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A CRITICAL POINT THEOREM FOR NONCONTINUOUS FUNCTIONALS

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Abstract

We present an abstract critical point theorem dropping any smoothness or continuity assumptions on the functional. The main tools are quantitative equivariant deformation properties. As a by-product, we prove an existence result of infinitely many critical values for an invariant and noncontinuous functional on a complete metric space. This result generalizes the nonsmooth and noncontinuous cases, the so-called "Fountain theorem".

1. Introduction

The recent developments of critical point theory for nonsmooth functionals motivated the introduction of weak slope for lower semicontinuous functions defined on metric spaces, see for instance [3] or [4]. Similar notion was introduced for continuous functions in [5]. Schechter in [7] proposed a new formulation for the mountain pass and saddle point theorems without the use of "auxiliary" sets for C^1 functions. This new formulation was extended to nonsmooth case in [2] by using continuity assumption and in [6] without using any continuity assumption.

In [1], the authors proved an abstract critical point theorem which guarantees the existence of infinitely many critical values of an even C^1 functional in a

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bounded range. This result has been used in [1] and [8], to prove the existence of a sequence of solutions with unbounded energy for the semilinear elliptic equation

$$-\Delta u = \lambda h(x)|u|^{q-2}u + \mu k(x)|u|^{q-2}u$$

in open bounded and unbounded domains for $\mu > 0$ and arbitrary λ.

The purpose of this paper is to show that similar critical point result can be proved in nonsmooth case and without using any continuity assumption. As in [1], our main tool is a quantitative equivariant deformation that we extend to nonsmooth case without using any continuity assumption. Thus our result can be applied to both continuous and noncontinuous functions. Also, as a by-product, we extend to the nonsmooth case, the so-called Fountain theorem (see [9]).

In Section 2, we prove the quantitative equivariant deformation theorem. Also, Section 3 deals with the main results.

2. Quantitative Equivariant Deformation

Let X be a complete metric space, $\Phi : X \to \mathbf{R}$ be a functional and G be a Lie group. An orthogonal finite dimensional representation V of G is said to be *admissible* if all equivariant and continuous function $h : \overline{U} \to V^{k+1}$, $k \geq 1$ admits a zero on ∂U, where U is an open and bounded equivariant neighborhood of 0 in V^{k+1}.

Definition 1. An open subset $U \subset X$ is said to be *G-invariant* or *invariant* if for all $g \in G$ and $u \in U$, $gu \in U$. A map $h : U \to V^k$ is said to be *equivariant* if $h(gu) = g(h(u))$. A function $\Phi : X \to \mathbf{R}$ is said to be *invariant* under the action G if for all $x \in X$, $g \in G$, $\Phi(gx) = \Phi(x)$.

Definition 2. Let $H : B(u; \delta) \times [0, \delta] \subset U \times \mathbf{R} \to X$ be an equivariant function, i.e., for all $g \in G$, and $\Phi : X \to \mathbf{R}$ be an invariant function. Then the weak slope of Φ at $u \in X$ denoted by $|d\Phi|(u)$ is the supremum of σ such that for all $v \in B(u, \delta)$ and $t \in [0, \delta]$,

(a) $d(H(v, t), v) \leq t$;

(b) $\Phi(H(v, t)) \leq \Phi(v) - \sigma t$.

Because of the equivariance of Φ, the weak slope $|d\Phi|(u)$ is invariant under the action G.

Definition 3. An element $u \in X$ is a critical point of Φ if $|d\Phi|(u) = 0$. A real number c is called a *critical value* of Φ if there exists a critical point $u \in X$ of Φ such that $\Phi(u) = c$.

Definition 4. A regular point of Φ is an element $x \in X$ such that there exists $\sigma > 0$ and $|d\Phi|(x) > \sigma$. Denote by $\mathrm{Reg}(\Phi)$ the set of regular points of Φ.

