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SOLUTIONS OF A CLASS OF VARIABLE-COEFFICIENT KORTEWEG-DE VRIES EQUATIONS

ZEID I. A. AL-MUHIAMEED

Department of Mathematics
Faculty of Science
Qassim University
Buraida, Saudi Arabia
e-mail: ksapr006@yahoo.com

Abstract

By using the variable-coefficient extended generalized hyperbolic function method, we present explicit solutions of a class of variable-coefficient Korteweg-de Vries (vcKdV) equations. The main idea of this method is to express solutions of these equations as polynomials in the solution of the Riccati equation that the generalized hyperbolic functions (GHFs) and generalized triangular functions (GTFs) satisfy.

1. Introduction

In recent decades, the study of nonlinear problems has been greatly intensified in many areas of science and technology. The investigation of exact solutions to nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear phenomena. In the past several decades, great progress has been made on the construction of exact solutions of NLEEs and many significant methods have been established such as inverse scattering method [21], Darboux transformation, Cole-Hopf transformation, Hirota method [1], Bäcklund transformation [1, 6], Painlevé method [1, 43], homogeneous balance method [16, 17, 19, 41, 42], tanh

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method [27, 34], the generalized hyperbolic function method [14, 20] and so on. Due to the availability of computer systems like Maple or Mathematica which allow us to perform some complicated and tedious algebraic calculations and differential calculations on a computer, at the same time, help us to find new exact solutions of NLEEs.

One of the most efficient methods for finding exact solutions of NLEEs is the tanh method [27, 34]. Recently, Fan [14] has proposed an extended tanh-function method. Fan et al. [15, 18], Yan [45] and Chen et al. [5, 29-32] further developed this method for a class of NLEEs. More recently, Elwakil et al. [11-13] modified the extended tanh-function method and obtained some new exact solutions. Gao and Tian [20] have presented a generalized hyperbolic function method by introducing coefficient functions. The finding of a new mathematical algorithm to construct exact solutions of NLEEs is important and might have significant impact on future research.

Recently, much attention has been paid to the vcKdV type models which are often used to describe various physical phenomena in the nature and actual physics and engineering [8, 10, 23, 25, 40]. The generalized vcKdV model with dissipative, perturbed and external-force terms can be written as

$$u_t + \mu_1(t)uu_x + \mu_2(t)u_{xxx} + \mu_3(t)u_x + \mu_4(t)u = \mu_5(t), \tag{1}$$

where the wave amplitude $u(x, t)$ is a function of the scaled 'space' x and scaled 'time' t, the real functions $\mu_1(t)$, $\mu_2(t)$, $\mu_3(t)$, $\mu_4(t)$ and $\mu_5(t)$ represent the coefficients of the nonlinear, dispersive, dissipative, perturbed and external-force terms, respectively. Many authors have studied equation (1) [8, 10, 23-25, 40].

The generalized vcKdV [3, 4, 22, 26, 35, 38, 39],

$$u_t + 2\beta(t)u + [\beta(t)x + \alpha(t)]u_x - 3c\gamma(t)uu_x + \gamma(t)u_{xxx} = 0. \tag{2}$$

The generalized variable coefficient modified Korteweg-de Vries (vcmKdV) equation reads as

$$u_t + \beta(t)u + [\beta(t)x - 4\alpha(t)]u_x + 6\gamma(t)u^2u_x - \gamma(t)u_{xxx} = 0, \tag{3}$$

which is of importance in mathematical physics field. The mKdV and cylindrical mKdV equation, etc. are special cases of equation (3) [9, 37, 44, 46, 47].

This paper is organized as follows: in the following section, we introduce the variable coefficient extended generalized hyperbolic function (vcEGHF) method to construct exact solutions for NLEEs. In Sections 3, 4 and 5, we apply this method to the vcKdV (1), the generalized vcKdV (2) and the generalized vcmKdV (3). Section 6 is a short summery discussion.

