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PSYCHOLOGICAL TESTS IN MEASUREMENT THEORY

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Abstract

The purpose of this paper is to show the measurement theoretical approach to a problem of analyzing scores of tests for students. The obtained score is assumed to be the sum of a true value and a measurement error caused by the test, in which a student's score is subject to a systematic error (= noise) depending on his/her health or psychological condition at the test. In such cases, statistical measurements are convenient since these two errors in measurement theory can be characterized in the different mathematical structures, respectively. As a result, we show that "reliability coefficient" = "correlation coefficient" in the clearer formulation.

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1. Introduction

Recently, in [5-15], we propose measurement theory, which is motivated by dynamical system theory and quantum mechanics, and is constructed in terms of operator algebras (cf. [17]) such as

$$\text{“measurment theory} \left(= \text{MT}\right)\text{”} = \underset{(\text{Linguistic rule 1})}{\left[\text{measurement}\right]} + \underset{(\text{Linguistic rule 2})}{\left[\text{causality}\right]}.$$

And we have the following classification:

$$\text{“MT”} = \begin{cases} \text{quantum measurement theory} \left(= \text{quantum mechanics} \left[19\right]\right) \\ \text{classical measurement theory,} \end{cases}$$

where the algebra is either non-commutative or commutative. Measurement theory is neither mathematics nor physics but quantitative language. As seen in [9], measurement theory is quite applicable, that is, it covers dynamical system theory, Fisher’s statistics, Bayesian statistics, control theory, information theory, quantum theory, etc. Therefore, we believe that measurement theory gives the framework to the language of ordinary science.

We note that the world-understanding (or precisely, the world-description) is composed of two methods, i.e., the linguistic method (i.e., quantitative language, idealism) and the mechanical method (i.e., physics, materialism), whose ultimate theory is respectively MT (i.e., measurement theory) and TOE (i.e., the theory of everything). Throughout the history of quantitative language, we have few quantitative keywords: “velocity” (cf. [11]), “causality”, “probability” (cf. [16]) and “measurement”. Note that dynamical system theory (= statistics, cf. [13]) [resp. measurement] is regarded as the quantitative language that is composed of “causality” and “probability” [resp. “causality” and “measurement”]. That is, in the history of quantitative language, we have the evolutions, decided by Occam’s razor, such as “velocity $\rightarrow$ causality” and “probability $\rightarrow$ measurement”.

The purpose of this paper is to treat a problem of analyzing scores of tests for students in measurement theory. The obtained score is assumed to be the sum of a true value and a measurement error caused by the test, in which a student’s score is subject to a systematic error depending on his/her health or psychological condition at the test. It is important to see that systematic and measurement errors can be respectively represented by the different mathematical structures in measurement theory. Therefore, we can avoid confusing the two errors in the measurement theory.

Thus, our main result (i.e., Theorem 4: "reliability coefficient" = "correlation coefficient") becomes clearer than the conventional one.

2. Measurement Theory

Focusing on classical measurements, we shall review the measurement theory introduced in [9].

2.1. Mathematical preparations

Throughout this paper, the symbol Ω denotes a locally compact Hausdorff space with the Borel field $\mathcal{B}_\Omega$. The space $C(\Omega)$ denotes the Banach-algebra:

$$C(\Omega) := \{f : \Omega \to \mathbb{R} \mid f \text{ is continuous and bounded on } \Omega\},$$

endowed with canonical structures. The functions 0 and 1 denote the zero function and the constant 1 function, respectively.

Define $\mathcal{M}(\Omega)$ by $\{\rho : \mathcal{B}_\Omega \to \mathbb{R} \mid \rho \text{ is a finite signed measure on } \Omega\}$. We shall introduce two subclasses of $\mathcal{M}(\Omega)$: the mixed state class $\mathcal{M}^m(\Omega)$ and the pure state class $\mathcal{M}^p(\Omega)$ defined by

$$\mathcal{M}^m(\Omega) := \{\rho \in \mathcal{M}(\Omega) \mid \rho(A) \geq 0, \text{ for } A \in \mathcal{B}_\Omega \text{ and } \rho(\Omega) = 1\},$$

$$\mathcal{M}^p(\Omega) := \{\delta_\omega \in \mathcal{M}(\Omega) \mid \omega \in \Omega\},$$

where $\delta_\omega : \mathcal{B}_\Omega \to \{0, 1\}$ is a point measure at $\omega \in \Omega$, i.e.,

$$\delta_\omega(A) = \begin{cases} 1, & \text{for } \omega \in A, \\ 0, & \text{for } \omega \notin A, \end{cases} \quad (A \in \mathcal{B}_\Omega).$$

The space $\mathcal{M}^p(\Omega)$ with the weak* topology can be identified with the Ω; $\mathcal{M}^p(\Omega)$ is called a *state space* under the identification.

Following Davies [2], we shall introduce a concept of observables. In all descriptions of this paper, the symbol X denotes a set, and $\mathcal{F}_X$ is the σ-subfield of the power set $\mathcal{P}(X) := \{\Xi \mid \Xi \subseteq X\}$.

