Far East Journal of Applied Mathematics
Volume 43, Number 1, 2010, Pages 49-52
Published Online: August 12, 2010
This paper is available online at http://pphmj.com/journals/fjam.htm

ON THE NICHOLS ALGEBRA ASSOCIATED TO $(q_{ij}) = \begin{pmatrix} \omega & -1 \\ -\omega^2 & \omega \end{pmatrix}$, OF TYPE A_2

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Abstract

We examine the defining relations of the Nichols algebra associated to $(q_{ij}) = \begin{pmatrix} \omega & -1 \\ -\omega^2 & \omega \end{pmatrix}$, of type A_2, by using the method introduced by Nichols [1] (see also [3]).

1. Introduction

Nichols algebras are graded braided Hopf algebras with the base field in degree 0 and which are coradically graded and generated by its primitive elements ([4-7]). Let V be a vector space and $c: V \otimes V \to V \otimes V$ be a linear isomorphism. Then (V, c) is called a *braided vector space*, if c is a solution of the braid equation, that is $(c \otimes id)(id \otimes c)(c \otimes id) = (id \otimes c)(c \otimes id)(id \otimes c)$. The pair (V, c) determines the Nichols algebras up to isomorphism. Let G be a group. A Yetter-Drinfeld module V over $\mathbb{K}G$ is a G-graded vector space $V = \oplus_{g \in G} V_g$, which is a

2010 Mathematics Subject Classification: 20F55.

Keywords and phrases: Nichols algebra, diagonal type, Cartan matrix.

Received May 19, 2010

G-module such that $g \cdot V_h \subset V_{ghg^{-1}}$, for all $g, h \in G$. The category ${}^G_G YD$ of $\mathbb{K}G$-Yetter-Drinfeld module is braided. For $V, W \in {}^G_G YD$, the braiding $c : V \otimes W \to W \otimes V$ is defined by $c(v \otimes w) = (g \cdot w) \otimes v$, $v \in V_g$, $w \in W$. Let V be a Yetter-Drinfeld module over G and let $T(V) = \oplus_{n \geq 0} T(V)(n)$ denote the tensor algebra of the vector space V. Let S be the set of all ideals and coideals I of $T(V)$ which are generated as ideals by $\mathbb{N}$-homogeneous elements of degree ≥ 2, and which are Yetter-Drinfeld submodules of $T(V)$. Let $I(V) = \sum_{I \in S} I$. Then $B(V) := T(V)/I(V)$ is called the *Nichols algebra* of $V \in {}^G_G YD$. In this article, we examine the defining relations of the Nichols algebra $B(V)$ associated to $(q_{ij}) = \begin{pmatrix} \omega & -1 \\ -\omega^2 & \omega \end{pmatrix}$, of type A_2.

2. Nichols Algebras of Cartan Type

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. Let G be an abelian group and V be a finite dimensional Yetter-Drinfeld module. Then the braiding is given by a nonzero scalar $q_{ij} \in \mathbb{K}$, $1 \leq i$, $j \leq \theta$, in the form $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, where $x_1, \ldots, x_\theta$, is a basis of V. If there is a basis such that $g \cdot x_i = \chi_i(g) x_i$ and $x_i \in V_{g_i}$, then V is called *diagonal type*. For the braiding, we have $c(x_i \otimes x_j) = \chi_j(g_i) x_j \otimes x_i$, for $1 \leq i$, $j \leq \theta$. Hence, we have $(q_{ij})_{1 \leq i,\ j \leq \theta} = (\chi_j(g_i))_{1 \leq i,\ j \leq \theta}$. Let $B(V)$ be the Nichols algebra of V. We can construct the Nichols algebra by $B(V) \cong T(V)/I$, where I denotes the sum of all ideals of $T(V)$ that are generated by homogeneous elements of degree ≥ 2 and that are coideals. If $B(V)$ is finite-dimensional, then the matrix (a_{ij}) defined by for all $1 \leq i \neq j \leq \theta$ by $a_{ii} := 2$ and $a_{ij} := -\min\{r \in \mathbb{N} \mid q_{ij} q_{ji} q_{ii}^r = 1 \text{ or } (r+1)_{q_{ii}} = 0\}$ is a generalized Cartan matrix fulfilling $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ or $ord\, q_{ii} = 1 - a_{ij}$. (a_{ij}) is called *Cartan matrix* associated to $B(V)$. To examine the defining relations of $B(V)$, we use the technique introduced by Nichols [1] and the following Proposition [3]. For all $1 \leq i \leq \theta$, let $\sigma_i : B(V) \to B(V)$ be the algebra automorphism given by the action of g_i. If $\sigma : B(V) \to B(V)$ is an algebra automorphism, an (id, σ)-derivation $D : B(V) \to B(V)$ is a $\mathbb{K}$-linear map such that $D(xy) = D(x)\sigma(y) + xD(y)$, for all $x, y \in B(V)$.

Proposition 2.1 [3]. (1) *For all* $1 \le i \le \theta$, *there exists a uniquely determined* (id, σ)*-derivation* $D_i : B(V) \to B(V)$ *with* $D_i(x_j) = \delta_{ij}$ (*Kronecker* δ), *for all* j.

