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m-PARTIAL ISOMETRIES ON HILBERT SPACES

ADEL SADDI and OULD AHMED MAHMOUD SID AHMED

Department of Mathematics
College of Education for Girls
King Khalid University
Sarat Ebeidah 61914, Abha, Saudi Arabia
e-mail: adel.saddi@fsg.rnu.tn

Department of Mathematics
College of Science
Aljouf University
Aljouf 2014, Saudi Arabia
e-mail: sidahmed@ju.edu.sa

Abstract

In this paper, we introduce a generalization of partial isometries to m-partial isometries on Hilbert space for $m = 1, 2, 3, \ldots$. Also, we study some of the basic algebraic and spectral properties and present some examples.

1. Introduction

The operator theory of partial isometries has been studied by several authors ([2], [8], [13]…). For example, in [10], Mbekhta and in [13], Schmoeger have characterized the class of partial isometries on Banach spaces. The class of m-isometric and in particular 2-isometric operators on a Hilbert space has been the object of some intensive study, especially by Agler and Stankus in [1], but also by Richter [12], Shimorin [14] and Hellings [9]. In this paper, we will give a

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generalization of partial isometries and m-isometries to m-partial isometries on Hilbert spaces. More precisely, we will study the bounded linear operator T on a complex Hilbert space H that satisfies the identity

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}T(T^*)^{m-k}T^{m-k} = 0 \qquad (*)$$

for some positive integer m. We will define an operator satisfying $(*)$ to be an m-partial isometry on H. The case when $m = 1$, represents the partial isometries class. If T is injective and it verifies $(*)$, then it is called an *m-isometry* that is deeply studied by Agler and Stankus in [1]. This paper is divided into four sections. In Section 2, we introduce the different notions and notations and we recall some definitions which we shall need in the sequel. We give also a characterization of partial isometries. In Section 3, we discuss the elementary operator theory of m-partial isometries. We observe that if T is an m-partial isometry for which $N(T)$ is a reducing subspace, then $T|_{N(T)^\perp}$ is an m-isometry and for all $n \in \mathbb{N}$, T is an $(m+n)$-partial isometry. Moreover, if S_T is defined by

$$S_T = \sum_{k=0}^{m-1}\binom{m-1}{k}(-1)^k T^*(T^*)^{m-1-k}T^{m-1-k}T,$$

then S_T is a positive operator, the kernel of S_T is invariant for T and $T|_{N(S_T)}$ is an $(m-1)$-partial isometry. Some spectral properties of an m-partial isometry are also studied, namely, if T is an m-partial isometry and reduces $N(T)$, then we prove that $\sigma_a(T) \subset \mathcal{C} \cup \{0\}$ and $\sigma(T) \subset \mathcal{C}$ or $\sigma(T) \subset \overline{\mathcal{D}}$, where $\mathcal{C}$ is the unit circle, $\mathcal{D}$ is the open unit disc and $\overline{\mathcal{D}}$ is the closed unit disc. In Section 4, we shall specialize to the case $m = 2$. We explore some properties of 2-partial isometries and obtain additional information. Finally, if T is finitely cyclic and reduces $N(T)$, then we prove that the operator S_T defined above is compact.

2. Notations and Preliminaries

Let H be a complex Hilbert space, and denoted by $\mathcal{L}(H)$ the set of bounded linear operators on H. Then for an operator $T \in \mathcal{L}(H)$, we write $N(T)$ for its kernel,

$R(T)$ for its range and T^* for its adjoint. Let $\rho(T)$, $\sigma(T)$, $\sigma_a(T)$ and $\sigma_p(T)$, respectively, denote the resolvent set, spectrum, the approximate point spectrum and point spectrum of the operator *T*. The spectral radius and the numerical radius of *T* will be denoted by $r(T)$ and $|w(T)|$, respectively. For any arbitrary operator $T \in \mathcal{L}(H)$, as usual $|T| = (T^*T)^{1/2}$ and consider the following standard definitions: *T* is normal if $T^*T = TT^*$, quasinormal if $TT^*T = T^*T^2$, quasi-isometry if $T^{*2}T^2 = T^*T$. If *A* is a subset of *H*, then we define $Span(A)$ to be the smallest subspace of *H* which contains *A*.