We call a triplet $(X, \mathcal{F}_X, F)$ an *observable* in $C(\Omega)$ if $F : \mathcal{F}_X \to C(\Omega)$ satisfies

(i) $0 \le F(\Xi) \le 1$ for $\Xi \in \mathcal{F}_X$, $F(\emptyset) = 0$ and $F(X) = 1$,

(ii) for any countable decomposition $\{\Xi_1, \Xi_2, \Xi_3, \dots\}$ of Ξ $(\Xi_k, \Xi \in \mathcal{F}_X$ $(k = 1, 2, 3, \dots))$, it holds that

$$F(\Xi)(\omega) = \lim_{N \to \infty} \sum_{n=1}^{N} F(\Xi_n)(\omega) \quad (\omega \in \Omega).$$

Remark 1. Note that any locally compact Hausdorff topological space has the Stone-Čech compactification. Furthermore, if Ω is compact, then the above σ-additivity (ii) can be interpreted in the sense of the norm topology (see Pettis's complete additivity theorem in [3]). As mentioned in [9], measurement theory has two formulations, i.e., C^*-algebraic formulation and W^*-algebraic formulation. In this paper, we devote ourselves to the C^*-algebraic formulation. In W^*-algebraic formulation, the state space Ω is not only assumed to be a locally compact Hausdorff space but also a σ-finite measure space $(\Omega, \mathcal{B}_\Omega, \nu)$ such that $\nu(D) > 0$ for any open set $D (\subseteq \Omega)$.

2.2. Linguistic rule in measurements

The linguistic rule to be presented below is analogous to a classical version of Born's probabilistic interpretation of quantum mechanics (see [2, 19]).

With any *system S*, an algebra $C(\Omega)$ can be associated in which a measurement theory of that system can be formulated. A *state* of the system S is represented by a pure state $\delta_\omega \in \mathcal{M}^p(\Omega)$ and an *observable* is represented by an observable $\mathbf{O} = (X, \mathcal{F}_X, F)$ in $C(\Omega)$. Moreover, the *measurement of an observable* $\mathbf{O}$ *for the system S with a state* δ_{ω_0} is denoted by $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_{\omega_0}]})$ (or in short, $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\omega_0]})$), so that we obtain a measured value in X by the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_{\omega_0}]})$.

Linguistic rule 1 (Measurements)**.** Let $\mathbf{O} = (X, \mathcal{F}_X, F)$ be an observable in $C(\Omega)$ and $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_{\omega_0}]})$ be a measurement of the observable $\mathbf{O}$ for the system with a state δ_{ω_0}. Then the probability that a measured value in X by $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_{\omega_0}]})$ belongs to a set $\Xi \in \mathcal{F}_X$ is given by $F(\Xi)(\omega_0)$.

2.3. Linguistic rule in statistical measurements

In our setting, we assert that most of the measurement problems is to infer an unknown state $\delta_\omega \in \mathcal{M}^p(\Omega)$. In order to know a state δ_ω, we take a measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_\omega]})$. When we want to emphasize that we have no information about δ_ω, the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_\omega]})$ is often denoted by $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]})$.

The measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]})$ for a statistical state $\rho (\in \mathcal{M}^m(\Omega))$ is denoted by $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\rho))$ (or in short, $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S(\rho))$), called a *statistical measurement*. This means that the probability that a measured value by $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S(\rho))$ belongs to $\Xi \in \mathcal{F}_X$ is given by

$$\langle \rho, F(\Xi) \rangle \left(\equiv \int_\Omega [F(\Xi)](\omega)\rho(d\omega) \right).$$

Summing up the above arguments, we have the following Linguistic rule.

Linguistic rule 1′ (Statistical measurements)**.** Let $\mathbf{O} = (X, \mathcal{F}_X, F)$ be an observable in $C(\Omega)$. Let $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S(\rho))$ be a statistical measurement of the observable $\mathbf{O}$ for the system with a statistical state ρ. Then the probability that a measured value in X by $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S(\rho))$ belongs to a set $\Xi \in \mathcal{F}_X$ is given by $\langle \rho, F(\Xi) \rangle$.

2.4. Linguistic rule (causality)

Let Ω_k $(k = 1, 2)$ be a locally compact Hausdorff space. The function $1_k \in C(\Omega_k)$ denotes the constant 1 function in Ω_k $(k = 1, 2)$. A continuous linear operator $\Phi_{1,2} : C(\Omega_2) \to C(\Omega_1)$ is called a *Markov operator*, if it satisfies

(i) $\Phi_{1,2} f_2 \geq 0$ for $f_2 \in C(\Omega_2)$ satisfying $f_2 \geq 0$,

(ii) $\Phi_{1,2} 1_2 = 1_1$,

(iii) there exists a bounded linear operator $\Phi^*_{1,2} : \mathcal{M}(\Omega_1) \to \mathcal{M}(\Omega_2)$ which satisfies

$$\int_{\Omega_1} (\Phi_{1,2} f)(\omega)\rho(d\omega) = \int_{\Omega_2} f(\omega)(\Phi^*_{1,2}\rho)(d\omega) \text{ for } f \in C(\Omega_2).$$

Note that the condition (iii) is not needed if Ω is compact. If $\Phi_{1,2} : C(\Omega_2) \to C(\Omega_1)$ is a Markov operator, note that the $(X, \mathcal{F}_X, \Phi_{1,2}F)$ is an observable in $C(\Omega_1)$ for any observable $(X, \mathcal{F}_X, F)$ in $C(\Omega_2)$, where $(\Phi_{1,2}F)(\Xi) = \Phi_{1,2}(F(\Xi))$, for $\Xi \in \mathcal{F}_X$.

