# ON THE LIE ALGEBRA STRUCTURES CLOSEST TO ALGEBRA STRUCTURES 

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#### Abstract

We define the Lie algebra structure closest to a given algebra structure and give a procedure to find the closest Lie structure. Furthermore, we demonstrate our strategy for the 3-dimensional Lie algebras over the field of real numbers.


## 1. Introduction

Let $\mathbb{R}$ be the field of real numbers and $V$ be the finite dimensional vector space over $\mathbb{R}$ with the fixed basis $\left\{x_{1}, \ldots, x_{n}\right\}$. A point $\left(c_{i j k}\right) \in \mathbb{R}^{n^{3}}$ defines a multiplication on $V$ by

$$
\left[x_{i}, x_{j}\right]=\sum_{k=1}^{n} c_{i j k} x_{k}, \quad(i, j=1, \ldots, n)
$$

and we have an algebra $A=\left(V,\left(c_{i j k}\right),[-,-]\right)$ with the underlying vector space $V$ and the set of structure constants $\left(c_{i j k}\right)$.

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For a pair of the sets $\left(c_{i j k}\right)$ and $\left(a_{i j k}\right)$ of structure constants, we define the distance $D\left(\left(c_{i j k}\right),\left(a_{i j k}\right)\right)$ by

$$
\begin{equation*}
D\left(\left(c_{i j k}\right),\left(a_{i j k}\right)\right)^{2}=\sum_{i, j, k}\left(c_{i j k}-a_{i j k}\right)^{2} \tag{1}
\end{equation*}
$$

If a set of structure constants gives a Lie algebra structure on $V$, then we have the following relations:

$$
\begin{align*}
& c_{i j k}=-c_{j i k} \quad(\text { for } i, j, k=1, \ldots, n)  \tag{2}\\
& \sum_{p=1}^{n}\left(c_{i j p} c_{p k q}+c_{j k p} c_{p i q}+c_{k i p} c_{p j q}\right)=0 \quad(i, j, k, q=1, \ldots, n) . \tag{3}
\end{align*}
$$

We denote by $\mathfrak{C}$ the algebraic set defined by the above polynomial equations (2) and (3). The closest structure $\left(c_{i j k}\right)$ to a given structure $\left(a_{i j k}\right)$ stays at $\mathfrak{C}$ and minimizes $D\left(\left(c_{i j k}\right),\left(a_{i j k}\right)\right)$.

In this paper, we demonstrate our idea for the 3-dimensional Lie algebras. We first express the algebraic set $\mathfrak{C}$ of the polynomials (2), (3) by the union of the seven algebraic sets represented by parameters, say, $\mathfrak{C}_{1}, \ldots, \mathfrak{C}_{7}$. For a given point $\left(a_{i j k}\right)$, we will find the point $\left(c_{i j k}\right)_{p}$ of $\mathfrak{C}_{p}$ which minimizes $D\left(\left(c_{i j k}\right)_{p},\left(a_{i j k}\right)\right)$ for each $p=1, \ldots, 7$ and choose the point $\left(c_{i j k}\right)_{0}$ among them with $D\left(\left(c_{i j k}\right)_{i},\left(a_{i j k}\right)\right)$ minimum. Then the set of Lie algebra structure constants $\left(c_{i j k}\right)_{0}$ gives the closest structure to an algebra structure with the set $\left(a_{i j k}\right)$ of structure constants. We employ Mathematica in several stages to find the desired points.

## 2. Algebraic Set of 3-dimensional Lie Algebras

By equations (2) and (3), we have

$$
\begin{align*}
& c_{231} c_{122}-c_{231} c_{313}-c_{121} c_{232}+c_{233} c_{311}=0  \tag{4}\\
& c_{312} c_{233}-c_{312} c_{121}-c_{232} c_{313}+c_{311} c_{122}=0  \tag{5}\\
& c_{123} c_{311}-c_{123} c_{232}-c_{313} c_{121}+c_{122} c_{233}=0 \tag{6}
\end{align*}
$$

These polynomials of the left hand sides already consist a Groebner basis of the ideal generated by themselves, with an ordering on the monomials

$$
c_{123}>c_{231}>c_{312}>c_{122}>c_{233}>c_{311}>c_{121}>c_{232}>c_{313} .
$$

In our previous paper [2], we discussed the closest associative algebra structures. To parameterize the algebraic set of associative algebras, we use the Groebner basis and the elimination-extension method. This method does not work in the Lie algebra case. We shall follow the Jacobson's method of determination of the low dimensional Lie algebras [1, pp. 11-14].