Consider the following standard definitions:

Definition 2.1. An operator $T \in \mathcal{L}(H)$ is called

1. An *isometry* if $\|Tx\| = \|x\|,\ \forall x \in H$.

2. A *partial isometry* if $\|Tx\| = \|x\|,\ \forall x \in N(T)^{\perp}$.

Remark 2.1. A partial isometry is an isometry if and only if it is injective.

If we consider the set $\mathcal{A} = \{T \in \mathcal{L}(H)/TT^*T = T\}$, then we have the following well-known characterization of partial isometries.

Proposition 2.1. *For* $T \in \mathcal{L}(H)$, *the following properties are equivalent*:

(i) $T \in \mathcal{A}$.

(ii) T^*T *is an orthogonal projection*.

(iii) *T is a partial isometry*.

We recall now the definition of an *m*-isometry on *H* introduced by Agler and Stankus in [1].

Definition 2.2. An operator $T \in \mathcal{L}(H)$ is called an *m-isometry* if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*m-k} T^{m-k} = 0.$$

3. *m*-partial Isometry

In this section, we generalize the notions of partial isometries and *m*-isometries to *m*-partial isometries.

Definition 3.1. An operator $T \in \mathcal{L}(H)$ is called an *m-partial isometry* if

$$T\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k} = 0.$$

Remark 3.1. (1) It is easy to see that $T \in \mathcal{L}(H)$ is an m-partial isometry if and only if

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}(x) = 0, \quad \forall x \in N(T)^{\perp},$$

which shows that the class of m-partial isometries generalizes those of m-isometries and partial isometries.

(2) A 1-isometry is an isometry and a 1-partial isometry is a partial isometry.

(3) An operator is an m-isometry if and only if it is an injective, m-partial isometry.

Example 3.1. Consider the operators $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} a & 0 \\ 1 & 0 \end{pmatrix}$ with $|a|^2 = \frac{1+\sqrt{5}}{2}$ acting on $H = \mathbb{C}^2$, then T and S are two 2-partial isometries but they are not 2-isometries on H.

Example 3.2. Set $H = \mathbb{C}^3$ and identify the operator T on H with the matrix

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then T is a quasinormal 2-partial isometry.

Example 3.3. Put $H = L^2([0, +\infty[, d\sigma(x))$ equipped with the norm $\| f \|_H^2 = \int_0^{+\infty} | f(x)|^2 d\sigma(x)$, and consider the operator V on H defined by

$$Vf(x) = \alpha(x) f(x^{\nu}),$$

where $\sigma(x) = x^{-\frac{1}{\nu}}$ and $\alpha(x) = \begin{cases} \sqrt{\nu}x^{(\nu-1)/2}, & x \in \mathbb{R}_+ \setminus \mathbb{N}, \\ 0, & x \in \mathbb{N}. \end{cases}$

If $\nu \in \mathbb{R}_+ \setminus \{0, 1, 2\}$, then the operator V is a partial isometry and it is not an isometry neither a 2-partial isometry on H.

Remark 3.2. (1) If $T \in \mathcal{L}(H)$, then $N(T^*T) = N(T)$ and if $k \in \mathbb{N}$, then $T^{*k}T^k(N(T)^{\perp}) \subset N(T)^{\perp}$.

(2) If $T \in \mathcal{L}(H)$ and satisfies $TT^*T = T$, then $R(T^*T) = R(T^*)$.

Proposition 3.1. *Let $T \in \mathcal{L}(H)$ be an m-partial isometry. If T is quasi-isometry or quasinormal, then T is a partial isometry.*

Proof. Let $T \in \mathcal{L}(H)$ such that $\sum_{k=0}^{m}(-1)^k\binom{m}{k}T(T^*)^{m-k}T^{m-k} = 0$. If T is quasi -isometry, then $T^{*k}T^k = T^*T$ for $k = 1, 2, 3, \ldots,$ and the equation above becomes $TT^*T = T$. Thus T is a partial isometry, and therefore, quasinormal. If now T is quasinormal, then for $k = 0, 1, 2, 3, \ldots,$ we get $T^{*k}T^k = (T^*T)^k$ and so $0 = \sum_{k=0}^{m}(-1)^k\binom{m}{k}T(T^*)^{m-k}T^{m-k} = T(I - T^*T)^m$. This implies that T^*T is a positive projection, hence T is a partial isometry.