Now, we can propose Linguistic rule 2 as follows:

Linguistic rule 2 (Causality)**.** The causal relation between systems is represented by a Markov operator $\Phi_{1,2} : C(\Omega_2) \to C(\Omega_1)$. Moreover, the observable O_2 in $C(\Omega_2)$ can be identified with the observable $\Phi_{1,2}\mathbf{O}_2$ in $C(\Omega_1)$, that is,

$$\Phi_{1,2}\mathbf{O}_2 \text{ in } C(\Omega_1) \xleftarrow[\text{identification}]{\Phi_{1,2}} \mathbf{O}_2 \text{ in } C(\Omega_2).$$

Remark 2. (i) In the above case, $T = \{1, 2\}$ is quite simple. However, in the W^*-algebraic formulation (cf. [9]), it is usual to consider that T is an infinite complete tree, that is, any subset $A(\subseteq T)$ that is bounded under below has an inf A.

(ii) Measurement theory is based on dualism, and therefore, "observer" and "observed object" must be always separated. This fact is sometimes confused. For example, the famous statement: "I think, therefore I exist" is not the statement in measurement theory since it includes the confusion. In fact, this statement has never been effectively used before in science.

2.5. Simultaneous measurement and parallel measurement

For each $k = 1, 2, \ldots, n,$ we consider an observable $\mathbf{O}_k := (X_k, \mathcal{F}_k, F_k)$ in $C(\Omega)$. Let $(\times_{k=1}^n X_k, \times_{k=1}^n \mathcal{F}_k)$ be the product measurable space of $(X_k, \mathcal{F}_X)$'s. An observable $\mathbf{O} := (\times_{k=1}^n X_k, \times_{k=1}^n \mathcal{F}_k, F)$ in $C(\Omega)$ is called the product observable of $\{\mathbf{O}_k : k = 1, 2, \ldots, n\}$ and denoted by $\times_{k=1}^n \mathbf{O}_k$ if it satisfies

$$[F(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_n)](\omega) = [F_1(\Xi_1)](\omega) \cdot [F_2(\Xi_2)](\omega) \cdots [F_n(\Xi_n)](\omega),$$

for all $\omega \in \Omega$, $\Xi_k \in \mathcal{F}_k$, $k = 1, 2, \ldots, n$. The measurement $\mathbf{M}_{C(\Omega)}(\times_{k=1}^n \mathbf{O}_k, S_{[\omega]})$ is called the *simultaneous measurement* of $\{\mathbf{O}_k\}_{k=1}^n$.

For each $k = 1, 2, \ldots, n,$ we consider an observable $\mathbf{O}_k := (X_k, \mathcal{F}_k, F_k)$ in $C(\Omega_k)$. An observable $\mathbf{O} := (\times_{k=1}^n X_k, \times_{k=1}^n \mathcal{F}_k, F)$ in $C(\times_{k=1}^n \Omega_k)$ is called the *parallel observable* of $\{\mathbf{O}_k : k = 1, 2, \ldots, n\}$ and denoted by $\otimes_{k=1}^n \mathbf{O}_k$ if it satisfies

$$[F(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_n)](\omega_1, \omega_2, \ldots, \omega_n) = [F_1(\Xi_1)](\omega_1) \cdot [F_2(\Xi_2)](\omega_2) \cdots [F_n(\Xi_n)](\omega_n),$$

for all $(\omega_1, \omega_2, \ldots, \omega_n) \in \times_{k=1}^n \Omega_k, \ \Xi_k \in \mathcal{F}_k, \ k = 1, 2, \ldots, n.$

The measurement $\mathbf{M}_{C(\times_{k=1}^n \Omega_k)}(\otimes_{k=1}^n \mathbf{O}_k, S_{[(\omega_1, \ldots, \omega_n)]})$, denoted by $\otimes_{k=1}^n \mathbf{M}_{C(\Omega_k)}(\mathbf{O}_k, S_{[\omega_k]})$, is called the *parallel measurement* of $\{\mathbf{M}_{C(\Omega_k)}(\mathbf{O}_k, S_{[\omega_k]})\}_{k=1}^n$. Moreover, the statistical measurement $\mathbf{M}_{C(\times_{k=1}^n \Omega_k)}$ $(\otimes_{k=1}^n \mathbf{O}_k, S(\otimes_{k=1}^n \rho_k))$, denoted by $\otimes_{k=1}^n \mathbf{M}_{C(\Omega_k)}(\mathbf{O}_k, S(\rho_k))$, is called the *parallel measurement* of $\{\mathbf{M}_{C(\Omega_k)}(\mathbf{O}_k, S(\rho_k))\}_{k=1}^n$.

Example 1 (Bell's inequality) Let $a = (a_1, a_2) \in \mathbb{R}^2$ such that $\| a \| = \sqrt{|a_1|^2 + |a_2|^2} = 1$, and let $b = (b_1, b_2) \in \mathbb{R}^2$ such that $\| b \| = 1$. Put $X = \{-1, 1\}$. Further, for any a, b such that $\| a \| = \| b \| = 1$, define the probability space $(X^2, 2^{X^2}, \nu_{ab})$ such that

$$P_{ab} \equiv \int_{X^2} x_1 \cdot x_2 [\nu_{ab}(dx_1 dx_2)](\omega_0) = a_1 b_1 + a_2 b_2$$

and, for any $x \in X$,

$$\begin{cases} \nu_{a^1 b^1}(\{x\} \times X) = \nu_{a^1 b^2}(\{x\} \times X), & \nu_{a^1 b^1}(X \times \{x\}) = \nu_{a^2 b^1}(X \times \{x\}), \\ \nu_{a^2 b^1}(\{x\} \times X) = \nu_{a^2 b^2}(\{x\} \times X), & \nu_{a^1 b^2}(X \times \{x\}) = \nu_{a^2 b^2}(X \times \{x\}). \end{cases}$$