We consider the following nonsmooth version of the quantitative equivariant deformation theorem:

Theorem 1. *Let* $\Phi : X \to \mathbf{R}$ *be an invariant function, S be a closed subset of X and U be an open neighborhood of S. Assume that U is invariant and* $\sigma > 0$ *such that* $|d\Phi|(u) > \sigma$ *for all* $u \in U$. *Then for* $\varepsilon > 0$, *there exists an equivariant continuous function* $\eta : X \times [0, 1] \to X$ *such that*

(i) $\eta(x, 0) = x$ for all $x \in X$;

(ii) $\eta(x, t) = x, \forall x \in X \setminus U$ *and for* $t \geq 0$;

(iii) $d(x, \eta(x, t)) \leq \varepsilon \cdot t \cdot d(x, X \setminus U)$;

(iv) $\Phi(x) - \Phi(\eta(x, t)) \geq \sigma d(x, \eta(x, t))$ *for all* $t \geq 0$.

Proof. Let $x \in U$. Then $x \in \mathrm{Reg}(\Phi)$ and consider $\varphi : U \times [0, 1] \to X$ as the corresponding regularity map at x (see [6]), then for all $u \in U$, φ is continuous and satisfies the following properties:

(a) $\varphi(u, 0) = u$;

(b) $\Phi(u) - \Phi(\varphi(u, t)) > 0, \forall t > 0$.

For all $g \in G$, define the function

$$\varphi_0(u, t) = g^{-1}\varphi(gu, t),$$

then $\varphi_0 : U \times [0, 1] \to X$ is continuous and satisfies

$$\varphi_0(u, 0) = g^{-1}\varphi(gu, 0) = g^{-1}gu = u \tag{1}$$

and for all $g' \in G,$ we have

$$\begin{aligned}
\varphi_0(g'u, t) &= g^{-1}\varphi(gg'u, t) \\
&= g'(g')^{-1}g^{-1}\varphi(gg'u, t) \\
&= g'(gg')^{-1}\varphi(gg'u, t) \qquad (2) \\
&= g'\varphi_0(u, t). \qquad (3)
\end{aligned}$$

From (1) and (2), $\varphi_0(\cdot, t)$ is equivariant for all $t \geq 0.$ On the other hand, for all $g \in G,$ $g^{-1} \in G$ and Φ being invariant, we have for all $v \in U,$ $\Phi(g^{-1}v) = \Phi(v)$ and by the invariance of $U,$ $g^{-1}v \in U.$ Thus

$$\begin{aligned}
\Phi(g^{-1}v) - \Phi(\varphi_0(g^{-1}v, t)) &= \Phi(v) - \Phi(g^{-1}\varphi(gg^{-1}v, t)) \\
&= \Phi(v) - \Phi(g^{-1}\varphi(v, t)) \\
&= \Phi(v) - \Phi(\varphi(v, t)) > 0. \qquad (4)
\end{aligned}$$

By equations (1) and (3), φ_0 satisfies the conditions (a) and (b) of a regularity map at *x* associated to $\Phi.$

Let us prove the existence of the deformation η. Our approach is closely related to the one proposed by Ioffe and Schwartzman in [5]. Denote by $R(u) = d(u, X \setminus U),$ for all $u \in U$ and define the function

$$\xi_u(t) = 2(\varepsilon \cdot R(u))^{-1}[\sup_{0 \leq \tau \leq 1} d(u, \varphi_0(u, \tau)) + t],$$

for all $\varepsilon > 0.$ For each $u \in Dom(\varphi),$ the function $t \to \xi_u(t)$ is strictly increasing, $\xi_u(0) = 0$ and for all $(u, t) \in U \times [0, 1],$ the mapping $(u, t) \to \xi_u(t)$ is an invariant continuous function because of the invariance of *U* and the equivariance of $\varphi_0.$ We notice that for all $t > 0,$ the inverse function ζ_u satisfies the same properties. Defining $\psi(u, t) = \varphi_0(u, \zeta_u(t)),$ then ψ is equivariant and satisfies the conditions (a) and (b) of regularity map at *x* associated to $\Phi,$ and

$$d(u, \psi(x, t)) \leq \left(\frac{\varepsilon}{2}\right)R(u) \cdot \xi_u(\zeta_u(t)) = \frac{\varepsilon}{2} R(u) \cdot t.$$

