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AN INEQUALITY AND ITS APPLICATIONS

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Abstract

In this paper, we derive an inequality for the coefficient of z^n in ${}_{r+1}\phi_r$ basic hypergeometric series. We use the inequality obtained in this paper to give a sufficient condition for the convergence of a bibasic series.

1. Introduction

q-series, which is also called *basic hypergeometric series*, plays a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials and physics, etc. Inequality technique is one of the useful tools in the study of special functions. There are many papers about it (see [1, 2, 4, 5, 6, 7]). In [7], the authors gave some inequalities for certain bibasic sums. In this paper, we give a new inequality about q-series. First, we recall some definitions, notations and known results which will be used in this paper. Throughout this paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(a;\, q)_0 = 1, \quad (a;\, q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad n = 1, 2, \ldots, \quad (a;\, q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k). \tag{1.1}$$

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We also adopt the following compact notation for multiple q-shifted factorial:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \tag{1.2}$$

where n is an integer or ∞.

The Fubini's theorem. *Suppose that* f_{ij} *is absolutely summable, that is,*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} | f_{ij} | < \infty,$$

then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} f_{ij}. \tag{1.3}$$

In [1], the basic hypergeometric series ${}_{r+1}\phi_r$, is introduced as follows:

$${}_{r+1}\phi_r(a_1, a_2, \ldots, a_{r+1}; b_1, b_2, \ldots, b_r; q, z) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(q, b_1, b_2, \ldots, b_r; q)_k} z^k. \tag{1.4}$$

2. Main Result and its Proof

We know that, estimating the value of

$$\left| \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} \right| \tag{2.1}$$

is an important problem in the study of q-series. In this section, we want to derive an upper bound of (2.1). In order to prove the main result, we need to introduce following lemma.

Lemma 2.1. *Let a and b be given complex numbers, satisfying* $b \neq \bar{b}$ *or* $b < 1$. *Then, for* $0 \leq x \leq 1$, *we have*

$$\left| \frac{1 - ax}{1 - bx} \right| \leq e^{Mx}, \tag{2.2}$$

where

$$M = \begin{cases} \alpha + | a | + | b |, & \bar{b} \neq b, \\ | a | + \dfrac{| b |}{1 - b}, & b < 1, \end{cases} \qquad \alpha = \frac{| b |^2}{| \mathrm{Im}(b) |}.$$

Proof. (i) When $b \neq \overline{b}$, let

$$f(x) = (1 - bx)(1 - \overline{b}x)e^{\alpha x}.$$

Since

$$\begin{aligned}\Delta &= (2|b|^2 - \alpha(b + \overline{b}))^2 - 4\alpha|b|^2(\alpha - (b + \overline{b}))\\ &= 4|b|^4 + \alpha^2(b + \overline{b})^2 - 4\alpha^2|b|^2\\ &= 4|b|^4 + \alpha^2(b - \overline{b})^2 = 0,\end{aligned}$$

hence

$$f'(x) = [\alpha|b|^2 x^2 + (2|b|^2 - \alpha(b + \overline{b}))x + \alpha - (b + \overline{b})]e^{\alpha x} \geq 0.$$

So $f(x)$ is monotonous increasing function with respect to $0 \leq x \leq 1$ and

$$f(x) \geq 1, \tag{2.3}$$

(2.3) is equivalent to

$$\frac{1}{(1 - bx)(1 - \overline{b}x)} \leq e^{\alpha x}. \tag{2.4}$$

Using (2.4), we have

$$\left|\frac{1 - ax}{1 - bx}\right| = \left|\frac{(1 - ax)(1 - \overline{b}x)}{(1 - bx)(1 - \overline{b}x)}\right| \leq \frac{(1 + |a|x)(1 + |\overline{b}|x)}{(1 - bx)(1 - \overline{b}x)} \leq e^{(\alpha + |a| + |b|)x}. \tag{2.5}$$

(ii) When $b < 1$, for $0 \leq x \leq 1$, we have

$$\frac{1}{1 - bx} \leq e^{\frac{b}{1-b}x}. \tag{2.6}$$

Using (2.6), we have

$$\left|\frac{1 - ax}{1 - bx}\right| \leq \frac{1 + |a|x}{1 - bx} \leq e^{\left(|a| + \frac{b}{1-b}\right)x} \leq e^{\left(|a| + \frac{|b|}{1-b}\right)x}. \tag{2.7}$$

Together with (2.5) and (2.7), (2.2) follows. $\square$

Theorem 2.2. *Let* a_i *and* b_i *be some complex numbers, satisfying* $b_i \neq \overline{b_i}$ *or* $b_i < 1$ *with* $i = 1, 2, \ldots, r$. *Then, for all nonnegative integer* n, *we have*

$$\left| \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} \right| \leq e^{\frac{1}{1-q}\sum_{i=1}^{r} M_i}, \tag{2.8}$$

where

$$M_i = \begin{cases} \alpha_i + |a_i| + |b_i|, & \overline{b_i} \neq b_i, \\ |a_i| + \dfrac{|b_i|}{1-b_i}, & b_i < 1, \end{cases} \qquad \alpha_i = \frac{|b_i|^2}{|\operatorname{Im}(b_i)|}.$$

