, n_k+t_{n_k}-1, k \ge 1\}$ for convenience.

Step I. In this part, we will construct a subset $E_\alpha(x_0) \subset E(x_0)$.

Fix $\alpha \in \mathbb{N}\backslash\{1\}$ and a sequence $\{n_k\} \subset \mathbb{N}$ satisfying $\lim_{k\to\infty} \frac{\log i_1(i_1-1)\cdots i_{t_{n_k}}(i_{t_{n_k}}-1)}{n_k} = \log B$ and

$$n_1 + \cdots + n_k + t_{n_k} < \frac{1}{k+1} n_{k+1}, \quad \forall k \ge 1. \tag{3.1}$$

We may assume that

$$(B-\varepsilon)^{n_k} \le i_1(i_1-1)\cdots i_{t_{n_k}}(i_{t_{n_k}}-1) \le (B+\varepsilon)^{n_k}, \quad \forall k \ge 1.$$

Let

$$E_\alpha(x_0) = \{x \in [0, 1) : x = [\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots], \sigma_{n_k+j} = i_j, k \ge 1, 1 \le j \le t_{n_k};$$

$$1 \le \sigma_j \le \alpha, j \notin \{n_k+1, n_k+2, \ldots, n_k+t_{n_k}, k \ge 1\}\}.$$

It is easy to see that $E_\alpha(x_0) \subseteq E(x_0)$. Let $s_\alpha(B)$ be the solution of $\sum_{k=2}^{\alpha} \left(\frac{1}{Bk(k-1)}\right)^s = 1$. Then we will prove that $\dim_H E_\alpha(x_0) \ge s_\alpha(B)$.

For any $n \ge 1$, define

$$D_n = \{(\sigma_1, \ldots, \sigma_n) \in \mathbb{N}^n : E_\alpha(x_0) \cap I_n(\sigma_1, \ldots, \sigma_n) \ne \varnothing\};$$

$$D = \bigcup_{n=0}^{\infty} D_n, \quad (D_0 := \varnothing).$$

For any $n \geq 1$ and $(\sigma_1, ..., \sigma_n) \in D_n$, we call

$$J(\sigma_1, ..., \sigma_n) := \bigcup_{\sigma_{n+1}} clI_{n+1}(\sigma_1, ..., \sigma_{n+1}) \tag{3.2}$$

a basic interval of order *n* with respect to $E_\alpha(x_0)$, where the union in (3.2) is taken over all σ_{n+1} such that $(\sigma_1, ..., \sigma_n, \sigma_{n+1}) \in D_{n+1}$ and *cl* stands for the closure. Then it follows

$$E_\alpha(x_0) = \bigcap_{n \geq 1} \bigcup_{(\sigma_1, ..., \sigma_n) \in D_n} J(\sigma_1, ..., \sigma_n). \tag{3.3}$$

From Lemma 2.2, if $n \notin \Gamma$, then we have

$$|J(\sigma_1, ..., \sigma_n)| = \frac{1}{a_1(a_1 - 1) \cdots a_n(a_n - 1)} \left(1 - \frac{1}{\alpha}\right). \tag{3.4}$$

Step II. For the lower bound, we define a probability measure supported on $E_\alpha(x_0)$.

Let $m_k = n_k - n_{k-1} - t_{n_{k-1}}$ with $k \geq 1$ and $n_0 = t_{n_0} := 0$. Now, we define a set function $\mu : \{J(\sigma), \sigma \in D \backslash D_0\} \to \mathbb{R}^+$ given as follows. In this paper, we always use q_{m_j} to denote

$$a_{n_{j-1}+t_{n_{j-1}}+1}(a_{n_{j-1}+t_{n_{j-1}}+1} - 1) \cdots a_{n_j}(a_{n_j} - 1), \quad j \in \mathbb{N}.$$

For any $n \geq 1$ and $(\sigma_1, ..., \sigma_n) \in D_n$, let

$$\mu(J(\sigma_1, ..., \sigma_n))$$

$$:= \begin{cases} \prod_{j=1}^{k} \left(\dfrac{1}{q_{m_j}(B + \varepsilon)^{m_j}} \right)^{s_\alpha(B+\varepsilon)}, & \text{if } n \in \{n_k, n_k + 1, n_k + 2, ..., n_k + t_{n_k}, k \geq 1\}, \\ \displaystyle\sum_{2 \leq \sigma_{n+1}, ..., \sigma_{n_k} \leq \alpha} \mu(J(\alpha_1, ..., \sigma_{n_k})), & \text{if } n_{k-1} + t_{n_{k-1}} < n < n_k, \text{for some } k \geq 1. \end{cases} \tag{3.5}$$

Until now, the set function $\mu : \{J(\sigma), \sigma \in D \backslash D_0\} \to \mathbb{R}^+$ is well defined. It is easy to check that for any $n \geq 1$ and $(\sigma_1, \ldots, \sigma_n) \in D_n$, we have

$$\mu(J(\sigma_1, \ldots, \sigma_n)) = \sum_{\sigma_{n+1}} \mu(J(\sigma_1, \ldots, \sigma_{n+1})),$$

where the summation is taken over all σ_{n+1} such that $(\sigma_1, \ldots, \sigma_n, \sigma_{n+1}) \in D_{n+1}$. Notice that

$$\sum_{(\sigma_1, \ldots, \sigma_{n_1}) \in D_{n_1}} \mu(J(\sigma_1, \ldots, \sigma_{n_1})) = 1,$$

by Kolmogorov extension theorem, the set function μ can be extended into a probability measure supported on $E_\alpha(x_0)$, which is still denoted by μ.