Notations. The set of structure constants $\boldsymbol{c}=\left(c_{i j k}\right)$ is of the form:

$$
\begin{aligned}
& \left(c_{121}, c_{122}, c_{123}, c_{231}, c_{232}, c_{233}, c_{311}, c_{312}, c_{313}\right. \\
& c_{111}, c_{112}, c_{113}, c_{221}, c_{222}, c_{223}, c_{331}, c_{332}, c_{333} \\
& \left.c_{211}, c_{212}, c_{213}, c_{321}, c_{322}, c_{323}, c_{131}, c_{132}, c_{133}\right)
\end{aligned}
$$

staying on $\mathbb{R}^{27}$. The first 9 entries determine all the entries of $\boldsymbol{c}$, we write

$$
\boldsymbol{c}=\left(c_{121}, \ldots, c_{313},-,-\right)
$$

for short, here

$$
\begin{aligned}
& \text { the first "-" }=(0, \ldots, 0) \\
& \text { the second "-" }=\left(-c_{121}, \ldots,-c_{313}\right) \text {. }
\end{aligned}
$$

Theorem 1. Let $\mathfrak{C}$ be the algebraic set defined by (2)-(3) in $\mathbb{R}^{27}$. Then the elements $\boldsymbol{c}=\left(c_{121}, \ldots, c_{313},-,-\right)$ of $\mathfrak{C}$ are expressed as

$$
\begin{aligned}
\mathfrak{C}^{\prime}= & \mathfrak{C}_{1} \cup \mathfrak{C}_{2} \cup \mathfrak{C}_{3} \cup \mathfrak{C}_{4} \cup \mathfrak{C}_{5} \cup \mathfrak{C}_{6} \cup \mathfrak{C}_{7} ; \\
\mathfrak{C}_{1}= & \{(\alpha, \beta, \gamma, p \alpha, p \beta, p \gamma, q \alpha, q \beta, q \gamma,---) \mid \alpha, \beta, \gamma, p, q \in \mathbb{R}\}, \\
\mathfrak{C}_{2}= & \{(0,0,0, \alpha, \beta, \gamma, p \alpha, p \beta, p \gamma,-,-) \mid \alpha, \beta, \gamma, p \in \mathbb{R}\}, \\
\mathfrak{C}_{3}= & \{(0,0,0,0,0,0, \alpha, \beta, \gamma,-,-) \mid \alpha, \beta, \gamma \in \mathbb{R}\}, \\
\mathfrak{C}_{4}= & \{(\alpha, q \alpha+p \beta, \beta, \gamma, q \gamma+p \delta, \delta, p \alpha+q \gamma, \\
& \left.\left.p^{2} \beta+p q(\alpha+\delta)+q^{2} \gamma, p \beta+q \delta,-,-\right) \mid \alpha, \beta, \gamma, \delta, p, q \in \mathbb{R}\right\},
\end{aligned}
$$

$$
\begin{aligned}
& \mathfrak{C}_{5}=\left\{\left(p \alpha, \beta, \alpha, p^{2} \alpha, p \beta, p \alpha, p \gamma, \delta, \gamma,-,-\right) \mid \alpha, \beta, \gamma, \delta, p \in \mathbb{R}\right\}, \\
& \mathfrak{C}_{6}=\{(0,0,0, \alpha, \beta, 0, \gamma, \delta, 0,-,-) \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}, \\
& \mathfrak{C}_{7}=\{(\alpha, \beta, \gamma, \delta, \varepsilon, \alpha, \varepsilon, \zeta, \beta,-,-) \mid \alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}\} .
\end{aligned}
$$

Proof. Let $V$ be the vector space over $\mathbb{R}$ with a basis $\left\{x_{1}, x_{2}, x_{3}\right\}$ and $L=\left(V,\left(c_{i j k}\right),[-,-]\right)$ be the 3-dimensional Lie algebra with the set of structure constants $\left(c_{i j k}\right)$. We figure out the structure constants separately for the cases $\operatorname{dim}[L, L]=1,2,3$.
(1) $\operatorname{dim}[L, L]=1:$ If $\left[x_{1}, x_{2}\right] \neq 0$, then we can write

$$
\begin{aligned}
& {\left[x_{1}, x_{2}\right]=c_{121} x_{1}+c_{122} x_{2}+c_{123} x_{3}(\neq 0),} \\
& {\left[x_{2}, x_{3}\right]=p\left[x_{1}, x_{2}\right],} \\
& {\left[x_{3}, x_{1}\right]=q\left[x_{1}, x_{2}\right],}
\end{aligned}
$$

for some $p, q \in \mathbb{R}$. Then the Jacobi identity is automatically satisfied. By setting $\alpha=c_{121}, \beta=c_{122}, \gamma=c_{123}$, we have $\boldsymbol{c} \in \mathfrak{C}_{1}$.