Definition 3.2. Let $T \in \mathcal{L}(H)$ and F be a subspace of H. Then we say that F is a *reducing subspace* for T if both F and $F^{\perp}$ are T-invariant or equivalently if F is invariant for both T and T^*.

Contrary to the case of quasinormal operators, in general, the kernel of an m-partial isometry is not reducing.

Theorem 3.1. *If $T \in \mathcal{L}(H)$ and $N(T)$ is a reducing subspace for T, then the following properties are equivalent:*

(i) *T is an m-partial isometry.*

(ii) $T|_{N(T)^{\perp}}$ *is an m-isometry.*

Proof. (i) $\Rightarrow$ (ii) Since T reduces $N(T)$, $T|_{N(T)^{\perp}}$ is an injective, m-partial isometry in $\mathcal{L}(N(T)^{\perp})$. Thus $T|_{N(T)^{\perp}}$ is an m-isometry.

(ii) $\Rightarrow$ (i) Let $x = x_1 + x_2,\ x_1 \in N(T),\ x_2 \in N(T)^{\perp}.$ Then

$$T\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}(x) = T\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}(x_2).$$

Since $T|_{N(T)^{\perp}}$ is an m-isometry, $\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}(x_2) = 0,$ so $T\sum_{k=0}^{m}(-1)^k$ $\cdot\binom{m}{k}T^{*m-k}T^{m-k}(x) = 0$ and the result is obtained.

Proposition 3.2. *Let $T, S \in \mathcal{L}(H)$ such that T is an m-partial isometry and S is an isometry with $TS = ST$ and $TS^* = S^*T$. Then TS is an m-partial isometry.*

Proof. Let $x \in H$. Then we have

$$\begin{aligned} &TS\sum_{k=0}^{m}(-1)^k\binom{m}{k}(TS)^{*m-k}(TS)^{m-k}(x) \\ &= TS\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}(S^{*m-k}S^{m-k}x) \\ &= T\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}(Sx) \\ &= 0 \end{aligned}$$

which finishes the proof.

Remark 3.3. For $T \in \mathcal{L}(H),$ T is a 1-partial isometry if and only if T^* is so. This equivalence is false for the class of m-partial isometries with $m \geq 2.$ For example, the operator $T = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\sqrt{6}}{3} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3)$ is a 2-partial isometry but T^* is not a 2-partial isometry.

Proposition 3.3. *If T and S are unitarily equivalent operators on H, then T is an m-partial isometry if and only if S is an m-partial isometry.*

Proof. Assume that there exists a unitary operator $V \in \mathcal{L}(H)$ such that $S = V^*TV$. Then

$$S\sum_{k=0}^{m}(-1)^k\binom{m}{k}S^{*m-k}S^{m-k}(x) = V^*TV\sum_{k=0}^{m}(-1)^k\binom{m}{k}(V^*TV)^{*m-k}(V^*TV)^{m-k}(x)$$

$$= V^*TVV^*\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}V(x)$$

$$= V^*T\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}V(x).$$

Thus the equivalence.

Remark 3.4. The following examples show that the classes of 1-partial isometries and 2-partial isometries are independent.

Example 3.4. Consider the operator $T = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix}$, acting on $\mathbb{C}^3$, then a direct computation shows that *T* is a 1-partial isometry but is not a 2-partial isometry.

Example 3.5. Consider the operator $S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{\frac{1+\sqrt{5}}{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, acting on $\mathbb{C}^3$, then a simple calculation shows that *S* is a 2-partial isometry but is not a 1-partial isometry.

Proposition 3.4. *Let* $T \in \mathcal{L}(H)$ *be an m-isometry. Then T is an* $(m+n)$-*isometry for all* $n = 0, 1, 2, \ldots$.