It is well known that the above ν_{ab} exists (see [18]). Let $\omega_0 \in \Omega$. Further, define the observable $\widetilde{\mathbf{O}}_{ab} := (X^2, 2^{X^2}, F_{ab})$ in $C(\Omega)$ such that $\nu_{ab}(\{x\} \times \{y\}) = [F_{ab}(\{x\} \times \{y\})](\omega_0)$ $(\forall x, y \in X)$. Now, putting $a^1 = (a_1^1, a_2^1)$, $a^2 = (a_1^2, a_2^2)$, $b^1 = (b_1^1, b_2^1)$ and $b^2 = (b_1^2, b_2^2)$, consider the following four measurements:

$$\mathbf{M}_{C(\Omega)}(\widetilde{\mathbf{O}}_{a^1b^1}, S_{[\omega_0]}), \quad \mathbf{M}_{C(\Omega)}(\widetilde{\mathbf{O}}_{a^1b^2}, S_{[\omega_0]}),$$

$$\mathbf{M}_{C(\Omega)}(\widetilde{\mathbf{O}}_{a^2b^1}, S_{[\omega_0]}), \quad \mathbf{M}_{C(\Omega)}(\widetilde{\mathbf{O}}_{a^2b^2}, S_{[\omega_0]}).$$

That is, consider the parallel measurement

$$\mathbf{M}_{C(\Omega\times\Omega\times\Omega\times\Omega)}\left(\bigotimes_{i,\, j=1,\, 2}\widetilde{\mathbf{O}}_{a^ib^j}, S_{[(\omega_0,\, \omega_0,\, \omega_0,\, \omega_0)]}\right).$$

If we put

$$a^1 = (0, 1), \quad b^1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad a^2 = (1, 0), \quad b^2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),$$

then we easily calculate that

$$|\, P_{a^1b^1} - P_{a^1b^2} \,| + |\, P_{a^2b^1} + P_{a^2b^2} \,| = 2\sqrt{2}. \tag{1}$$

We may say that Bell's inequality (cf. [1, 18]) is broken even in classical measurements (as well as in quantum measurements), though (1) is not Bell's inequality.

3. Psychological Tests

In this section, we take a problem of tests of measuring intelligence, for example mathematical intelligence, of students. Through the measurement theory, we study the reliability of tests.

3.1. Cases 1 and 2

We shall start with a simple problem; the measurement-theoretical representation of tests for one student. Put $X_{\mathbb{R}} = \mathbb{R}$ and $\Omega_{\mathbb{R}} = \mathbb{R}$.

Case 1 (Tests for one student)**.** Let $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}$ be a set of n students. The set Θ will be regarded as a state space. We identify Θ with $\mathcal{M}^p(\Theta)$ by the identification: $\Theta \ni \theta_i \leftrightarrow \delta_{\theta_i} \in \mathcal{M}^p(\Theta)$. The mathematical intelligence of a student is generally depending on his/her health or psychological conditions at the test. Therefore, in our measurement theory, the intelligence of θ_i is assumed to be represented by a statistical state $\Phi^*\delta_{\theta_i} \in \mathcal{M}^m(\Omega_{\mathbb{R}})$ $(i = 1, 2, ..., n)$, where Φ^* :

$\mathcal{M}^m(\Theta) \to \mathcal{M}^m(\Omega_{\mathbb{R}})$ is the dual Markov operator of $\Phi : C(\Omega_{\mathbb{R}}) \to C(\Theta)$. Let $\mathbf{O} = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F)$ be an observable in $C(\Omega_{\mathbb{R}})$. As seen in the previous section, a test for a student θ_i can be represented by a statistical measurement $\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^*\delta_{\theta_i}))$. Linguistic rule 1′ asserts that the probability that the score of θ_i by $\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^*\delta_{\theta_i}))$ belongs to a set $\Xi \in \mathcal{F}_{X_{\mathbb{R}}}$ is given by $\langle \Phi^*\delta_{\theta_i}, F(\Xi) \rangle$.

For each $i = 1, 2, ..., n$, we consider a measurement $\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^*\delta_{\theta_i}))$ defined in the above problem. We define the variance σ_i^2 of statistical state $\Phi^*\delta_{\theta_i}$ by

$$\sigma_i^2 = \int_{\Omega_{\mathbb{R}}} (\omega - \mu_i)^2 (\Phi^*\delta_{\theta_i})(d\omega), \tag{2}$$

where μ_i is the expectation of $\Phi^*\delta_{\theta_i}$, that is,

$$\mu_i = \int_{\Omega_{\mathbb{R}}} \omega(\Phi^*\delta_{\theta_i})(d\omega). \tag{3}$$

The positive constant σ_i is regarded as a kind of *systematic errors* (see [3]).

We will next consider tests for a group of *n* students.

Case 2 (Tests for *n* students)**.** Let an observable $\mathbf{O} = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F)$ and a measurement $\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^*\delta_{\theta_i}))$ $(i = 1, 2, ..., n)$ be as in Case 1. Here, we consider a parallel measurement $\mathbf{M}_{C(\Omega_{\mathbb{R}}^n)}(\hat{\mathbf{O}}, S(\hat{\rho}))$, where $\hat{\mathbf{O}} = \otimes_{i=1}^n \mathbf{O}$ and $\hat{\rho} = \otimes_{i=1}^n \Phi^*\delta_{\theta_i}$. The parallel measurement $\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^*\delta_{\theta_i}))$ denoted in the above is called a *group test*. Linguistic rule 1′ in Subsection 2.3 asserts that the probability that the score in $X_{\mathbb{R}}^n$ obtained by a group test $\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^*\delta_{\theta_i}))$ belongs to a set $\times_{i=1}^n \Xi_i \in \mathcal{F}_{X_{\mathbb{R}}^n}$ is given by $\times_{i=1}^n \langle \Phi^*\delta_{\theta_i}, F(\Xi_i) \rangle =: \hat{P}_1(\times_{i=1}^n \Xi_i)$. Here, note that $(X_{\mathbb{R}}^n, \mathcal{F}_{X_{\mathbb{R}}^n}, \hat{P}_1)$ is a probability space.