If $\left[x_{1}, x_{2}\right]=0$ and $\left[x_{2}, x_{3}\right] \neq 0$, then we can write

$$
\begin{aligned}
& {\left[x_{1}, x_{2}\right]=0,} \\
& {\left[x_{2}, x_{3}\right]=c_{231} x_{1}+c_{232} x_{2}+c_{233} x_{3}(\neq 0),} \\
& {\left[x_{3}, x_{1}\right]=p\left[x_{2}, x_{3}\right]}
\end{aligned}
$$

for some $p \in \mathbb{R}$. Then the Jacobi identity holds. We set $\alpha=c_{231}, \beta=c_{232}$, $\gamma=c_{233}$ and have $\boldsymbol{c} \in \mathfrak{C}_{2}$. When $\left[x_{1}, x_{2}\right]=0$ and $\left[x_{2}, x_{3}\right]=0$, we must have $\left[x_{3}, x_{1}\right] \neq 0$. These lead to $\boldsymbol{c} \in \mathfrak{C}_{3}$.
(2) $\operatorname{dim}[L, L]=2$ : Assume that $\left[x_{1}, x_{2}\right] \neq 0$. If $\left[x_{1}, x_{2}\right]$ and $\left[x_{2}, x_{3}\right]$ are linearly independent, then we can write

$$
\begin{aligned}
& {\left[x_{1}, x_{2}\right]=c_{121} x_{1}+c_{122} x_{2}+c_{123} x_{3}(\neq 0),} \\
& {\left[x_{2}, x_{3}\right]=c_{231} x_{1}+c_{232} x_{2}+c_{233} x_{3}(\neq 0),} \\
& {\left[x_{3}, x_{1}\right]=p\left[x_{1}, x_{2}\right]+q\left[x_{2}, x_{3}\right],}
\end{aligned}
$$

for some $p, q \in \mathbb{R}$. The Jacobi identity $\left[\left[x_{1}, x_{2}\right], x_{3}\right]+\left[\left[x_{2}, x_{3}\right], x_{1}\right]+\left[\left[x_{3}, x_{1}\right], x_{2}\right]$ $=\left(q c_{231}+p c_{233}-c_{232}\right)\left[x_{1}, x_{2}\right]+\left(c_{122}-p c_{123}-q c_{121}\right)\left[x_{2}, x_{3}\right]$ gives us $q c_{231}+$ $p c_{233}-c_{232}=0, \quad q c_{121}+p c_{123}-c_{122}=0$. By putting $\alpha=c_{121}, \quad \beta=c_{123}$, $\gamma=c_{231}, \delta=c_{233}$, we have $\boldsymbol{c} \in \mathfrak{C}_{4}$. If $\left[x_{1}, x_{2}\right]$ and $\left[x_{2}, x_{3}\right]$ are linearly dependent, then $\left[x_{1}, x_{2}\right.$ ] and $\left[x_{3}, x_{1}\right]$ must be linearly independent and we may write $\left[x_{2}, x_{3}\right]=p\left[x_{1}, x_{2}\right]$ for some $p \in \mathbb{R}$. It follows from the Jacobi identity that $\boldsymbol{c} \in \mathfrak{C}_{5}$.

Assume that $\left[x_{1}, x_{2}\right]=0$. We may write

$$
\begin{aligned}
& {\left[x_{1}, x_{2}\right]=0} \\
& {\left[x_{2}, x_{3}\right]=c_{231} x_{1}+c_{232} x_{2}+c_{233} x_{3}(\neq 0),} \\
& {\left[x_{3}, x_{1}\right]=c_{311} x_{1}+c_{312} x_{2}+c_{313} x_{3}(\neq 0)}
\end{aligned}
$$