Proof. It suffices to prove the result for $n = 1$, we have

$$\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \| T^{m+1-j}(x) \|^2$$

$$= \| T^{m+1}(x) \|^2 + \sum_{j=1}^{m} (-1)^j \binom{m+1}{j} \| T^{m+1-j}(x) \|^2 - (-1)^m \| x \|^2$$

$$= \| T^{m+1}(x) \|^2 + \sum_{j=1}^{m} (-1)^j \left(\binom{m}{j} + \binom{m}{j-1} \right) \| T^{m+1-j}(x) \|^2 - (-1)^m \| x \|^2$$

$$= \| T^{m}(Tx) \|^2 + \sum_{j=1}^{m} (-1)^j \binom{m}{j} \| T^{m-j}(Tx) \|^2$$

$$+ \sum_{j=1}^{m} (-1)^j \binom{m}{j-1} \| T^{m+1-j}(x) \|^2 - (-1)^m \| x \|^2$$

$$= 0 - \sum_{j=0}^{m} (-1)^j \binom{m}{j} \| T^{m-j}(x) \|^2 = 0.$$

Hence *T* is an $(m+1)$-isometry.

In the following, we generalize the previous result for *m*-partial isometries.

Proposition 3.5. *Let $T \in \mathcal{L}(H)$ be an m-partial isometry such that $N(T)$ is a reducing subspace for T. Then T is an $(m+n)$-partial isometry for $n = 0, 1, 2, \ldots$.*

Proof. If *T* is an *m*-partial isometry, then $T|_{N(T)^{\perp}}$ is an *m*-isometry, we deduce from Proposition 3.5 that $T|_{N(T)^{\perp}}$ is an $(m+n)$-isometry, hence by Theorem 3.1, *T* is an $(m+n)$-partial isometry.

Example 3.6. The operator $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3)$ is a 1-partial isometry and a 2-partial isometry and $N(T)$ is a reducing subspace for *T*.

Remark 3.5. If $N(T)$ is not reducing for *T*, then it is not necessary that an *m*-partial isometry is also an $(m+n)$-partial isometry.

For example, the operator $T = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3)$ is a 1-partial isometry with $N(T)$ is not reducing for *T* and *T* is not a 2-partial isometry.

Notation. For $T \in \mathcal{L}(H)$ and $m \geq 1$, let $S_T = T^*\Delta_T T$, where

$$\Delta_T = \sum_{k=0}^{m-1}(-1)^k \binom{m-1}{k} T^{*m-1-k}T^{m-1-k}.$$

The following results illustrate the interest of S_T.

Proposition 3.6. *Let $T \in \mathcal{L}(H)$ be an m-partial isometry such that $N(T)$ is a reducing subspace for T. Then $S_T \geq 0$.*

Proof. For $x \in H$, we get $\langle S_T x | x\rangle = \langle \Delta_T Tx | Tx\rangle$. According to [1, Proposition 1.5], we have $\Delta_{(T|_{N(T)^\perp})} \geq 0$. Since *T* reduces $N(T)$, $Tx \in N(T)^\perp$ and $\langle S_T x | x\rangle = \langle \Delta_T Tx | Tx\rangle \geq 0$. Hence the result.

Theorem 3.2. *Let $T \in \mathcal{L}(H)$ be an m-partial isometry such that $m \geq 2$ and $N(T)$ is a reducing subspace for T. Then we have the following properties*:

1. *$N(S_T)$ is an invariant subspace for T.*

2. *$T|_{N(S_T)}$ is an $(m-1)$-partial isometry.*

3. *If G is an invariant subspace for T and $T|_G$ is an $(m-1)$-partial isometry, then $G \subset N(S_T)$.*

Proof. Note first that *T* is an *m*-partial isometry if and only if $(S_T - \Delta_T)T^* = 0$.

1. Let $x \in N(S_T)$. Then $\langle S_T Tx | Tx\rangle = \langle T^*\Delta_T T^2 x | Tx\rangle = \langle T^{*2}\Delta_T T^2 x | x\rangle$. Since $Tx \in N(T)^\perp$, we obtain $\langle S_T Tx | Tx\rangle = \langle T^*\Delta_T Tx | x\rangle = \langle S_T x | x\rangle = 0$. The positivity of S_T implies that $N(S_T)$ is invariant for *T*.