Let $W_1 : X_{\mathbb{R}}^n \to \mathbb{R}$ be a statistics, i.e., a measurable function on the n-dimensional space $X_{\mathbb{R}}^n$. Then $\mathcal{E}_{\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O},\, S(\Phi^* \delta_{\theta_i}))}[W_1]$, the expectation of W_1 concerning $\hat{P}_1$, is defined by

$$\mathcal{E}_{\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O},\, S(\Phi^* \delta_{\theta_i}))}[W_1] = \int_{X_{\mathbb{R}}} \cdots \int_{X_{\mathbb{R}}} W_1(x_1, x_2, \ldots, x_n) \hat{P}_1(dx_1 dx_2 \cdots dx_n).$$

Definition 1 (Expectation and variance)**.** Let $\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))$ be a group test. By $\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))$, we will obtain a measured value $(x_1, x_2, \ldots, x_n) \in X_{\mathbb{R}}^n$ as in Case 1. We define the expectation $\mathrm{Av}[\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))]$ of $\frac{1}{n}(x_1 + x_2 + \cdots + x_n)$ and the variance $\mathrm{Var}[\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))]$ of $(x_1, x_2, \ldots, x_n)$ as follows:

$$\begin{aligned}\mathrm{Av}[\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))] &:= \frac{1}{n}\sum_{i=1}^n \mathcal{E}_{\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O},\, S(\Phi^* \delta_{\theta_i}))}[x_i] \\ &= \frac{1}{n}\sum_{i=1}^n \int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} x F(dx)(\omega) \right) (\Phi^* \delta_{\theta_i})(d\omega),\end{aligned}$$

$$\mathrm{Var}[\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))]$$

$$:= \frac{1}{n}\sum_{j=1}^n \mathcal{E}_{\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O},\, S(\Phi^* \delta_{\theta_i}))}[(x_j - \mathrm{Av}[\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))])^2]$$

$$= \frac{1}{n}\sum_{j=1}^n \int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} (x - \mathrm{Av}[\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))])^2 F(dx)(\omega) \right) (\Phi^* \delta_{\theta_j})(d\omega).$$

Let μ_i be an expectation of $\Phi^* \delta_{\theta_i}$, i.e., $\mu_i = \int_{\Omega_{\mathbb{R}}} \omega (\Phi^* \delta_{\theta_i})(d\omega)$. Here, we define $\overline{\mu}$ by the mean of true values defined by

$$\overline{\mu} := \frac{1}{n}\sum_{i=1}^n \mu_i = \frac{1}{n}\sum_{i=1}^n \int_{\Omega_{\mathbb{R}}} \omega (\Phi^* \delta_{\theta_i})(d\omega). \tag{4}$$

3.2. Reliability coefficients

Definition 2 (Unbiased observable)**.** We call $\mathbf{O} = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F)$ an unbiased observable in $C(\Omega_{\mathbb{R}})$, if F satisfies the following condition:

$$\int_{X_{\mathbb{R}}} xF(dx)(\omega) = \omega \quad (\forall \omega \in \Omega_{\mathbb{R}}). \tag{5}$$

Here, we have the following theorem.

Theorem 1. *Let* $\mathbf{O} = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F)$ *be an unbiased observable in* $C(\Omega_{\mathbb{R}})$. *Let* $\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))$ *be a group test. Then, we see the following conditions* H_1^1 *and* H_1^2.

$$\mathrm{H}_1^1 : \frac{1}{n} \sum_{i=1}^{n} \int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} (x - \omega) F(dx)(\omega) \right) (\Phi^* \delta_{\theta_i})(d\omega) = 0,$$

$$\mathrm{H}_1^2 : \frac{1}{n} \sum_{i=1}^{n} \int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} (\omega - \overline{\mu})(x - \omega) F(dx)(\omega) \right) (\Phi^* \delta_{\theta_i})(d\omega) = 0,$$

where $\overline{\mu} = \frac{1}{n} \sum_{i=1}^{n} \mu_i$ (see (4)).

Proof. Since $\mathbf{O} = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F)$ is an unbiased observable, we see $\int_{X_{\mathbb{R}}} (x - \omega) F(dx)(\omega) = 0$, which results in H_1^1 and H_1^2. □

From the condition H_1^1, we see

$$\begin{aligned} \mathrm{Av}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))] &= \frac{1}{n} \sum_{i=1}^{n} \int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} xF(dx)(\omega) \right) (\Phi^* \delta_{\theta_i})(d\omega) \\ &= \frac{1}{n} \sum_{i=1}^{n} \int_{\Omega_{\mathbb{R}}} \omega (\Phi^* \delta_{\theta_i})(d\omega) \\ &= \overline{\mu}, \end{aligned} \tag{6}$$

and we see the variance Δ_E^2 of students' intelligence

$$\Delta_E^2 = \frac{1}{n}\sum_{i=1}^{n}\int_{\Omega_{\mathbb{R}}} (\omega - \overline{\mu})^2 (\Phi^* \delta_{\theta_i})(d\omega). \tag{7}$$