By the Jacobi identity $\left[\left[x_{1}, x_{2}\right], x_{3}\right]+\left[\left[x_{2}, x_{3}\right], x_{1}\right]+\left[\left[x_{3}, x_{1}\right], x_{2}\right]=c_{233}\left[x_{3}, x_{1}\right]-$ $c_{313}\left[x_{2}, x_{3}\right]$, we have $c_{233}=c_{313}=0$. It shows that $\boldsymbol{c} \in \mathfrak{C}_{6}$.
(3) $\operatorname{dim}[L, L]=3$ : It follows from the Jacobi identity

$$
\begin{aligned}
& {\left[\left[x_{1}, x_{2}\right], x_{3}\right]+\left[\left[x_{2}, x_{3}\right], x_{1}\right]+\left[\left[x_{3}, x_{1}\right], x_{2}\right] } \\
= & \left(c_{311}-c_{232}\right)\left[x_{1}, x_{2}\right]+\left(c_{122}-c_{313}\right)\left[x_{2}, x_{3}\right]+\left(c_{233}-c_{121}\right)\left[x_{3}, x_{1}\right],
\end{aligned}
$$

that $c_{311}=c_{232}, c_{122}=c_{313}, \quad c_{233}=c_{121}$. This shows that $\boldsymbol{c} \in \mathfrak{C}_{7}$.

## 3. Closest Lie Algebra Structures

Let $\left(a_{i j k}\right)$ be a point in $\mathbb{R}^{27}$. This point gives us the multiplication on the vector space $\mathbb{R} x_{1}+\mathbb{R} x_{2}+\mathbb{R} x_{3}$ with a basis $\left\{x_{1}, x_{2}, x_{3}\right\}:$

$$
x_{i} x_{j}=\sum_{k=1}^{3} a_{i j k} x_{k}, \quad(i, j=1,2,3)
$$

We shall find the point $\left(c_{i j k}\right)$ on $\mathfrak{C}$ closest to the point $\left(a_{i j k}\right)$, so that $\left(c_{i j k}\right)$ minimizes the distance $D\left(\left(c_{i j k}\right),\left(a_{i j k}\right)\right)$ stated in the introduction.

Theorem 2. Let $\mathfrak{C}_{1}, \ldots, \mathfrak{C}_{7}$ be the algebraic set given in Theorem 1 and $\left(a_{i j k}\right)$ in $\mathbb{R}^{27}$. Then the point $\boldsymbol{c}_{i}$ on $\mathfrak{C}_{i}$ closest to the point $\left(a_{i j k}\right)$ is of the following forms:
(1) $\boldsymbol{c}_{3}=\left(0,0,0,0,0,0, \frac{a_{311}-a_{131}}{2}, \frac{a_{312}-a_{132}}{2}, \frac{a_{313}-a_{133}}{2},-,-\right)$.
(2)

$$
\begin{aligned}
& \boldsymbol{c}_{6}=\left(0,0,0, \frac{a_{231}-a_{321}}{2}, \frac{a_{232}-a_{322}}{2}, 0, \frac{a_{311}-a_{131}}{2}, \frac{a_{312}-a_{132}}{2}, 0,-,-\right) . \\
& \text { (3) } \boldsymbol{c}_{7}=\left(\frac{a_{121}+a_{233}-a_{211}-a_{323}}{4}, \frac{a_{122}+a_{313}-a_{212}-a_{133}}{4},\right. \\
& \frac{a_{123}-a_{213}}{2}, \frac{a_{231}-a_{321}}{2}, \frac{a_{232}+a_{311}-a_{322}-a_{131}}{4}, \\
& \frac{a_{121}+a_{233}-a_{211}-a_{323}}{4}, \frac{a_{232}+a_{311}-a_{322}-a_{131}}{4}, \\
&\left.\frac{a_{312}-a_{132}}{2}, \frac{a_{122}+a_{313}-a_{212}-a_{133}}{4},-,-\right) .
\end{aligned}
$$

(4) Real numbers $\alpha, \beta, \gamma, \delta, p, q$ determining $\boldsymbol{c}_{1}, \boldsymbol{c}_{2}, \boldsymbol{c}_{4}, \boldsymbol{c}_{5}$ are chosen as follows: We denote by $\langle X, Y\rangle=\operatorname{tr}\left({ }^{t} X Y\right)$ for the square matrices $X, Y$.