Step III. We now give the estimation of $\mu(J(\sigma_1, \ldots, \sigma_n))$ for each $(\sigma_1, \ldots, \sigma_n) \in D_n$.

Fix $0 < t < s_\alpha(B + \varepsilon)$, take $\tau = \dfrac{s_\alpha(B+\varepsilon) - t}{2}$. We claim that there is an integer N such that $n \geq N$ and $(\sigma_1, \ldots, \sigma_n) \in D_n$ implies

$$\mu(J(\sigma_1, \ldots, \sigma_n)) \leq c \cdot |J(\sigma_1, \ldots, \sigma_n)|^{t-2\tau}, \tag{3.6}$$

where $c > 0$ is an absolute constant.

We will distinguish two cases to establish this. Choose k_0 sufficiently large such that

$$\frac{t}{t+\tau} \leq \frac{m_k}{n_k}, \quad 1 - \frac{1}{\alpha} \leq 2^{n_k \tau}, \quad \forall k > k_0. \tag{3.7}$$

Take $c_0 = \alpha^{2n_{k_0}} (B + \varepsilon)^{(n_1 + \cdots + n_{k_0})}$. Then we have

$$\prod_{j=1}^{k_0} \left(\frac{1}{q_{m_j}(B+\varepsilon)^{m_j}} \right)^{s_{m_j, (B+\varepsilon)}(\alpha)} \leq 1 \leq c_0 \prod_{j=1}^{k_0} \left(\frac{1}{q_{m_j}(B+\varepsilon)^{n_j}} \right)^t. \tag{3.8}$$

For any $n > n_{k_0}$ and $(\sigma_1, \ldots, \sigma_n) \in D_n$, we estimate $\mu(J(\sigma_1, \ldots, \sigma_n))$.

Case I. $n \in \{n_k, n_k + 1, n_k + 2, \ldots, n_k + t_{n_k}\}$ for some $k \geq k_0$,

$$\mu(J(\sigma_1, \ldots, \sigma_n))$$

$$= \prod_{j=1}^{k} \left(\frac{1}{q_{m_j}(B+\varepsilon)^{m_j}} \right)^{s_\alpha(B+\varepsilon)} \quad \text{(by (3.5))}$$

$$= \prod_{j=1}^{k_0} \left(\frac{1}{q_{m_j}(B+\varepsilon)^{m_j}} \right)^{s_\alpha(B+\varepsilon)} \cdot \prod_{j=k_0+1}^{k} \left(\frac{1}{q_{m_j}(B+\varepsilon)^{m_j}} \right)^{s_\alpha(B+\varepsilon)}$$

$$\leq c_0 \prod_{j=1}^{k_0} \left(\frac{1}{q_{m_j}(B+\varepsilon)^{n_j}} \right)^{t} \cdot \prod_{j=k_0+1}^{k} \left(\frac{1}{q_{m_j}(B+\varepsilon)^{m_j}} \right)^{t+\tau} \quad \text{(by (3.8))}$$

$$\leq c_0 \prod_{j=1}^{k_0} \left(\frac{1}{q_{m_j}(B+\varepsilon)^{n_j}} \right)^{t} \cdot \prod_{j=k_0+1}^{k} \left(\frac{1}{q_{m_j}} \right)^{t} \cdot \prod_{j=k_0+1}^{k} \left(\frac{1}{(B+\varepsilon)^{n_j}} \right)^{t} \quad \text{(by (3.7))}$$

$$= c_0 \prod_{j=1}^{k} \left(\frac{1}{q_{m_j}(B+\varepsilon)^{n_j}} \right)^{t} \leq c_0 |\, J(\sigma_1, \ldots, \sigma_{n_k + t_{n_k}})\,|^{t-\tau} \quad \text{(by (3.4) and (3.7))}$$

$$\leq c_0 |\, J(\sigma_1, \ldots, \sigma_n)\,|^{t-\tau}, \tag{3.9}$$

where $q_{n_k} = \sigma_1(\sigma_1 - 1) \cdots \sigma_{n_k}(\sigma_{n_k} - 1)$.

Case II. $n_{k-1} + t_{n_{k-1}} < n < n_k$ for some $k \geq k_0$.