For $\omega \in \Omega$, we define the Δ_ω^2 by

$$\Delta_\omega^2 = \int_{X_{\mathbb{R}}} (x - \omega)^2 F(dx)(\omega), \tag{8}$$

where Δ_ω^2 is a kind of *measurement errors* (see [3]). Put

$$\begin{aligned}\Delta_{GT}^2 &:= \frac{1}{n}\sum_{i=1}^{n}\int_{\Omega_{\mathbb{R}}} \Delta_\omega^2 (\Phi^* \delta_{\theta_i})(d\omega)\\ &= \frac{1}{n}\sum_{i=1}^{n}\int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} (x - \omega)^2 F(dx)(\omega)\right)(\Phi^* \delta_{\theta_i})(d\omega). \end{aligned} \tag{9}$$

From what we have seen, we can get the following theorem.

Theorem 2 (Variance of measured values)**.** *Let* $\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))$ *be a group test, where* $\mathbf{O} = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F)$ *is an observable in* $C(\Omega_{\mathbb{R}})$. *Under the assumption* H_1^1 *and* H_1^2 *in Theorem* 1, *it holds that*

$$\mathrm{Var}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))] = \Delta_E^2 + \Delta_{GT}^2. \tag{10}$$

Proof. The variance $\mathrm{Var}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))]$ is calculated as follows:

$$\begin{aligned}&\mathrm{Var}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^* \delta_{\theta_i}))]\\ &= \frac{1}{n}\sum_{i=1}^{n}\int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} (x_i - \overline{\mu})^2 F(dx_i)(\omega_i)\right)(\Phi^* \delta_{\theta_i})(\omega_i)\, d\omega_i\\ &= \frac{1}{n}\sum_{i=1}^{n}\int_{\Omega_{\mathbb{R}}} (\omega - \overline{\mu})^2 (\Phi^* \delta_{\theta_i})(d\omega)\\ &\quad + \frac{1}{n}\sum_{i=1}^{n}\int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} (x - \omega)^2 F(dx)(\omega)\right)(\Phi^* \delta_{\theta_i})(d\omega)\end{aligned}$$

$$+\frac{2}{n}\sum_{i=1}^{n}\int_{\Omega_{\mathbb{R}}}\left(\int_{X_{\mathbb{R}}}(\omega-\overline{\mu})(x-\omega)F(dx)(\omega)\right)(\Phi^{*}\delta_{\theta_i})(d\omega).$$

On the condition H_1^2, the third term is equal to 0. Thus, we get equality (10). □

Now, we can define the reliability coefficient as follows.

Definition 3 (Reliability coefficients)**.** Let $\otimes_{i=1}^{n}\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^{*}\delta_{\theta_i}))$ be a group test, where $\mathbf{O} = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F)$ is an observable in $C(\Omega_{\mathbb{R}})$. Under the assumption that the pair $(\{\Phi^{*}\delta_{\theta_i}\}_{i=1}^{n}, \mathbf{O})$ satisfies conditions H_1^1 and H_1^2, the *reliability coefficient* $\mathrm{RC}[\otimes_{i=1}^{n}\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^{*}\delta_{\theta_i}))]$ of a group test $\otimes_{i=1}^{n}\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^{*}\delta_{\theta_i}))$ is defined by

$$\mathrm{RC}[\otimes_{i=1}^{n}\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^{*}\delta_{\theta_i}))] = \frac{\Delta_E^2}{\mathrm{Var}[\otimes_{i=1}^{n}\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^{*}\delta_{\theta_i}))]}. \qquad (11)$$

From Theorem 2, we see the reliability coefficient $\mathrm{RC}[\otimes_{i=1}^{n}\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^{*}\delta_{\theta_i}))]$ ranges from 0 to 1.

3.3. Correlation coefficients

Concerning Definition 3, we must note that we cannot directly get the variance Δ_E^2 of mathematical intelligence of *n* students from the measured data in $X_{\mathbb{R}}^{n}$. Thus, we shall focus on the problem how to estimate the reliability coefficient. Here, we recall one typical method, say the split-half method.

Split-half method. Divide the test into equivalent halves and measure the internal consistency of a test, that is, calculate the correlation coefficient between these two sets. With psychological tests, a usual procedure is to obtain scores on the odd and even items.

Under the discussions of Case 2, we shall introduce a characterization of split-half method through our measurement theory. We adopt the same notation as used before. Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, $X_{\mathbb{R}} = \Omega_{\mathbb{R}} = \mathbb{R}$ and $\Phi^{*} : \mathcal{M}(\Theta) \to \mathcal{M}(\Omega_{\mathbb{R}})$ be as in Case 2.

Definition 4 (Group simultaneous tests)**.** Let $\mathbf{O}_1 = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F_1)$ and $\mathbf{O}_2 = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F_2)$ be observables in $C(\Omega_{\mathbb{R}})$ and $\mathbf{O}_1 \times \mathbf{O}_2$ be the product observable of $\mathbf{O}_1$ and $\mathbf{O}_2$. Then the parallel simultaneous measurement

$$\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i})),$$

is called a *group simultaneous test* of $\mathbf{O}_1$ and $\mathbf{O}_2$.