Step 1. Find $X(p, q), \boldsymbol{b}(p, q)$ in cases of $\mathfrak{C}_{1}$ and $\mathfrak{C}_{4}$, and $X(p), \boldsymbol{b}(p)$ in cases of $\mathfrak{C}_{2}$ and $\mathfrak{C}_{5}$, by using $X_{1}, \ldots, X_{4}$ given in the following table:

$$
\begin{aligned}
& X(p, q):=2\left(\left\langle X_{i}, X_{j}\right\rangle\right) \text { for } \mathfrak{C}_{1}, \mathfrak{C}_{4} \\
& X(p):=2\left(\left\langle X_{i}, X_{j}\right\rangle\right) \text { for } \mathfrak{C}_{2}, \mathfrak{C}_{5} \\
& \boldsymbol{b}(p, q):=\left(\left\langle\tilde{A}, X_{i}\right\rangle\right) \text { for } \mathfrak{C}_{1}, \mathfrak{C}_{4} \\
& \boldsymbol{b}(p):=\left(\left\langle\tilde{A}, X_{i}\right\rangle\right) \text { for } \mathfrak{C}_{2}, \mathfrak{C}_{5}
\end{aligned}
$$

here

$$
\tilde{A}=\left(\begin{array}{lll}
a_{211}-a_{121} & a_{212}-a_{122} & a_{213}-a_{123} \\
a_{321}-a_{231} & a_{322}-a_{232} & a_{323}-a_{233} \\
a_{131}-a_{311} & a_{132}-a_{312} & a_{133}-a_{313}
\end{array}\right)
$$

|  | $X_{1}$ | $X_{2}$ | $X_{3}$ | $X_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathfrak{C}_{1}$ | $\left(\begin{array}{lll}1 & 0 & 0 \\ p & 0 & 0 \\ q & 0 & 0\end{array}\right)$ | $\left(\begin{array}{lll}0 & 1 & 0 \\ 0 & p & 0 \\ 0 & q & 0\end{array}\right)$ | $\left(\begin{array}{lll}0 & 0 & 1 \\ 0 & 0 & p \\ 0 & 0 & q\end{array}\right)$ |  |
| $\mathfrak{C}_{2}$ | $\left(\begin{array}{lll}0 & 0 & 0 \\ 1 & 0 & 0 \\ p & 0 & 0\end{array}\right)$ | $\left(\begin{array}{lll}0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p & 0\end{array}\right)$ | $\left(\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & p\end{array}\right)$ |  |
| $\mathfrak{C}_{4}$ | $\left(\begin{array}{ccc}1 & q & 0 \\ 0 & 0 & 0 \\ p & p q & 0\end{array}\right)$ | $\left(\begin{array}{ccc}0 & p & 1 \\ 0 & 0 & 0 \\ 0 & p^{2} & p\end{array}\right)$ | $\left(\begin{array}{ccc}0 & 0 & 0 \\ 1 & q & 0 \\ q & q^{2} & 0\end{array}\right)$ | $\left(\begin{array}{ccc}0 & 0 & 0 \\ 0 & p & 1 \\ 0 & p q & q\end{array}\right)$ |
| $\mathfrak{C}_{5}$ | $\left(\begin{array}{ccc}p & 0 & 1 \\ p^{2} & 0 & p \\ 0 & 0 & 0\end{array}\right)$ | $\left(\begin{array}{lll}0 & 1 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0\end{array}\right)$ | $\left(\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ p & 0 & 1\end{array}\right)$ | $\left(\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}\right)$ |

Step 2. Express $\alpha, \beta, \gamma, \delta$ in terms of $p, q$ by

$$
\begin{aligned}
& { }^{t}(\alpha \beta \gamma)=-X(p, q)^{-1} \boldsymbol{b}(p, q) \text { for } \mathfrak{C}_{1}, \\
& { }^{t}(\alpha \beta \gamma)=-X(p)^{-1} \boldsymbol{b}(p) \text { for } \mathfrak{C}_{2}, \\
& { }^{t}(\alpha \beta \gamma \delta)=-X(p, q)^{-1} \boldsymbol{b}(p, q) \text { for } \mathfrak{C}_{4}, \\
& { }^{t}(\alpha \beta \gamma \delta)=-X(p)^{-1} \boldsymbol{b}(p) \text { for } \mathfrak{C}_{5} .
\end{aligned}
$$

Step 3. Find $p, q$ minimizing the following $f(p, q), f(p)$,

$$
\begin{aligned}
& f(p, q)=-^{t} \boldsymbol{b}(p, q) X(p, q)^{-1} \boldsymbol{b}(p, q)+K, \\
& f(p)=-^{t} \boldsymbol{b}(p) X(p)^{-1} \boldsymbol{b}(p)+K,
\end{aligned}
$$

here $K=\sum_{i, j, k} a_{i j k}^{2}$.
Note. The explicit forms of $X(p, q)$ and $X(p)$ are found in the proof.