Proof. When $n = 0$, it is obvious that (2.8) holds; when $n \geq 1$, for $0 \leq x \leq 1$ and $1 \leq i \leq r$, by Lemma 2.1, we have

$$\left| \frac{1 - a_i q^k}{1 - b_i q^k} \right| \leq e^{M_i q^k}, \quad (k = 0, 1, 2, \ldots).$$

Consequently,

$$\left| \frac{(a_i; q)_n}{(b_i; q)_n} \right| \leq e^{M_i (1 + q + \cdots + q^{n-1})} \leq e^{\frac{1}{1-q} M_i}, \quad (i = 1, 2, \ldots, r).$$

Thus, (2.8) follows. □

3. Some Applications of the Inequality

Convergence is an important problem in the study of q-series. In this section, we use the inequality obtained in this paper to give a sufficient condition for the convergence of a bibasic series.

Theorem 3.1. *Let* z, a_i, b_j *be some complex numbers, satisfying* $|z| < 1$, $b_j \neq \overline{b_j}$ *or* $b_j < 1$ *with* $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$. *Let* $\{c_n\}$ *be a complex sequence satisfying*

$$\sum_{n=0}^{\infty} |c_n| < \infty.$$

Then, the series

$$\sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_s; q)_k} z^k \tag{3.1}$$

is absolutely convergent.

Proof. Not losing generality, suppose $s < r$. By (2.8), we have

$$\sum_{n=0}^{\infty} | c_n | \sum_{k=0}^{n} \left| \frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_s; q)_k} \right| | z |^k$$

$$\leq e^{\frac{1}{1-q}\sum_{i=1}^{r} M_i} \sum_{n=0}^{\infty} | c_n | \sum_{k=0}^{n} | z |^k$$

$$\leq \frac{1}{1 - | z |} e^{\frac{1}{1-q}\sum_{i=1}^{r} M_i} \sum_{n=0}^{\infty} | c_n | < \infty, \tag{3.2}$$

where $b_i = 0$ in M_i, when $i > s$. Thus, the series (3.1) is absolutely convergent. □

Theorem 3.2. *Let z, a_i, b_j be some complex numbers, satisfying $| z | < 1$, $b_j \neq \overline{b}_j$ or $b_j < 1$ with $i = 1, 2, \ldots, r + 1$, $j = 1, 2, \ldots, r$. Then, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{k=1}^{n} k \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(b_1, b_2, \ldots, b_r, q; q)_k} z^k \tag{3.3}$$

is absolutely convergent and

$$1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{k=1}^{n} k \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(b_1, b_2, \ldots, b_r, q; q)_k} z^k$$

$$= {}_{r+1}\phi_r(a_1, a_2, \ldots, a_{r+1}; b_1, b_2, \ldots, b_r; q, z). \tag{3.4}$$

Proof. By (2.8), we have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{k=1}^{n} k \left| \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(b_1, b_2, \ldots, b_r, q; q)_k} z^k \right|$$

$$\leq e^{\frac{1}{1-q}\sum_{i=1}^{r+1} M_i} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{k=1}^{n} k | z |^k$$

$$\leq e^{\frac{1}{1-q}\sum_{i=1}^{r+1} M_i} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \cdot \frac{| z |}{(1 - | z |)^2} < \infty, \tag{3.5}$$

where $b_{r+1} = q$, in M_{r+1}, M_i is defined as before $(i = 1, \ldots, r + 1)$.

From (3.5), we know that, the series in (3.3) is absolutely convergent and by (1.3), we have

$$1+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{k=1}^{n}k\,\frac{(a_1,\, a_2,\, \ldots,\, a_{r+1};\, q)_k}{(b_1,\, b_2,\, \ldots,\, b_r,\, q;\, q)_k}\,z^k$$

$$=1+\sum_{k=1}^{\infty}k\,\frac{(a_1,\, a_2,\, \ldots,\, a_{r+1};\, q)_k}{(b_1,\, b_2,\, \ldots,\, b_r,\, q;\, q)_k}\,z^k\sum_{n=k}^{\infty}\frac{1}{n(n+1)}$$

$$=1+\sum_{k=1}^{\infty}\frac{(a_1,\, a_2,\, \ldots,\, a_{r+1};\, q)_k}{(b_1,\, b_2,\, \ldots,\, b_r,\, q;\, q)_k}\,z^k. \tag{3.6}$$

Together with (1.4) and (3.6), (3.4) follows. □

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