2. Since $N(S_T)$ is invariant for T, $\Delta_T|_{N(S_T)} = P_{N(S_T)}\Delta_T|_{N(S_T)}$. So, for $x \in H$, we have

$$\begin{aligned} S_T(x) = 0 &\Rightarrow TS_T(x) = 0 \\ &\Leftrightarrow T\Delta_T(x) = 0 \\ &\Rightarrow T|_{N(S_T)}\Delta_T|_{N(S_T)} = 0. \end{aligned}$$

Thus $T|_{N(S_T)}$ is an $(m-1)$-partial isometry.

3. Let G be an invariant subspace for T and $T|_G$ is an $(m-1)$-partial isometry. If $x \in G$, then $Tx \in G \cap N(T)^{\perp} \subset N(T|_G)^{\perp}$. Thus $\langle S_T x \mid x\rangle = \langle T^*\Delta_T Tx \mid x\rangle = \langle T^*\Delta_{(T|_G)}Tx \mid x\rangle = 0$, so $S_T x = 0$. This means that $G \subset N(S_T)$.

Note that if $T \in \mathcal{L}(H)$ is an m-partial isometry and reduces $N(T)$, then $N(\Delta_T) \subset N(T)^{\perp}$, $N(S_T) = N(T) \oplus N(\Delta_T)$ and $T^*S_T T = S_T$.

In the next proposition, we discuss maximal reducing subspaces on which a given m-partial isometry is an $(m-1)$-partial isometry.

Proposition 3.7. *Let $T \in \mathcal{L}(H)$ be an m-partial isometry such that $m \geq 2$ and $N(T)$ is a reducing subspace for T. Then there exists a unique subspace $F \subset H$ that is maximal with respect to the following properties*:

(1) *F is a reducing subspace for T.*

(2) *$T|_F$ is an $(m-1)$-partial isometry.*

Proof. The existence of the subspace F that is maximal with respect to (1) and (2) follows from Zorn's lemma. To prove uniqueness, it suffices to establish that if F_1 and F_2 satisfy (1) and (2), then so also $F = \overline{F_1 + F_2}$ satisfies (1) and (2), and that F satisfies (1) is immediate. To see that F satisfies (2), first observe that $T|_{F_1}$ and $T|_{F_2}$ are $(m-1)$-partial isometries. Theorem 3.2 implies that $F_1 \subset N(S_T)$ and $F_2 \subset N(S_T)$ and also, $F \subset N(S_T)$. Thus (2) holds for F.

In the following results, we examine some spectral properties of m-partial isometries.

In [1, Lemma 1.21], the authors proved that if T is an m-isometry, then $\sigma_a(T) \subset \mathcal{C}$. This is not true for an m-partial isometry. For example, on $\mathbb{C}^2$ the matrix operator $S = \begin{pmatrix} a & 0 \\ 1 & 0 \end{pmatrix}$, where $|a|^2 = \frac{1+\sqrt{5}}{2}$ is a 2-partial isometry, with $\sigma(S) = \{0, a\}$.

However, if in addition, assume that T reduces $N(T)$, then we obtain the following result:

Proposition 3.8. *Let $T \in \mathcal{L}(H)$ be an m-partial isometry such that $N(T)$ is a reducing subspace for T. Then $\sigma_a(T) \subset \mathcal{C} \cup \{0\}$.*

Proof. Let $\lambda \in \sigma_a(T)$. Then there exists a sequence $(x_n)_{n\geq 1} \subset H$, with $\| x_n \| = 1$ such that $(T - \lambda I)x_n \to 0$. By induction for each integer $k \geq 0$, we have $(T^k - \lambda^k I)x_n \to 0$. Since $R(T) \subset N(T)^{\perp}$, from Remark 3.1(1), for all $n \geq 1$, we have

$$0 = \left\langle \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k T x_n \,|\, x_n \right\rangle$$

$$= \left\langle \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k (T - \lambda) x_n \,|\, x_n \right\rangle + \lambda \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \| T^k x_n \|^2$$

$$= \left\langle \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k (T - \lambda) x_n \,|\, x_n \right\rangle$$

$$+ \lambda \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} (\| T^k - \lambda^k) x_n \|^2 + 2\mathfrak{Re}\langle (T^k - \lambda^k) x_n \,|\, \lambda x_n \rangle + |\lambda|^{2k})$$

as $(T^k - \lambda^k I)x_n \to 0$, we obtain

$$\lambda(1 - |\lambda|^2)^m = 0.$$

Then $\lambda = 0$ or $|\lambda| = 1$.