Proof. The statements (1), (2) and (3) hold obviously. To prove (4), let us denote by

$$
\begin{aligned}
& A_{+}=\left(\begin{array}{lll}
a_{121} & a_{122} & a_{123} \\
a_{231} & a_{232} & a_{233} \\
a_{311} & a_{312} & a_{313}
\end{array}\right), \quad A_{-}=\left(\begin{array}{lll}
a_{211} & a_{212} & a_{213} \\
a_{321} & a_{322} & a_{323} \\
a_{131} & a_{132} & a_{133}
\end{array}\right), \\
& A_{0}=\left(\begin{array}{lll}
a_{111} & a_{112} & a_{113} \\
a_{221} & a_{222} & a_{223} \\
a_{331} & a_{332} & a_{333}
\end{array}\right), \quad C=\left(\begin{array}{lll}
c_{121} & c_{122} & c_{123} \\
c_{231} & c_{232} & c_{233} \\
c_{311} & c_{312} & c_{313}
\end{array}\right) .
\end{aligned}
$$

Then we have $\tilde{A}=A_{-}-A_{+}$and

$$
\begin{aligned}
D(\boldsymbol{c}, \boldsymbol{a})^{2} & =\left\langle C-A_{+}, C-A_{+}\right\rangle+\left\langle-C-A_{-},-C-A_{-}\right\rangle+\left\langle O-A_{0}, O-A_{0}\right\rangle \\
& =2\langle C, C\rangle+2\left\langle A_{-}-A_{+}, C\right\rangle+\left\langle A_{+}, A_{+}\right\rangle+\left\langle A_{-}, A_{-}\right\rangle+\left\langle A_{0}, A_{0}\right\rangle
\end{aligned}
$$

Let us consider the case of $\mathfrak{C}_{1}$. In terms of $X_{1}, X_{2}, X_{3}$ of the table above, we can write

$$
C=\left(\begin{array}{ccc}
\alpha & \beta & \gamma \\
p \alpha & p \beta & p \gamma \\
q \alpha & q \beta & q \gamma
\end{array}\right)=\alpha X_{1}+\beta X_{2}+\gamma X_{3}
$$

Then we have

$$
\begin{aligned}
& \langle C, C\rangle=(\alpha \beta \gamma)\left(\begin{array}{ccc}
\left\langle X_{1}, X_{1}\right\rangle & \left\langle X_{1}, X_{2}\right\rangle & \left\langle X_{1}, X_{3}\right\rangle \\
\left\langle X_{2}, X_{1}\right\rangle & \left\langle X_{2}, X_{2}\right\rangle & \left\langle X_{2}, X_{3}\right\rangle \\
\left\langle X_{3}, X_{1}\right\rangle & \left\langle X_{3}, X_{2}\right\rangle & \left\langle X_{3}, X_{3}\right\rangle
\end{array}\right)\left(\begin{array}{l}
\alpha \\
\beta \\
\gamma
\end{array}\right), \\
& \left\langle A-A_{+}, C\right\rangle=\left(\left\langle\tilde{A}, X_{1}\right\rangle\left\langle\tilde{A}, X_{2}\right\rangle\left\langle\tilde{A}, X_{3}\right\rangle\right)\left(\begin{array}{l}
\alpha \\
\beta \\
\gamma
\end{array}\right)
\end{aligned}
$$

hence

$$
D(\boldsymbol{c}, \boldsymbol{a})^{2}={ }^{t} \boldsymbol{x} X(p, q) \boldsymbol{x}+2^{t} \boldsymbol{b}(p, q) \boldsymbol{x}+K
$$

here

$$
\begin{aligned}
& X(p, q)=2\left(\left\langle X_{i}, X_{j}\right\rangle\right) \\
& \quad=2 \operatorname{diag}\left(1+p^{2}+q^{2}, 1+p^{2}+q^{2}, 1+p^{2}+q^{2}\right) \\
& \boldsymbol{b}(p, q)={ }^{t}\left(\left\langle\tilde{A}, X_{1}\right\rangle\left\langle\tilde{A}, X_{2}\right\rangle\left\langle\tilde{A}, X_{3}\right\rangle\right) \\
& \boldsymbol{x}={ }^{t}(\alpha \beta \gamma) \\
& K=\left\langle A_{+}, A_{+}\right\rangle+\left\langle A_{-}, A_{-}\right\rangle+\left\langle A_{0}, A_{0}\right\rangle .
\end{aligned}
$$

