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COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPS IN FUZZY METRIC SPACES

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Abstract

The aim of this paper is to introduce the notion of occasionally weakly compatible (owc) maps and prove common fixed point theorems for single valued maps without considering the completeness of the space and continuity of maps in fuzzy metric space. Our results extend, generalize and unify several results existing in the literature.

1. Introduction

It was a turning point in the development of mathematics when Zadeh [30] introduced the concept of fuzzy set. This laid the foundation of fuzzy mathematics. Consequently, the last three decades were very productive for fuzzy mathematics. Several authors like Deng [11], Erceg [13], Kaleva and Seikkala [19] and Kramosil and Michalek [21] have introduced the concept of fuzzy metric space in different ways.

The concepts of weak commutativity, compatibility, non-compatibility and weak compatibility were frequently used to prove fixed point theorems for single and set valued maps satisfying certain conditions in different spaces.

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Recently, in a paper submitted before 2006 but published only in 2008, Al-Thagafi and Shahzad [2] weakened the notion of weakly compatible maps by introducing a new concept of occasionally weakly compatible (owc) maps. This concept is the most general among all the commutativity concepts and has opened a new venue for many mathematicians. This newly defined concept has also fascinated many authors like Bouhadjera and Godet-Thobie [6, 7], Abbas and Rhoades [1], Gairola and Rawat [14], Chandra and Bhatt [8], etc.

The main purpose of our paper is to introduce the concept of occasionally weakly compatible (owc) maps in fuzzy metric space and to prove common fixed point theorems for single valued maps under strict contractive condition.

Our improvements in this paper are five-fold as:

(i) Relaxed the continuity of maps completely,

(ii) Completeness of the space removed,

(iii) Minimal type contractive condition used,

(iv) The condition $\lim_{t\to\infty} M(x, y, t) = 1$ not used in our results,

(v) Weakened the concept of compatibility by a more general concept of occasionally weak compatible (owc) maps.

We first give some preliminaries and definitions.

2. Preliminaries

Definition 2.1. A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is continuous *t-norm* if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$ for all $a \in [0, 1]$,

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

Definition 2.2. The triplet $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous *t-norm* and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following:

(FM-1) $M(x, y, t) > 0$,

(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, \bullet) : (0, \infty) \to (0, 1]$ is left continuous for all $x, y, z \in X$ and $s, t > 0$.

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all x, y in X.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Example 1 (Induced fuzzy metric space)**.** Let (X, d) be a metric space, denote $a * b = ab$ for all $a, b \in [0, 1]$ and let M_d be fuzzy set on $X^2 \times [0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, *)$ is a fuzzy metric space. We call this *fuzzy metric* induced by a metric d.

Example 2. Let $X = N$. Define $a * b = \max\{0, a + b - 1\}$ for all $a, b \in [0, 1]$ and let M be a fuzzy set on $X^2 \times [0, \infty)$ as follows:

$$M(x, y, t) = \left\{\begin{array}{l} \frac{x}{y} \text{ if } x \leq y \\ \frac{y}{x} \text{ if } y \leq x \end{array}\right\} \text{ for all } x, y \in X.$$

Then $(X, M, *)$ is a fuzzy metric space.

Note that in the above example, there exists no metric d on X, satisfying $M(x, y, t) = \frac{t}{t + d(x, y)}$, where $(X, M, *)$ is defined in above example. Also, note that the above function M is not a fuzzy metric with the *t-norm* defined as $a * b = \min\{a, b\}$.

Let f, g be self maps of a fuzzy metric space $(X, M, *)$ and $C(f, g) = \{x \in X;\ f(x) = g(x)\}$. Then the pair (f, g) is called

(1) *weakly commuting* if $M(fgx, gfx, t) \geq M(fx, gx, t)$ for all $t > 0$.

(2) *weakly compatible* if $f(g(x)) = g(f(x))$ for all $x \in C(f, g)$.

(3) *occasionally weakly compatible* (owc) if $f(g(x)) = g(f(x))$ for some $x \in C(f, g)$.

The following example shows that weakly compatible maps form a proper subclass of the owc maps.

Example 3. Let $X = [0, \infty)$ with $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $t > 0$. Then $(X, M, *)$ is a fuzzy metric space.

Let $f, g : X \to X$ be maps defined by

$$f(x) = \begin{Bmatrix} 2, & 0 \leq x \leq 1 \\ 2x, & x > 1 \end{Bmatrix}, \qquad g(x) = \begin{Bmatrix} x + 1, & 0 \leq x \leq 1 \\ 4, & x > 1 \end{Bmatrix}.$$

Here '1' and '2' are two coincidence points of f and g, i.e., $C(f, g) = \{1, 2\}$. $f(1) = 2 = g(1)$, $f(2) = 4 = g(2)$.

But $fg(1) = f(2) = 4 = gf(1) = g(2)$ and $fg(2) = f(4) = 8 \neq gf(2) = g(4) = 4$.

Thus (f, g) is a pair of occasionally weak compatible (owc) maps but they are not weak compatible.

3. Main Result

Now, we prove our main results.

Theorem 1. *Let $(X, M, *)$ be a fuzzy metric space. Let $f, g : X \to X$ be occasionally weakly compatible (owc) maps satisfying*

$$M(fx, fy, kt) \geq \min\begin{Bmatrix} M(gx, gy, t), M(gx, fy, t) \\ M(gy, fx, t), M(gy, fy, t) \end{Bmatrix} \tag{1.1}$$

for every $x, y \in X$, $t > 0$. Then f and g have unique common fixed point in X.

Proof. Since the f and g are occasionally weakly compatible (owc) maps, there exists an element u in X such that $fu = gu, \quad fgu = gfu$.