Corollary 3.1. *If T is an m-partial isometry and reduces* $N(T)$, *then* $r(T) = 1$. *In particular*, $\sigma(T) \subset \mathcal{C}$ *or* $\sigma(T) = \overline{\mathcal{D}}$.

Proof. It is known (see for example [3, 5]) that the convex envelopes of all spectra coincide. Thus from Proposition 3.8, it follows that $r(T) = 1$.

On the other hand, $\rho(T) \cap \mathcal{D}$ is both open and closed subset of the domain $\mathcal{D}$. Consequently, we find $\sigma(T) \subset \mathcal{C}$ or $\sigma(T) = \overline{\mathcal{D}}$.

We have also the following properties:

Proposition 3.9. *Let T be an m-partial isometry such that* $N(T)$ *is a reducing subspace for T. Then*

(i) $\lambda \in \sigma_a(T) \setminus \{0\}$ *implies* $\overline{\lambda} \in \sigma_a(T^*)$.

(ii) $\lambda \in \sigma_p(T) \setminus \{0\}$ *implies* $\overline{\lambda} \in \sigma_p(T^*)$.

(iii) *Eigenvectors of T corresponding to distinct eigenvalues are orthogonal*.

Proof. (i) Let $\lambda \in \sigma_a(T) \setminus \{0\}$. Then choose a sequence $(x_n)_{n \geq 1} \subset H$ such that $\| x_n \| = 1$ and $(T - \lambda)x_n \to 0$, so for all $k \geq 0$, we have $T^{*k}T^k(T - \lambda)x_n \to 0$ and $(T^k - \lambda^k)x_n \to 0$. On the other hand,

$$\begin{aligned} T^{*k}T^k(T - \lambda)x_n &= T^{*k}T^{k+1}x_n - \lambda T^{*k}T^k x_n \\ &= T^{*k}T^{k+1}x_n - \lambda T^{*k}(T^k - \lambda^k)x_n - \lambda T^{*k}\lambda^k x_n \to 0. \end{aligned}$$

Since *T* is an *m*-partial isometry, we deduce that

$$\lambda \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} (\lambda T^*)^k \to 0.$$

Hence $\left(T^* - \frac{1}{\lambda} I\right)^m x_n \to 0$. Finally, the operator $\left(T^* - \frac{1}{\lambda} I\right)$ is not bounded below. From Proposition 3.9, we conclude that $\overline{\lambda} = \frac{1}{\lambda} \in \sigma_a(T^*)$.

(ii) Let $\lambda \in \sigma_p(T) \setminus \{0\}$. Then there exists $x \in H \setminus \{0\}$ such that $Tx = \lambda x$, using a similar argument as in (i), we get $(T^* - \overline{\lambda} I)^m x = 0$ from which it follows that $\overline{\lambda} \in \sigma_p(T^*)$.

(iii) Let λ and μ be distinct eigenvalues of T. Suppose $Tx = \lambda x$ and $Ty = \mu y$. Then

$$0 = \left\langle \left(\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} T^{*k} T^{k} \right) Tx \mid y \right\rangle$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \langle (T^{*k} T^{k}) Tx \mid y \rangle$$

$$= \lambda \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \langle T^{k} x \mid T^{k} y \rangle$$

$$= \lambda \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (\lambda \overline{\mu})^{k} \langle x \mid y \rangle$$

$$= \lambda (\lambda \overline{\mu} - 1)^{m} \langle x \mid y \rangle.$$

If $\mu = 0$, then $\langle x \mid y \rangle = 0$, if $\lambda\mu \neq 0$, then $\overline{\mu} = \frac{1}{\mu}$ and we find also that $\langle x \mid y \rangle = 0$.

The following corollary gives a more detailed description of the spectrum in a special case.