Linguistic rule 1′ asserts that the probability that a score in $X_{\mathbb{R}}^{2n}$ obtained by a group simultaneous test $\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))$ belongs to a set $\times_{i=1}^{n}(\Xi_i^{(1)} \times \Xi_i^{(2)}) \in \mathcal{F}_{X_{\mathbb{R}}^{2n}}$ is given by

$$\times_{\theta_i \in \Theta} \langle \Phi^* \delta_{\theta_i}, (F_1 \times F_2)(\Xi_i^{(1)} \times \Xi_i^{(2)}) \rangle =: \hat{P}_2(\times_{i=1}^{n}(\Xi_i^{(1)} \times \Xi_i^{(2)})).$$

Here note that $(X_{\mathbb{R}}^{2n}, \mathcal{F}_{X_{\mathbb{R}}^{2n}}, \hat{P}_2)$ is a probability space.

Let $W_2 : X_{\mathbb{R}}^{2n} \to \mathbb{R}$ be a statistics. Then $\mathcal{E}_{\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))}[W_2]$, the expectation of W_2 concerning $\hat{P}_2$, is defined by

$$\mathcal{E}_{\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))}[W_2]$$
$$= \int_{X_{\mathbb{R}}^{2n}} W_2(x_1^{(1)}, x_1^{(2)}, \ldots, x_n^{(1)}, x_n^{(2)}) \hat{P}_2(dx_1^{(1)} dx_1^{(2)} \cdots dx_n^{(1)} dx_n^{(2)}).$$

In the same manner as Definition 1, we use the following notations.

Definition 5 (Expectation, variance and covariance)**.** Let $\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}$ $(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))$ be a group simultaneous test as in Definition 4. By the $\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))$, we will obtain a measured value $(x_1^{(1)}, x_1^{(2)}, \ldots, x_n^{(1)}, x_n^{(2)}) \in X_{\mathbb{R}}^{2n}$, and introduce the following notation:

$$\mathrm{Av}^{(k)}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))]$$
$$:= \mathcal{E}_{\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))}\left[\frac{1}{n}\sum_{i=1}^{n} x_i^{(k)}\right] \quad (k = 1, 2),$$

$$\mathrm{Var}^{(k)}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))]$$

$$:= \mathcal{E}_{\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))}$$

$$\times \left[\frac{1}{n} \sum_{j=1}^{n} (x_j^{(k)} - \mathrm{Av}^{(k)}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))])^2 \right] \quad (k = 1, 2),$$

$$\mathrm{Cov}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))]$$

$$:= \mathcal{E}_{\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))}$$

$$\times \left[\frac{1}{n} \sum_{j=1}^{n} (x_j^{(1)} - \mathrm{Av}^{(1)}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))]) \right.$$

$$\left. (x_i^{(2)} - \mathrm{Av}^{(2)}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))]) \right].$$

We see

$$\mathrm{Av}^{(k)}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))] = \mathrm{Av}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_k,\, S(\Phi^* \delta_{\theta_i}))]$$

and

$$\mathrm{Var}^{(k)}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))] = \mathrm{Var}[\otimes_{i=1}^{n} \mathbf{M}_{\mathbf{O}_k}^{\Phi^* \delta_{\theta_i}}] \quad (k = 1, 2).$$

Here, recall that $\mathrm{Av}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_k,\, S(\Phi^* \delta_{\theta_i}))] = \overline{\mu} \quad (k = 1, 2),$ under the assumption H_1^1 in Theorem 1. That is, we see

$$\mathrm{Var}^{(k)}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))]$$

$$= \mathcal{E}_{\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))} \left[\frac{1}{n} \sum_{j=1}^{n} (x_j^{(k)} - \overline{\mu})^2 \right],$$

$$\mathrm{Cov}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))]$$

$$= \mathcal{E}_{\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2,\, S(\Phi^* \delta_{\theta_i}))} \left[\frac{1}{n} \sum_{i=1}^{n} (x_i^{(1)} - \overline{\mu})(x_i^{(2)} - \overline{\mu}) \right].$$

In the similar way in Subsection 3.2, we assume that the pair $(\{\Phi^*\delta_{\theta_i}\}_{i=1}^n, \mathbf{O}_k)$ $(k = 1, 2)$ satisfies conditions H_1^1 and H_1^2.

As in (9), we define

$$\Delta_{GT}^{(k)} = \left(\frac{1}{n} \sum_{i=1}^{n} \int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} (x - \omega)^2 F_k(dx)(\omega) \right) (\Phi^*\delta_{\theta_i})(d\omega) \right)^{1/2} \quad (k = 1, 2).$$

As we have seen in (10), we get

$$\mathrm{Var}^{(k)}[\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^*\delta_{\theta_i}))] = \Delta_E^2 + (\Delta_{GT}^{(k)})^2 \quad (k = 1, 2). \tag{12}$$

Definition 6 (Equivalence condition of unbiased observables)**.** Two unbiased observables $\mathbf{O}_1 = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F_1)$ and $\mathbf{O}_2 = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F_2)$ in $C(\Omega_{\mathbb{R}})$ are called to be *equivalent* if

$$\Delta_\omega^{(1)} = \Delta_\omega^{(2)} \quad (\forall \omega \in \Omega_{\mathbb{R}}), \tag{13}$$

where $\Delta_\omega^{(k)} = \left(\int_{X_{\mathbb{R}}} (x - \omega)^2 F_k(dx)(\omega) \right)^{1/2}$ $(k = 1, 2)$.

Equivalence condition of unbiased observables induces an equivalency of group averages of error variances.

