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NEW REPRESENTATION OF SOLITON SOLUTIONS FOR NONLINEAR EQUATION

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Abstract

A new representation of n-soliton solution to the modified Vakhnenko equation is constructed by using the Hirota method. The method can be applied to other nonlinear integrable equations as well.

1. Introduction

Nonlinear integrable equations are widely used to describe many important natural phenomena and dynamic processes in physics, mechanics, chemistry, biology, etc. In order to better understand these phenomena and further apply them in the practical life, it is an important task to seek new multi-soliton solutions of nonlinear integrable equations. Moreover, various useful methods for obtaining exact solutions to nonlinear integrable equations have been developed, such as the inverse scattering approach [1-3], Darboux transformation [4-6], the Hirota method [7-9], the Wronskian technique [10-13] and so on. Among them, the Hirota method is an efficient way to obtain soliton solutions for nonlinear evolution equations [14, 15].

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Recently, Chen presented a direct method, which can obtain new multi-soliton solutions of nonlinear integrable equations [14, 15]. In this paper, we use the method to study nonlinear integrable equations. A new representation of *n*-soliton solution to the modified Vakhnenko equation is given by using the extended Hirota method.

2. A New Representation of *n*-solition Solution to the Modified Vakhnenko Equation

The modified Vakhnenko equation is a completely integrable model, which describes the nonlinear propagation of deformation wave in a flexible long string [16],

$$\frac{\partial}{\partial x}(\Lambda^2 u - u^2 - u) - \Lambda u = 0, \tag{1}$$

where $\Lambda := \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$.

To obtain new *n*-soliton solution of equation (1), we transform the independent variables

$$x = T + \int_{-\infty}^{X} U(X', T)dX' + x_0, \quad t = X, \tag{2}$$

where $u(x, t) = U(X, T)$ and x_0 is a constant. According to equation (2), equation (1) becomes

$$U_{XXT} - 2UU_T + U_X\int_X^{\infty} U_T(X', T)dX' - U_T - U_X = 0. \tag{3}$$

Letting $W_X = U$, equation (3) becomes

$$W_{XXXT} - 2W_X W_{XT} - W_{XX}W_T - W_{XT} - W_{XX} = 0. \tag{4}$$

We introduce an auxiliary variable Y and take $W = -4(\ln f)_X$. Then equation (4) can be expressed as the following bilinear equations:

$$D_X(D_Y + D_X^3) f \cdot f = 0,$$

$$\left[D_X(D_T + D_X - D_T D_X^2) + \frac{1}{3}D_T(D_Y + D_X^3)\right] f \cdot f = 0, \tag{5}$$

where D is the Hirota bilinear operator [7].

We expand f as the series

$$f(X, T, Y) = 1 + f^{(1)}\varepsilon + f^{(2)}\varepsilon^2 + f^{(3)}\varepsilon^3 + \cdots. \tag{6}$$

Substituting equation (6) into equation (5) yields

$$f_{XY}^{(1)} + f_{XXXX}^{(1)} = 0, \quad f_{TX}^{(1)} - f_{TXXX}^{(1)} + f_{XX}^{(1)} + \frac{1}{3}(f_{TY}^{(1)} + f_{TXXX}^{(1)}) = 0, \tag{7}$$

$$f_{XY}^{(2)} + f_{XXXX}^{(2)} = -\frac{1}{2} D_X (D_Y + D_X^3) f^{(1)} \cdot f^{(1)}, \tag{8a}$$

$$f_{TX}^{(2)} - f_{TXXX}^{(2)} + f_{XX}^{(2)} + \frac{1}{3}(f_{TY}^{(2)} + f_{TXXX}^{(2)})$$

$$= -\frac{1}{2}\left[D_X (D_T + D_X - D_T D_X^2) + \frac{1}{3} D_T (D_Y + D_X^3)\right] f^{(1)} \cdot f^{(1)}, \ldots. \tag{8b}$$

In order to obtain new n-soliton solution of equation (1), we take

$$f^{(1)} = \sum_{j=1}^{n} \eta_j e^{\xi_j}, \quad \xi_j = \omega_j T + k_j X + p_j Y + \xi_j^{(0)},$$

$$\eta_j = \alpha_j T + \beta_j X + \gamma_j Y + \delta_j, \tag{9}$$

where ω_j, k_j, p_j, $\xi_j^{(0)}$, α_j, β_j, γ_j and δ_j are arbitrary real constants.