Corollary 3.2. *Let $T \in \mathcal{L}(H)$. If both T and T^* are m-partial isometries such that T reduces $N(T)$ and T^* reduces $N(T^*)$, then $\sigma(T) \subset \mathcal{C} \cup \{0\}$.*

Proof. It follows from $\sigma(T) = \sigma_a(T) \cup \{\overline{\lambda}, \lambda \in \sigma_p(T^*)\}$ and Proposition 3.9.

Note also that, if both T and T^* are m-partial isometries satisfying $N(T) = N(T^*)$, then $\sigma(T) \subset \mathcal{C} \cup \{0\}$. The result follows from [1, Corollary 1.22] since both $T\,|_{N(T)^{\perp}}$ and $T^*\,|_{N(T)^{\perp}}$ are m-isometries.

4. 2-partial Isometry

In this section, we prove some results for the class of 2-partial isometries.

Theorem 4.1. *Let* $T \in \mathcal{L}(H)$ *be a* 2*-partial isometry such that* $N(T)$ *is a reducing subspace for T. Then any power of T is also a* 2*-partial isometry*.

Proof. Let $k \in \mathbb{N}$. If T is a 2-partial isometry, then $T|_{N(T)^\perp}$ is a 2-isometry and from [11, Theorem 2.1], $T^k|_{N(T)^\perp}$ is a 2-isometry. Hence T^k is a 2-partial isometry.

Theorem 4.2. *A nonisometric unilateral weighted shift T with weights* $(\lambda_n)_{n\geq 1}$ *is a* 2*-partial isometry if and only if*

$$\lambda_n(|\lambda_n|^2|\lambda_{n+1}|^2 - 2|\lambda_n|^2 + 1) = 0 \quad \textit{for each } n.$$

Proof. Suppose that T is a 2-partial isometry. If $(e_n)_{n\geq 1}$ is an orthonormal base for H, then $Te_n = \lambda_{n+1}e_{n+1}$ and $T^*e_{n+1} = \overline{\lambda}_{n+1}e_n$. Hence $\lambda_n(|\lambda_n|^2|\lambda_{n+1}|^2 - 2|\lambda_n|^2 + 1) = 0$ for each n. The converse assertion is obvious.

Corollary 4.1. *Let T be an injective nonisometric unilateral weighted shift with weights* $(\lambda_n)_{n\geq 1}$. *If T is a* 2*-partial isometry*, *then the following assertions hold*:

(i) $|\lambda_n|^2|\lambda_{n+1}|^2 - 2|\lambda_n|^2 + 1 = 0$ *for each n*.

(ii) $(|\lambda_n|)_n$ *is a strictly decreasing sequence of real numbers converging to* 1.

(iii) $1 < |\lambda_n| < \sqrt{2}$ *for each* $n > 1$.

Proof. (i) is a consequence of Theorem 4.2, since T is injective.

(ii) Suppose that $|\lambda_{k+1}| \geq |\lambda_k|$ for some k. Then by (i), we find that $0 \geq (1 - |\lambda_k|)^2$ or $|\lambda_k| = 1$. But this will contradict $|\lambda_n| \neq 1$ for each n, since T is nonisometric.

(iii) Rewriting equation (i) as

$$|\lambda_{n+1}|^2 - 2 + \frac{1}{|\lambda_n|^2} = 0,$$

hence $|\lambda_n| < \sqrt{2}$ for each $n \geq 2$ and by (ii), we get $|\lambda_n| > 1$.

Theorem 4.3. *Let* $T \in \mathcal{L}(H)$ *be a* 2*-partial isometry such that* $N(T)$ *is a reducing subspace for T. If there exists a constant* $M > 0$ *satisfying*

$$\| T^n |_{N(T)^\perp} \| \leq M, \quad \forall n \in \mathbb{N}^*,$$

then T is a partial isometry and quasi-isometry. In particular, T is quasinormal.

Proof. If T is a 2-partial isometry, then $T |_{N(T)^\perp}$ is a 2-isometry, and by [11, Theorem 2.4], $T |_{N(T)^\perp}$ is an isometry. In particular, Proposition 2.1 gives $TT^*T = T$. Using equation $T(T^{*2}T^2 - 2T^*T + I) = 0$, we deduce the result.