Since $X(p, q)$ is positive definite for any real numbers $p, q, D\left(\left(c_{i j k}\right),\left(a_{i j k}\right)\right)^{2}$ has the minimum value at $\boldsymbol{x}=\boldsymbol{x}(p, q)$ such that $X(p, q) \boldsymbol{x}(p, q)+\boldsymbol{b}(p, q)=\mathbf{0}$ for each $p, q$. Such an $x(p, q)$ satisfies

$$
\begin{aligned}
D\left(\left(c_{i j k}\right),\left(a_{i j k}\right)\right)^{2} & ={ }^{t} \boldsymbol{x}(p, q) X(p, q) \boldsymbol{x}(p, q)+2^{t} \boldsymbol{b}(p, q) \boldsymbol{x}(p, q)+K \\
& ={ }^{t} \boldsymbol{x}(p, q) \boldsymbol{b}(p, q)+K \\
& ={ }^{t} \boldsymbol{b}(p, q) X(p, q)^{-1} \boldsymbol{b}(p, q)+K \\
& =f(p, q)
\end{aligned}
$$

We can find $\boldsymbol{c}_{2}, \boldsymbol{c}_{4}$ and $\boldsymbol{c}_{5}$ by using the same procedure as above. Here we give only the list of $X(p, q), X(p)$. In the cases of $\mathfrak{C}_{2}, \mathfrak{C}_{4}$ and $\mathfrak{C}_{5}$,

$$
\begin{aligned}
& X(p)=2 \operatorname{diag}\left(1+p^{2}, 1+p^{2}, 1+p^{2}\right) \\
& X(p, q)=2\left(\begin{array}{cccc}
\left(1+p^{2}\right)\left(1+q^{2}\right) & p q\left(1+p^{2}\right) & p q\left(1+q^{2}\right) & p^{2} q^{2} \\
p q\left(1+p^{2}\right) & \left(1+p^{2}\right)^{2} & p^{2} q^{2} & p q\left(1+p^{2}\right) \\
p q\left(1+q^{2}\right) & p^{2} q^{2} & \left(1+q^{2}\right)^{2} & p q\left(1+q^{2}\right) \\
p^{2} q^{2} & p q\left(1+p^{2}\right) & p q\left(1+q^{2}\right) & \left(1+p^{2}\right)\left(1+q^{2}\right)
\end{array}\right), \\
& X(p)=2 \operatorname{diag}\left(\left(1+p^{2}\right)^{2}, 1+p^{2}, 1+p^{2}, 1\right),
\end{aligned}
$$

respectively.

## 4. Examples (Demonstration)

Recall that a point $\left(a_{i j k}\right)$ in $\mathbb{R}^{27}$ gives us the multiplication on the vector space $\mathbb{R} x_{1}+\mathbb{R} x_{2}+\mathbb{R} x_{3}$ with a basis $\left\{x_{1}, x_{2}, x_{3}\right\}: x_{i} x_{j}=\sum_{k=1}^{3} a_{i j k} x_{k}$.

Example 1. Our point $\boldsymbol{a}=\left(a_{i j k}\right)$ in $\mathbb{R}^{27}$ is $\boldsymbol{a}=(1,0,-1,-1,1,0,0,0,1,0,0$, $0,0,0,0,0,0,0,-1,0,1,1,-1,0,0,0,-1)$ and the corresponding multiplication table is

$$
\begin{array}{lll}
x_{1} x_{2}=x_{1}-x_{3} & x_{2} x_{1}=-x_{1} x_{2} & x_{1} x_{1}=0 \\
x_{2} x_{3}=-x_{1}+x_{2} & x_{3} x_{2}=-x_{2} x_{3} & x_{2} x_{2}=0 \\
x_{3} x_{1}=x_{3} & x_{1} x_{3}=-x_{3} x_{1} & x_{3} x_{3}=0 .
\end{array}
$$