Theorem 3. *Let* $\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^*\delta_{\theta_i}))$ *be a group simultaneous test as in Definition* 4. *Suppose two unbiased observables* $\mathbf{O}_1 = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F_1)$ *and* $\mathbf{O}_2 = (X_{\mathbb{R}}, \mathcal{F}_{X_{\mathbb{R}}}, F_2)$ *are equivalent. Then we see* $\Delta_{GT}^{(1)} = \Delta_{GT}^{(2)}$.

Proof. From definition (9) of group average of error variances, we see

$$\mathrm{H}_2 \colon \Delta_{GT}^{(1)} = \left(\frac{1}{n} \sum_{i=1}^{n} \int_{\Omega_{\mathbb{R}}} (\Delta_\omega^{(k)})^2 (\Phi^*\delta_{\theta_i})(d\omega) \right)^{1/2} = \Delta_{GT}^{(2)} \ (k = 1, 2).$$

That is,

$$\mathrm{Var}^{(1)}[\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^*\delta_{\theta_i}))]$$

$$= \mathrm{Var}^{(2)}[\otimes_{i=1}^n \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^*\delta_{\theta_i}))].$$

Then we see $\Delta_{GT}^{(1)} = \Delta_{GT}^{(2)}$. □

Under these assumptions, we shall calculate the correlation coefficient of the measured values by $\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))$.

Theorem 4 (Correlation coefficients)**.** *Let* $\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))$ *be a group simultaneous test in Definition* 4 *and assume that* $(\{\Phi^* \delta_{\theta_i}\}_{i=1}^{n}, \mathbf{O}_k)$ $(k = 1, 2)$ *satisfies conditions* H_1^1, H_1^2 *and* H_2. *Then it holds that*

$$\mathrm{RC}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_k, S(\Phi^* \delta_{\theta_i}))]$$

$$= \frac{\mathrm{Cov}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))]}{\sqrt{\mathrm{Var}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1, S(\Phi^* \delta_{\theta_i}))]} \cdot \sqrt{\mathrm{Var}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))]}}$$

$$(k = 1, 2), \quad (14)$$

where

$$\mathrm{RC}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_k, S(\Phi^* \delta_{\theta_i}))]$$

$$= \Delta_E^2 / \mathrm{Var}^{(k)}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))] \ (k = 1, 2).$$

Proof. Using $x_i - \overline{\mu} = (\omega - \overline{\mu}) + (x_i - \omega)$, we see

$$\mathrm{Cov}[\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mathcal{E}_{\otimes_{i=1}^{n} \mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1 \times \mathbf{O}_2, S(\Phi^* \delta_{\theta_i}))}[(x_i^{(1)} - \overline{\mu})(x_i^{(2)} - \overline{\mu})]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} \int_{X_{\mathbb{R}}} (x_i^{(1)} - \overline{\mu})(x_i^{(2)} - \overline{\mu})(F_1(dx_i^{(1)}) \cdot F_2(dx_i^{(2)}))(\omega) \right) (\Phi^* \delta_{\theta_i})(d\omega)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_{\Omega_{\mathbb{R}}} (\omega - \overline{\mu})^2 (\Phi^* \delta_{\theta_i})(d\omega)$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} \int_{X_{\mathbb{R}}} (x^{(1)} - \omega)(x^{(2)} - \omega)(F_1(dx^{(1)}) \cdot F_2(dx^{(2)}))(\omega) \right) (\Phi^* \delta_{\theta_i})(d\omega)$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \int_{\Omega_{\mathbb{R}}} \left(\int_{X_{\mathbb{R}}} (x^{(1)} - \omega)(\omega - \overline{\mu}) F_1(dx^{(1)})(\omega) \right) (\Phi^* \delta_{\theta_i})(d\omega)$$

$$+\frac{1}{n}\sum_{i=1}^{n}\int_{\Omega_{\mathbb{R}}}\left(\int_{X_{\mathbb{R}}}(\omega-\overline{\mu})(x^{(2)}-\omega)F_2(dx^{(2)})(\omega)\right)(\Phi^*\delta_{\theta_i})(d\omega).$$

The second term is equal to 0 by H_2, and the third and the fourth terms are also 0 by H_1^2. Thus, we see $\mathrm{Cov}[\otimes_{i=1}^{n}\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1\times\mathbf{O}_2, S(\Phi^*\delta_{\theta_i}))]=\Delta_E^2$, which completes the proof. □

4. Conclusions

In this paper, we proposed a measurement-theoretical understanding of psychological tests and a split-half method and found the following correspondences:

Usual test $\leftrightarrow$ Group test,
$\otimes_{i=1}^{n}\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}, S(\Phi^*\delta_{\theta_i}))$

Split-half method $\leftrightarrow$ Group simultaneous test.
$\otimes_{i=1}^{n}\mathbf{M}_{C(\Omega_{\mathbb{R}})}(\mathbf{O}_1\times\mathbf{O}_2, S(\Phi^*\delta_{\theta_i}))$

In Theorem 4 (in Subsection 3.3), we clarified the well-known theorem: "reliability coefficient" = "correlation coefficient" in terms of measurement theory.

We have to take good care of the concept of "errors" when we treat psychological tests. Though we generally classify these errors as either random or systematic, we may have some difficulties to make a strict distinction of two differences in conventional statistical approaches. By contrast, our approach characterizes tests as an observable, the ability of students as a statistical state and the score of tests as a measured value. Also, the measurement errors and the systematic ones are, respectively, characterized in (8) and in (2). Therefore, we can avoid confusing the two errors in the measurement theory. It is reasonable to consider that the psychological test is a kind of measurement. Thus, we assert that the measurement-theoretical approach has some advantage of formulating the methods of psychological tests.

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