For each $x \in \mathrm{Reg}(\Phi)$, take $r(x) > 0$ such that $r(x) < \frac{1}{2} R(x)$ and $B(x; r(x)) \subset U$. Then the interiors of the balls $B(x; r(x))$ form an open covering of the metric space $Y = \mathrm{Reg}(\Phi)$ which is a therefore paracompact space. Hence there exists a finite subcovering $(\mathrm{int}(B(x_i; r(x_i))))_{i \in I}$ of Y, where I is finite. Let $(\mu_i)_{i \in I}$ be a partition of unity subordinate to this subcovering. Set $I(x) = \{i \in I : \mu_i(x) > 0\}$ and φ_{0i} is the equivariant regularity map at x_i and define

$$\begin{aligned} \psi_i &= \varphi_{0i}(u, t \cdot \mu_i(x)), \quad \text{if} \quad u \in B(x_i; r(x_i)) \\ &= u \text{ otherwise.} \end{aligned}$$

Because of the invariance of U, $\mu_i(u)$ is invariant and by the equivariance of φ_0, ψ_i is equivariant for all $i \in I$. Note that, ψ_i is defined on $U \times [0, 1]$ and satisfies the conditions (a) and (b) of regularity map at x, $\psi_i(u, t) \neq u$ for all $t > 0$, $\mu_i > 0$ and $d(u, \psi_i(u, t)) \leq \frac{\varepsilon}{2} R(u) \cdot \mu_i(u)$ for all $u \in U$.

Fixing arbitrary $x \in U$ and ordering $I(x) = \{i_1, i_2, ..., i_n\}$, by using the following iteration:

$$u_1 = x, \quad u_2 = \psi_{i_1}(x, t \cdot \mu_{i_1}(x)), ..., u_{k+1} = \psi_{i_k}(u_k, t \cdot \mu_{i_k}(x))$$

for all $k = 1, 2, ..., n$, we define η by

$$\eta(x, t) = u_{n+1}.$$

Thus $\eta : X \times [0, 1] \to X$ is continuous equivariant and satisfies (i)-(iv).

3. Existence of Infinite Sequence of Critical Values

Let X be a metric space with an isometric G-action, $\Phi : X \to \mathbf{R}$ be an invariant function under the G-action, and $H : K_\delta \to X$, where $K_\delta = B(u; \delta) \times [0, \delta]$.

Assume the following hypotheses:

(A_1) There exists an admissible representation V of G such that $X = \oplus_{j \in I} X(j)$, where $I = \mathbf{N}$ or $I = \mathbf{Z}$ and $X(j) \simeq V$, $\forall j \in I$.

(A_2) $\forall k \geq 0,$ $X_k = \overline{\oplus_{j \geq k} X(j)},$ $X^k = \oplus_{j \leq k} X(j),$ $X_k^l = \oplus_{j=k}^l X(j)$ and $X_{-n}^n = \oplus_{j=-n}^n X(j).$

(A_3) There exists $c \in \mathbf{R}$ such that each sequence $(u_n) \subset X_{-n}^n$ with $\Phi(u_n) \to c$ and $|d\Phi|(u_n) \to 0,$ as $n \to \infty$ admits a subsequence converging in $X_{-n}^n.$

Definition 5. A sequence (u_n) satisfying the condition (A_3) on X is a Palais-Smale sequence at level c or $(PS)_c$ -sequence in short.

Definition 6. A function $\Phi : X \to \mathbf{R}$ satisfies the $(PS)_c$ condition if all $(PS)_c$ -sequence admits a converging subsequence.

We give the following version of the Palais-Smale-Cerami (PSC) condition:

Definition 7. Let $\Phi : X \to \mathbf{R}$ be a continuous function. Then a sequence (u_n) in X is called a *PSC sequence* if $(\Phi(u_n))$ is bounded and $(1 + \| u_n \|) | d\Phi | (u_n) \to 0.$ Also, Φ satisfies the PSC condition if all its PSC sequences are precompact.