This multiplication does not satisfy the Jacobi identity, that is, $\left(x_{1} x_{2}\right) x_{3}+\left(x_{2} x_{3}\right) x_{1}$ $+\left(x_{3} x_{1}\right) x_{2}=-x_{2}$. By Theorem 2, we have the following point $\boldsymbol{c}_{i}$ on $\mathfrak{C}_{i}$ closest to the point $\boldsymbol{a}$ and the minimum value $D\left(\boldsymbol{c}_{i}, \boldsymbol{a}\right)^{2}$ :

$$
\begin{aligned}
& c_{1}=(0.978694,-0.43556,-0.784851 \text {, } \\
& \text {-0.784851, 0.349292, 0.629402, } \quad ; \quad D\left(\boldsymbol{c}_{1}, \boldsymbol{a}\right)^{2}=3.50604 \text {. } \\
& \text {-0.43556, 0.193842, 0.349292, -, -) } \\
& \boldsymbol{c}_{2}=(0 ., 0 ., 0 .,-1 ., 1 ., 0 ., 0 ., 0 ., 0 .,-,-) \quad ; \quad D\left(\boldsymbol{c}_{2}, \boldsymbol{a}\right)^{2}=6 . \\
& \boldsymbol{c}_{3}=(0,0,0,0,0,0,0,0,1,-,-) \quad ; \quad D\left(\boldsymbol{c}_{3}, \boldsymbol{a}\right)^{2}=8 . \\
& \boldsymbol{c}_{4}=(1.10653,-0.0525276,-1.02323 \text {, } \\
& \text {-1.02323, 0.741678, -0.152996, } \quad ; \quad D\left(\boldsymbol{c}_{4}, \boldsymbol{a}\right)^{2}=0.727296 . \\
& -0.0525276,-0.434549,0.741678,-,-) \\
& \boldsymbol{c}_{5}=(0.75,-0.5,-0.75,-0.75,0.5,0.75 \text {, } \\
& -0.5,0 ., 0.5,-,-) \quad ; \quad D\left(\boldsymbol{c}_{5}, \boldsymbol{a}\right)^{2}=3.5 . \\
& \boldsymbol{c}_{6}=(0,0,0,-1,1,0,0,0,0,-,-) \quad ; \quad D\left(\boldsymbol{c}_{6}, \boldsymbol{a}\right)^{2}=6 . \\
& \boldsymbol{c}_{7}=(1 / 2,1 / 2,-1,-1,1 / 2,1 / 2,1 / 2,0,1 / 2,-,-) ; \quad D\left(\boldsymbol{c}_{7}, \boldsymbol{a}\right)^{2}=3 .
\end{aligned}
$$

We find $\boldsymbol{c}_{1}, \boldsymbol{c}_{2}, \boldsymbol{c}_{4}, \boldsymbol{c}_{5}$ by using Mathematica. Since $D\left(\boldsymbol{c}_{4}, \boldsymbol{a}\right)$ is the smallest among the $D\left(\boldsymbol{c}_{i}, \boldsymbol{a}\right)$ 's, the closest point to $\boldsymbol{a}$ is $\boldsymbol{c}_{4}$ and the corresponding multiplication table is

$$
\begin{aligned}
& {\left[x_{1}, x_{2}\right]=1.10653 x_{1}-0.0525276 x_{2}-1.02323 x_{3}} \\
& {\left[x_{2}, x_{3}\right]=-1.02323 x_{1}+0.741678 x_{2}-0.152996 x_{3}} \\
& {\left[x_{3}, x_{1}\right]=-0.0525276 x_{1}-0.434549 x_{2}+0.741678 x_{3} .}
\end{aligned}
$$

Example 2. This example presents the scheme:
(1) Take a point $\boldsymbol{c}=(0,0,0,0,-1,0,1,0,0,-,-)$ on $\mathfrak{C}_{6}$.
(2) Choose a perturbed point
$\boldsymbol{a}=\boldsymbol{c}+(0,0,0,0,0,0,0,0,0.1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-0.1)$
in $\mathbb{R}^{27}$.
(3) Find the closest point $\boldsymbol{c}^{\prime}$ on $\mathfrak{C}$ to $\boldsymbol{a}$. We note that $\boldsymbol{c}^{\prime}$ is on $\mathfrak{C}_{5}$.

$$
\begin{aligned}
& \boldsymbol{c} \text { of (1) } \quad \boldsymbol{a} \text { of (2) }
\end{aligned}
$$

$$
\begin{aligned}
& \boldsymbol{c}^{\prime} \text { of (3) } \\
& \Rightarrow \begin{array}{l}
{\left[x_{1}, x_{2}\right]^{\prime}=-0.0499994 x_{2}} \\
{\left[x_{2}, x_{3}\right]^{\prime}=-0.9974942 x_{2}}
\end{array} \\
& {\left[x_{3}, x_{1}\right]^{\prime}=1.00249 x_{1}+0.05025 x_{3} .}
\end{aligned}
$$

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