(2) $\bigcap_{i=1}^{\theta} ker(D_i) = \mathbb{K}1$.

Let $B(V)$ be a Nichols algebra with Cartan matrix $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ of type A_2, i.e., the braiding matrix (q_{ij}) fulfills $q_{12}q_{21} = q_{11}^{-1}$ or $q_{11} = -1$, and $q_{12}q_{21} = q_{22}^{-1}$.

From the results of Helbig [2], if $q_{12}q_{21} = q_{11}^{-1} = q_{22}^{-1}$ ($\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$) and $N := ordq_{11} \ge 3$, then $B(V) = T(V)/([x_1x_1x_2], [x_1x_2x_2], x_1^N, [x_1x_2]^N, x_2^N)$ with basis $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1} \mid 0 \le r_2, r_{12}, r_1 < N\}$ and $\dim_{\mathbb{K}} B(V) = N^3$. Using this, we obtain the following.

Proposition 2.2. *Let* $(q_{ij}) = \begin{pmatrix} \omega & -1 \\ -\omega^2 & \omega \end{pmatrix}$, (*type* A_2) (*where* ω *is a primitive cube root of unity, i.e.*, $\omega^3 = 1$, $\omega^2 + \omega + 1 = 0$). *Then the Nichols algebra* $B(V)$ *is described as follows*:

Generators: x_1, x_2.

Relations: $x_1^3 = 0$, $x_2^3 = 0$,

$$x_1^2x_2 - \omega^2x_1x_2x_1 + \omega x_2x_1^2 = 0,$$

$$x_2^2x_1 - \omega x_2x_1x_2 + \omega^2x_1x_2^2 = 0,$$

$$x_1x_2x_1x_2^2 - \omega x_2x_1x_2x_1x_2 + \omega^2x_2^2x_1x_2x_1 = 0,$$

$$x_2x_1x_2x_1^2 - \omega^2x_1x_2x_1x_2x_1 + \omega x_1^2x_2x_1x_2 = 0,$$

$$3\omega x_2^2x_1x_2x_1^2 + x_1x_2x_1x_2x_1x_2 - 2x_2x_1x_2x_1x_2x_1 = 0,$$

$$3\omega^2x_1^2x_2x_1x_2^2 - 2x_1x_2x_1x_2x_1x_2 + x_2x_1x_2x_1x_2x_1 = 0.$$

Its basis is given as follows:

$$\{1,\ x_1,\ x_2,\ x_1^2,\ x_2^2,\ x_1x_2,\ x_2x_1,\ x_1x_2x_1,\ x_2x_1^2,\ x_2x_1x_2,\ x_1x_2^2,$$

$$(x_1x_2)^2,\ x_1x_2^2x_1,\ x_2x_1x_2^2,\ (x_2x_1)^2,\ x_1x_2x_1^2,\ x_2(x_1x_2)^2,\ (x_1x_2)^2x_1,$$

$$(x_2x_1)^2x_1,\ (x_1x_2)^2x_2,\ (x_2x_1)^3,\ (x_2x_1)^2x_2^2,\ (x_1x_2)^2x_1^2,\ (x_1x_2)^3\}.$$

Hence the Hilbert polynomial of $B(V)$ is

$$P(t) = 1 + 2t + 4t^2 + 4t^3 + 5t^4 + 4t^5 + 4t^6 + 2t^7 + t^8.$$

Proof. From Proposition 2.1, we see that $D_1(x_1^2x_2) = \omega^2x_1x_2$, $D_1(x_1x_2x_1) = -\omega x_2x_1 + x_1x_2$, $D_1(x_2x_1^2) = -\omega^2x_2x_1$, $D_2(x_1^2x_2) = x_1^2$, $D_2(x_1x_2x_1) = -\omega^2x_1^2$, $D_2(x_2x_1^2) = \omega x_1^2$, and from these $D_i(x_1^2x_2 - \omega^2x_1x_2x_1 + \omega x_2x_1^2) = 0$ $(i = 1, 2)$. So, we obtain the relation $x_1^2x_2 - \omega^2x_1x_2x_1 + \omega x_2x_1^2 = 0$. The other relations are similarly obtained. □

References

[1] W. D. Nichols, Bialgebras of type one, Comm. Algebra 6 (1978), 1521-1552.

[2] M. Helbig, On the lifting of Nichols algebras, preprint.

[3] N. Andruskiewitsch and H.-J. Schneider, Pointed Hopf algebras, New Directions in Hopf Algebras, Vol. 43, MSRI Publications, Cambridge Univ. Press, 2002, pp. 1-68.

[4] N. Andruskiewitsch and H.-J. Schneider, Finite quantum groups and Cartan matrices, Adv. Math. 154 (2000), 1-45.

[5] I. Heckeuberger, Examples of finite-dimensional rank 2 Nichols algebras of diagonal type, Compos. Math. 143 (2007), 165-190.

[6] M. Graña, On Nichols algebras of low dimension, Contemp. Math. 267 (2000), 111-134.

[7] A. Milinski and H.-J. Schneider, Pointed indecomposable Hopf algebras over Coxeter groups, Contemp. Math. 267 (2000), 215-236.