Let $B_k \subset X_k$ and $A_k \subset X^k$ such that $A_k \cap B_k = \varnothing,$ and there exists $\gamma \in C(B_k, X)$ a continuous and equivariant function with $\gamma(B_k) \cap A_k \neq \varnothing.$ Denote $\Gamma_k = \{\gamma \,|\, \gamma(B_k) \cap A_k \neq \varnothing\}.$

Also, define

$$b_k = \inf_{u \in B_k} \Phi(u) \quad \text{and} \quad a_k = \inf_{\gamma \in \Gamma_k} \sup_{u \in B_k} \Phi(\gamma(u)).$$

Thus $b_k \leq a_k,$ since B_k links $A_k.$ On the other hand, we may assume that

$$d(A_k, B_k) > 0.$$

Let

$$B'_k = \{v \in B_k \,|\, \Phi(v) < a_k\}.$$

Then we note that $B'_k = \varnothing$ if and only if $b_k = a_k.$ Denote by $d'_k = d(A_k, B'_k),$ then for all nonempty set A, if we write $d(A, \varnothing) = \infty,$ then $d'_k = \infty$ if $B'_k = \varnothing.$

Assume that

(A_4) $\sup_{B_k} \Phi \circ \gamma$ is attained at a point not in $B_k.$

(A_5) There exist $\alpha_k > 0$ and $T > 0$ such that $0 < T_k < d'_k$ and $\dfrac{a_k - b_k}{T_k} < \alpha_k.$

Our main result can be formulated as follows:

Theorem 2. *Assume that* Φ *satisfies the hypotheses* (A_1)-(A_5). *Then there exists* k_0 *such that for all* $k \geq k_0$, Φ *admits a critical value* $c_k \in [b_k, a_k]$.

Corollary 1. *Under the hypotheses of Theorem* 2, *if* $B_k' = \emptyset$, *then there exists a sequence* $u_k \subset X$ *such that*

(i) $u_n \in cl\left(\Phi^{-1}\left[c_k - \frac{1}{n^2}, c_k + \frac{1}{n^2}\right]\right)$;

(ii) $|d\Phi|(u_n) \leq \frac{1}{n}$;

(iii) $d(u_n, B_n) \leq \frac{1}{n}$.

The following corollary is a generalization of the Fountain theorem (see [9, Theorem 3.6]).

Corollary 2. *Under the hypotheses of Theorem* 2, *if* Φ *satisfies the* $(PS)_c$ *condition for all* $c \in \mathbf{R}$, *and if* $d_k' \to \infty$ *and* $\lim_{k\to\infty} \inf_{B_k} \Phi(u) = \infty$, *then* Φ *has an infinite sequence of critical values*.

Proof. If $d_k' \to \infty$ as $k \to \infty$, then $a_k - b_k \to 0$. Since $b_k \leq c_k \leq a_k$ if $\inf_{B_k} \Phi(u) = b_k \to \infty$ as $k \to \infty$, then $c_k \to \infty$ and by Theorem 2, Φ has an infinite sequence of critical values.

Proof of Theorem 2. Without loss of generality, we assume that $-\infty < b_k$, $a_k < \infty$. Under the hypotheses (A_4), (A_5) and by the definition of A_k and B_k, using Theorem 1 of [6] or Theorem 2.1 of [7], we obtain that for all $\varepsilon > 0$ such that $\varepsilon < \min(d_k' - T_k, \alpha_k \cdot d(A_k, B_k \backslash B_k')/2)$ and $\varepsilon \in \left]0, \frac{T_k - \alpha_k - (a_k - b_k)}{2}\right[$, there exists $u \in X$ such that

(1) $b_k - \varepsilon \leq \Phi(u) \leq a_k + \varepsilon$;

(2) $|d\Phi|(u) \leq \alpha_k$;

(3) $dist(u, B_k') < T_k$, $dist(u, B_k \backslash B_k') < \frac{\varepsilon}{\alpha_k}$.

Assume

$$D_{k,n} \subset X_{-k}^{n}, \quad k \leq n$$

and define

$$\Lambda_{k,n} = \{\lambda \in C(D_{k,n}, X_{-n}^{n}) | \lambda(gu) = g\lambda(u), \forall g \in G, \text{ and } \lambda(u) = u, \forall u \in A_k\}$$

and

$$c_k^n = \inf_{\lambda \in \Lambda_{k,n}} \sup_{u \in D_{k,n}} \Phi(\lambda(u)).$$

We can prove that $b_k \leq c_k^n \leq a_k$ for all $n \geq k$. Infact, by (1), (2) and (3) above, c_k^n is an almost critical value, that is, there exists a sequence $(u_l)_{l \geq 1} \subset X_{-n}^{n}$ with $\Phi(u_l) \to c_k^n$ and $|d\Phi|(u_l) \to 0$ as $l \to \infty$. Also, by (A_3) and (1), c_k^n converges along the subsequence to a critical value $c_k \in [b_k, a_k]$ as $n \to \infty$.