Corollary 4.2. *Let* $T \in \mathcal{L}(H)$ *be a* 2*-partial isometry such that* $N(T)$ *is a reducing subspace for T. If* $r(T |_{N(T)^\perp}) = | w(T |_{N(T)^\perp}) |$, *then T is a partial isometry.*

Theorem 4.4 [11, Theorem 2.10]**.** *The spectrum of a* 2*-isometry is the closed unit disc provided it is nonunitary.*

In the next for $T \in \mathcal{L}(H)$ we denote by *P* the canonical projector on $N(T)$. We will generalize the result below for 2-partial isometry.

Theorem 4.5. *Let* $T \in \mathcal{L}(H)$ *be a* 2*-partial isometry and reduce* $N(T)$. *If* $T^*T - I + P$ *is not the null operator, then the spectrum of T is the closed unit disc.*

Proof. Let *T* be a 2-partial isometry on *H*. If *T* is injective, then it is a 2-isometry and the result is given by Theorem 4.4. If *T* is not injective, since $N(T)$ is a reducing subspace for *T*, then for $\lambda \in \mathbb{C}^*$, the operator $S = \lambda I - T$ is invertible if and only if $S |_{N(T)}$ and $S |_{N(T)^\perp}$ are invertible. Hence $\sigma(T) = \sigma(T |_{N(T)^\perp}) \cup \{0\}$. Now the fact that $T^*T - I + P$ is not null, shows that $T |_{N(T)^\perp}$ is a nonunitary 2-isometry. Consequently, the result follows from the preceding theorem.

Corollary 4.3. *If T is a* 2*-partial isometry and reduces* $N(T)$, *then each isolated point of its spectrum is an eigenvalue.*

Proof. If the spectrum of *T* has an isolated point, then from the above theorem, we deduce that *T* is unitary and hence the result follows.

Definition 4.1. Let $T \in \mathcal{L}(H)$. Then T is said to be *finitely cyclic* if there exist an integer p and vectors $x_1, x_2, \ldots, x_p$ in H such that

$$\overline{Span}\{T^k x_j;\, 1 \leq j \leq p,\, k = 0, 1, 2, \ldots\} = H.$$

For a finitely cyclic operator, we prove the following result:

Theorem 4.6. *If $T \in \mathcal{L}(H)$ is a finitely cyclic, 2-partial isometry and it reduces $N(T)$, then S_T is compact.*

Proof. If T is finitely cyclic, then for all $k \geq 1$, T^k is finitely cyclic and so $R(T^k)^{\perp}$ is finite dimensional. If P_k denotes the orthogonal projection of H onto $R(T^{k-1})$, then $S_T - P_k S_T P_k$ is finite rank. Let $x \in H$, since T is a 2-partial isometry and reduces $N(T)$, $T^* S_T T = S_T$, and thus

$$\begin{aligned}\langle P_k S_T P_k T^{k-1}x \,|\, T^{k-1}x\rangle &= \langle S_T T^{k-1}x \,|\, T^{k-1}x\rangle \\ &= \langle T^{*k-1} S_T T^{k-1}x \,|\, x\rangle \\ &= \langle S_T x \,|\, x\rangle \\ &= \langle \Delta_T Tx \,|\, Tx\rangle,\end{aligned}$$

since $Tx \in N(T)^{\perp}$ and $T|_{N(T)^{\perp}}$ is a 2-isometry, we deduce from [1, Proposition 1.24] that

$$\begin{aligned}\langle \Delta_T Tx \,|\, Tx\rangle &= \frac{1}{k}(\| T^{k+1}x \|^2 - \| Tx \|^2) \\ &\leq \frac{1}{k}\| T^{k+1}x \|^2 \\ &= \frac{1}{k}\| T^2 T^{k-1}x \|^2 \\ &\leq \frac{\| T \|^2}{k}\| T^{k-1}x \|^2.\end{aligned}$$

Hence $\langle P_k S_T P_k T^{k-1}x \,|\, T^{k-1}x\rangle \leq \dfrac{\| T \|^2}{k}\| T^{k-1}x \|^2$, so $\lim\limits_{k \to +\infty} \| P_k S_T P_k \| = 0$ and $S_T = \lim\limits_{k \to +\infty} (S_T - P_k S_T P_k)$ is compact.

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