Let $\ell' = n_k - n$. By the definition of μ, similar to the proof of (3.9), we have

$$\mu(J(\sigma_1, \ldots, \sigma_n))$$

$$= \sum_{2 \leq \sigma_{n+1}, \ldots, \sigma_{n_k} \leq \alpha} \mu(J(\sigma_1, \ldots, \sigma_n, \sigma_{n+1}, \ldots, \sigma_{n_k}))$$

$$= \prod_{j=1}^{k-1} \left(\frac{1}{q_{m_j}(B+\varepsilon)^{m_j}} \right)^{s_\alpha(B+\varepsilon)} \sum_{2 \leq \sigma_{n+1}, \ldots, \sigma_{n_k} \leq \alpha} \left(\frac{1}{q_{m_k}(B+\varepsilon)^{m_k}} \right)^{s_\alpha(B+\varepsilon)}$$

$$\leq \frac{c_0}{(a_1(a_1-1)\cdots a_n(a_n-1))^{t-\tau}} \sum_{2\leq a_1,\ldots,a_{\ell'}\leq\alpha} \left(\frac{1}{(B+\varepsilon)^{\ell'} q_{\ell'}}\right)^{s_\alpha(B+\varepsilon)}$$

$$= \frac{c_0}{(a_1(a_1-1)\cdots a_n(a_n-1))^{t-\tau}} \left(\sum_{k=2}^{\alpha}\left(\frac{1}{(B+\varepsilon)q_{\ell'}}\right)^{s_\alpha(B+\varepsilon)}\right)^{\ell'}$$

$$= \frac{c_0}{(a_1(a_1-1)\cdots a_n(a_n-1))^{t-\tau}}$$

$$\leq c_0 |J(\sigma_1, \ldots, \sigma_n)|^{t-2\tau} \quad \text{(by (3.4) and (3.7))}, \tag{3.10}$$

where $q_{\ell'} = \sigma_1(\sigma_1 - 1)\cdots\sigma_{\ell'}(\sigma_{\ell'} - 1)$.

Step IV. In this part, we will estimate the measure of $B(x, r)$.

For any $x \in E_\alpha(x_0)$, there exists an infinite sequence $\{\sigma_1, \sigma_2, \ldots\}$ with $\sigma_{n_k+j} = i_j$, $k \geq 1$, $1 \leq j \leq t_{n_k}$; $1 \leq \sigma_j \leq \alpha$, $j \notin \{n_k+1, n_k+2, \ldots, n_k+t_{n_k}, k \geq 1\}$ such that $x \in J(\sigma_1, \ldots, \sigma_n)$, for all $n \geq 1$. Let $r_0 = \min_{\sigma\in D_{n_{k_0}}} |J(\sigma)|$. Then, for any $0 < r < r_0$, there exists an integer $n \geq n_{k_0}$ such that

$$|J(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})| \leq r < |J(\sigma_1, \ldots, \sigma_n)|. \tag{3.11}$$

Now, we distinguish two cases to estimate the measure of $B(x, r)$.

Case I. $n \in \Gamma$, i.e., $n_k \leq n < n_k + t_{n_k}$, for some $k \geq 1$.

In this case, the ball $B(x, r)$ can intersect only one basic interval of order n, which is just $J(\sigma_1, \ldots, \sigma_n)$ and can intersect at most one basic interval of order $n+1$. From the dimension of the measure μ and (3.6), we have

$$\mu(B(x, r)) \leq \mu(J(\sigma_1, \ldots, \sigma_n)) = \mu(J(\sigma_1, \ldots, \sigma_{n+1}))$$

$$\leq c_0|J(\sigma_1, \ldots, \sigma_{n+1})|^{t-2\tau} \leq c_0|r|^{t-2\tau}. \tag{3.12}$$

Case II. $n \notin \Gamma$.

The ball $B(x, r)$ can intersect at most three basic intervals of order n.

By the dimension of the measure μ, for any $\xi \neq \eta \in \{1, 2, ..., \alpha\}$, we have $\frac{\mu(J(a_1, ..., a_n, \xi))}{\mu(J(a_1, ..., a_n, \eta))} \leq \alpha^2$. So

$$\mu(J(\sigma_1, ..., \sigma_n, \sigma_{n+1})) \geq \frac{1}{\alpha^3}\mu(J(\sigma_1, ..., \sigma_n)).$$

From (3.6) and (3.11), we obtain

$$\begin{aligned}\mu(B(x, r)) &\leq 3\alpha^5\mu(J(\sigma_1, ..., \sigma_{n+1})) \\ &\leq 3c_0 \cdot \alpha^5 |J(\sigma_1, ..., \sigma_{n+1})|^{t-2\tau} \leq 3c_0\alpha^5 \cdot r^{t-2\tau}. \end{aligned} \tag{3.13}$$

Combining these two cases with Lemma 3.1, we can get

$$\dim_H E_\alpha(x_0) \geq t - 2\tau = 2t - s_\alpha(B + \varepsilon).$$

Since $t < s_\alpha(B + \varepsilon)$ is arbitrary, we have

$$\dim_H E(x_0) \geq \dim_H E_\alpha(x_0) \geq s_\alpha(B + \varepsilon).$$

Notice that $\lim_{\alpha \to \infty} s_\alpha(B + \varepsilon) = s(B + \varepsilon)$ and Lemma 2.3, Theorem 1.1 is proved. □

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