When $n = 1$, by using the following formula:

$$D_t^s D_x^r (\eta_1 \eta_2 \cdots \eta_h e^{\xi_1}) \cdot (\eta_{h+1}\eta_{h+2} \cdots \eta_m e^{\xi_2})$$

$$= e^{\xi_1 + \xi_2} \prod_{p=1}^{h} (\eta_p + \alpha_p \partial_{k_1} + \beta_p \partial_{\omega_1})$$

$$\times \prod_{q=h+1}^{m} (\eta_q + \alpha_q \partial_{k_2} + \beta_q \partial_{\omega_2})(\omega_1 - \omega_2)^s (k_1 - k_2)^r, \tag{10}$$

and equations (7) and (8), we can work out

$$f^{(1)} = \eta_1 e^{\xi_1}, \quad f^{(2)} = -\frac{\beta_1^2}{4k_1^2} e^{2\xi_1}, \quad f^{(n)} = 0 \ (n \geq 3), \tag{11}$$

$$\omega_1 = \frac{k_1}{k_1^2 - 1}, \quad p_1 = -k_1^3, \quad \alpha_1 = -\frac{k_1^2 + 1}{(k_1^2 - 1)^2}\beta_1, \quad \gamma_1 = -3k_1^2\beta_1. \tag{12}$$

Hence, we have

$$W(X, T) = -4\left[\ln\left(1 + \eta_1 e^{\xi_1} - \frac{\beta_1^2}{4k_1^2}e^{2\xi_1}\right)\right]_X,$$

$$U(X, T) = -4\left[\ln\left(1 + \eta_1 e^{\xi_1} - \frac{\beta_1^2}{4k_1^2}e^{2\xi_1}\right)\right]_{XX}. \tag{13}$$

Thus, new one-soliton solution of equation (1) can be given. For $n = 2$ in equation (9), $f^{(1)} = \eta_1 e^{\xi_1} + \eta_2 e^{\xi_2}$. From equations (7) and (8), we obtain

$$f^{(2)} = -\frac{\beta_1^2}{4k_1^2}e^{2\xi_1} - \frac{\beta_2^2}{4k_2^2}e^{2\xi_2} + \left[\frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}\eta_1\eta_2 - 4k_1\beta_2\frac{(k_1 - k_2)}{(k_1 + k_2)^3}\eta_1\right.$$

$$\left. + 4k_2\beta_1\frac{(k_1 - k_2)}{(k_1 + k_2)^3}\eta_2 + \frac{4\beta_1\beta_2(k_1^2 - 4k_1k_2 + k_2^2)}{(k_1 + k_2)^4}\right]e^{\xi_1 + \xi_2}, \tag{14a}$$

$$f^{(3)} = -\left[\frac{\beta_2^2(k_1 - k_2)^4}{4k_2^2(k_1 + k_2)^4}\eta_1 - \frac{2\beta_1\beta_2^2(k_1 - k_2)^3}{k_2(k_1 + k_2)^5}\right]e^{\xi_1 + 2\xi_2}$$

$$-\left[\frac{\beta_1^2(k_1 - k_2)^4}{4k_1^2(k_1 + k_2)^4}\eta_2 + \frac{2\beta_2\beta_1^2(k_1 - k_2)^3}{k_1(k_1 + k_2)^5}\right]e^{\xi_2 + 2\xi_1}, \tag{14b}$$

$$f^{(4)} = \frac{\beta_1^2\beta_2^2(k_1 - k_2)^8}{16k_1^2k_2^2(k_1 + k_2)^8}e^{2\xi_1 + 2\xi_2}, \quad f^{(n)} = 0\ (n \geq 5), \tag{14c}$$

$$\omega_j = \frac{k_j}{k_j^2 - 1}, \quad p_j = -k_j^3, \quad \alpha_j = -\frac{k_j^2 + 1}{(k_j^2 - 1)^2}\beta_j, \quad \gamma_j = -3k_j^2\beta_j,\ j = 1, 2. \tag{14d}$$

Take $\varepsilon = 1$, then the series (6) is truncated and becomes $f = 1 + f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)}$. Thus, we can obtain new two-soliton solution of equation (1).

In general, we have

$$f = \sum_{\mu=0,1,2} \left\{ \prod_{j=1}^{n} \left(\frac{\beta_j}{2ik_j} \right)^{\mu_j(\mu_j - 1)} (\beta_j \partial_{k_j} + \delta_j)^{\mu_j(2-\mu_j)} \right.$$

$$\left. \times \exp\left(\sum_{j=1}^{n} \mu_j \xi_j + \sum_{1 \le j < l}^{(n)} \mu_j \mu_l A_{jl} \right) \right\}, \tag{15a}$$

$$e^{A_{jl}} = \frac{(k_j - k_l)^2}{(k_j + k_l)^2}, \quad \omega_j = \frac{k_j}{k_j^2 - 1}, \quad p_j = -k_j^3,$$

$$\alpha_j = -\frac{k_j^2 + 1}{(k_j^2 - 1)^2} \beta_j, \quad \gamma_j = -3k_j^2 \beta_j, \tag{15b}$$

the first summation is taken over all possible combinations of $\mu_j = 0, 1, 2$ $(j = 1, 2, ..., n)$. Then new n-soliton solution of equation (1) can be obtained.

When $\eta_j = 1$ $(j = 1, 2, ..., n)$, equation (15) is nothing but the soliton solution of the modified Vakhnenko equation in [16]. Therefore, we give the new representation of soliton solution of the modified Vakhnenko equation (1).

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