From the definitions of $D_{k,n}$ and c_k^n, we have $c_k^n \geq b_k$. It remains to prove the inequality $c_k^n \leq a_k$ for all $n \geq k$. We suppose it is false. Then there exists $\varepsilon > 0$ such that $a_k < c_k^n - 2\varepsilon$. Applying Theorem 1 with $S = \gamma(B_k)$, and defining

$$\beta'(u) = \beta(\gamma(u)) = \eta(1, \lambda(\gamma(u))),$$

then by the definition of γ, there exists $u \in B_k$ such that $\gamma(u) \in A_k$. For such $u \in B_k$, we have $\lambda(\gamma(u)) = \gamma(u)$ and $\eta(1, \lambda(\gamma(u))) = \gamma(u)$. Moreover, β' is equivariant and belongs to $\Lambda_{k,n}$. Therefore,

$$\begin{aligned} \sup_{u \in D_{k,n}} \Phi(\beta'(u)) &= \sup_{u \in D_{k,n}} \Phi(\beta \circ \gamma(u)) \\ &= \sup_{u \in D_{k,n}} \Phi(\eta(1, \lambda(\gamma(u)))) \\ &= \sup_{v \in A_k} \Phi(\eta(1, v)) \\ &= \sup_{v \in A_k} \Phi(v) < c_k^n - 2\varepsilon, \end{aligned}$$

which contradicts the definition of c_k^n.

Remark. (1) The results obtained in Theorem 2 do not require the function Φ to be C^1. Moreover, the main tool that is, the quantitative equivariant deformation theorem does not require Φ to be continuous. Thus, the results obtained in Theorem 2 can be applied even to noncontinuous functions.

(2) Let d be the metric defined on X, $A \subset X$ be a nonempty set and $\beta : [0, +\infty[\to]0, +\infty[$ be a continuous function. Then Theorem 4.1 of [2] implies that there exists a metric $\tilde{d}$ on X topologically equivalent to d such that, for any subset B of X, we have

$$\tilde{d}(B, A) \geq \int_0^{d(B, A)} \beta(t)\,dt,$$

and if $\int_0^\infty \beta(t)\,dt = \infty$, then $(X, \tilde{d})$ is complete if and only if (X, d) is complete. Let

$$B_k' = \{v \in B_k \mid \Phi(v) < a_k\},$$

and consider a sequence (u_k) in X such that $(|d\Phi|(u_k))$ is bounded. Assume

$$\tilde{d}(B_k', A_k) \geq \int_0^{d(B_k', A_k)} \beta(t)\,dt$$

and define

$$|\tilde{d}\Phi|(u_k) = \frac{|d\Phi|(u_k)}{\beta(d_k')},$$

where $d_k' = d(B_k', A_k)$. Then, if $\beta(d_k') \to \infty$ as $d_k' \to \infty$, which is possible if $B_k' = \emptyset$ or $\|u_k\| \to \infty$, we have $|\tilde{d}\Phi|(u_k) \to 0$ under the metric $\tilde{d}$. If (u_k) is such that

$$c_k - \varepsilon \leq \Phi(u_k) \leq c_k + \varepsilon,$$

for ε as in the proof of Theorem 2, then Corollary 2 implies that Φ has an infinite sequence of critical values. We observe that in this case, $|d\Phi|(u_k)$ need not to be very small.

An example of function β satisfying Remark 2 is given by: $\beta(d_k') = (\max_{u \in A_k} d(u, B_k') + 1)^\mu$ or $\beta(t) = (1 + t^2)^\mu$ for $\mu > 0$. If $\mu \geq 0$ and $|\tilde{d}\Phi|(u) \leq \varepsilon$ for all $\varepsilon > 0$, then we obtain a Cerami condition.

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