We claim that fu is the unique common fixed point of f and g. We first assert that fu is a fixed point of f.

For this, if $ffu \neq fu$, then from (1.1), we have

$$M(fu, ffu, kt) \geq \min\begin{Bmatrix} M(gu, gfu, t), M(gu, ffu, t) \\ M(fu, fu, t), M(gfu, ffu, t) \end{Bmatrix}$$

$$= \min\begin{Bmatrix} M(fu, ffu, t), M(fu, ffu, t) \\ M(ffu, ffu, t), 1 \end{Bmatrix}$$

$$= M(fu, ffu, t).$$

This gives $fu = ffu$ and so $ffu = fgu = gfu = fu$.

Thus fu is a common fixed point of f and g.

For uniqueness, suppose that $u, v \in X$ such that $fu = gu = u$ and $fv = gv = v$ and $u \neq v$. Then from (1.1), we have

$$M(fu, fv, kt) \geq \min\begin{Bmatrix} M(gu, gv, t), M(gu, fv, t) \\ M(gv, fu, t), M(gv, fv, t) \end{Bmatrix}.$$

But $fu = gu = u$ and $fv = gv = v$, so

$$M(u, v, kt) \geq \min\begin{Bmatrix} M(u, v, t), M(u, u, t) \\ M(v, u, t), M(u, v, t) \end{Bmatrix},$$

i.e., $M(u, v, kt) \geq M(u, v, t)$. Thus $u = v$. Therefore, f and g have a unique common fixed point.

Now, we give an example which satisfies all the conditions of the above theorem.

Example 4. Let $(X, M, *)$ be a fuzzy metric space in which $X = [0, \infty)$, $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \dfrac{t}{t + d(x, y)}$, $\forall t > 0$.

Define the maps $f, g : X \to X$ by setting

$$f(x) = 2, \quad g(x) = \frac{x+2}{2}.$$

Then *f* and *g* are occasionally weakly compatible (owc) maps.

Also, the contractive condition (1.1) is satisfied as L.H.S. is always 1.

Thus all the conditions of our theorem are satisfied and '2' is the unique common fixed point of *f* and *g*.

Theorem 2. *Let* $(X, M, *)$ *be a fuzzy metric space with* $t * t \geq t$. *Let* $f, g : X \to X$ *be occasionally weakly compatible* (*owc*) *maps satisfying*

$$M(fx, fy, kt) \geq \min\begin{Bmatrix} M(gx, gy, t) * M(gx, fy, t) \\ M(gy, fx, t) * M(gy, fy, t) \end{Bmatrix} \tag{1.1}$$

for every $x, y \in X$, $t > 0$. *Then f and g have unique common fixed point in X.*

Proof. Since the maps *f* and *g* are occasionally weakly compatible (owc) maps, there exists an element *u* in *X* such that $fu = gu$, $fgu = gfu$.

We claim that *fu* is the unique common fixed point of *f* and *g*. We first assert that *fu* is a fixed point of *f*.

For this, if $ffu \neq fu$, then from (1.1), we have

$$\begin{aligned} M(fu, ffu, kt) &\geq \min\begin{Bmatrix} M(gu, gfu, t) * M(gu, ffu, t) \\ M(gfu, fu, t) * M(gfu, ffu, t) \end{Bmatrix} \\ &= \min\begin{Bmatrix} M(fu, ffu, t) * M(fu, ffu, t) \\ M(ffu, fu, t) * 1 \end{Bmatrix} \\ &= M(fu, ffu, t). \end{aligned}$$

This gives $fu = ffu$ and so $ffu = fgu = gfu = fu$.

Thus *fu* is a common fixed point of *f* and *g*.

For uniqueness, suppose that $u, v \in X$ such that $fu = gu = u$ and $fv = gv = v$ and $u \neq v$. Then from (1.1), we have

$$M(fu, fv, kt) \geq \min\begin{Bmatrix} M(gu, gv, t) * M(gu, fv, t) \\ M(gv, fu, t) * M(gv, fv, t) \end{Bmatrix}.$$

But $fu = gu = u$ and $fv = gv = v$, so

$$M(u, v, kt) \geq \min\begin{Bmatrix} M(u, v, t) * M(u, u, t) \\ M(v, u, t) * M(u, v, t) \end{Bmatrix},$$

i.e., $M(u, v, kt) \geq M(u, v, t)$. Thus $u = v$. Therefore, f and g have a unique common fixed point.

Now, we prove the projection of above theorems, from fuzzy metric space to metric space.

Theorem 3. *Let f and g be occasionally weakly compatible maps on a metric space* (X, d). *If there exists* $k \in (0, 1)$ *such that*

$$d(fx, fy) \leq [\max\{d(gx, gy), d(gx, fy), d(gy, fx), d(gy, fy)\}]$$

for all $x, y \in X$, *then f and g have a unique common fixed point in X.*

Proof. The proof follows from Theorem 1. Considering the induced fuzzy metric space $(X, M, *)$, where $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$.

Theorem 4. *Let f and g be occasionally weakly compatible maps on a metric space* (X, d). *If there exists* $k \in (0, 1)$ *such that*

$$d(fx, fy) \leq k[\max\{d(gx, gy) + d(gx, fy), d(gy, fx) + d(gy, fy)\}]$$

for all $x, y \in X$, *then f and g have a unique common fixed point in X.*

Proof. The proof follows from Theorem 2. Considering the induced fuzzy metric space $(X, M, *)$, where $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$.

Remark 1. In view of Theorems 3 and 4, it is clear that some results of [5-8] and [1] are special cases of our main results in fuzzy metric space.

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