2. The Variable Coefficient Extended Generalized Hyperbolic Function Method

The main idea of this method is to express the solutions of NLEEs as a polynomials in the solution of the Riccati equation that the GHFs and GTFs satisfy. Consider a given variable-coefficient nonlinear partial differential equation

$$H(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0. \tag{4}$$

Suppose that $u(x, t)$ can be expressed by a finite power series of $F(\xi)$,

$$u(x, t) = \sum_{i=0}^{n} a_i(t) F^i(\xi(x, t)), \quad \xi \equiv \xi(x, t) = f(t)x + g(t), \tag{5}$$

where n is the highest degree of the series, which can be determined by balancing the highest derivative term with the nonlinear term(s) in equation (4) and $a_i(t)$, $f(t)$ and $g(t)$ are arbitrary functions of t to be determined. The function $F(\xi)$ satisfies the Riccati equation

$$F'(\xi) = A + BF^2(\xi), \quad ' \equiv \frac{d}{d\xi}, \tag{6}$$

where A and B are constants. Substituting (5) with (6) into (4), then the left-hand side of equation (4) can be converted into a polynomial in $F(\xi)$. Setting each coefficient of the polynomial to zero yields system of PDEs for $a_0(t), a_1(t), \ldots, a_n(t)$, $f(t)$ and $g(t)$. Solving this system, then $a_0(t), a_1(t), \ldots, a_n(t)$, $f(t)$ and $g(t)$ can be expressed by A and B. Substituting these results into (5), then a general formula solution of equation (4) can be obtained. Choose properly A and B in ODE (6) such that the corresponding solution $F(\xi)$ is one of the GHF and GTF given below. Some definitions and properties of the GHFs and GTFs are given in Appendix A.

Case 1. If $A = k$ and $B = k$, then (6) possesses a solution $\tan_{pqk}(\xi)$.

Case 2. If $A = k$ and $B = -k$, then (6) possesses a solution $\cot_{pqk}(\xi)$.

Case 3. If $A = k$ and $B = -k$, then (6) possesses solutions $\tanh_{pqk}(\xi)$, $\coth_{pqk}(\xi)$.

Case 4. If $A = \frac{k}{2}$, $B = -\frac{k}{2}$, $p = l$, $\left(\text{or } p = \frac{1}{l}\right)$ and $q = \frac{1}{l}$, $(\text{or } q = l)$, then (6) possesses a solution $\frac{\tanh_{pqk}(\xi)}{1 \pm \operatorname{sech}_{pqk}(\xi)}$.

Case 5. If $A = \frac{k}{2}$, $B = \frac{k}{2}$, $p = l$, $\left(\text{or } p = \frac{1}{l}\right)$ and $q = \frac{1}{l}$, $(\text{or } q = l)$, then (6) possesses a solution $\tan_{pqk}(\xi) \pm \sec_{pqk}(\xi)$, $\frac{\tan_{pqk}(\xi)}{1 \pm \sec_{pqk}(\xi)}$.

Case 6. If $A = k$, $B = 4k$, $p = l$, $\left(\text{or } p = \frac{1}{l}\right)$ and $q = \frac{1}{l}$, $(\text{or } q = l)$, then (6) possesses a solution $\frac{\tan_{pqk}(\xi)}{1 + \tan^2_{pqk}(\xi)}$.

Case 7. If $A = k$, $B = -4k$, $p = l$, $\left(\text{or } p = \frac{1}{l}\right)$ and $q = \frac{1}{l}$, $(\text{or } q = l)$, then (6) possesses a solution $\frac{\tanh_{pqk}(\xi)}{1 + \tanh^2_{pqk}(\xi)}$, where l is an arbitrary constant.

Now, we can apply the vcEGHF method to a class of vcKdV equations.

3. Exact Analytic Solutions of the vcKdV Equation (1)

Now, we can apply the vcEGHF method to the vcKdV equation (1), balancing the highest derivative term u_{xxx} with the nonlinear term uu_x, gives $n = 2$. Therefore, the solution of equation (1) can be expressed as

$$u(x, t) = a_0(t) + a_1(t)F(\xi) + a_2(t)F^2(\xi), \tag{7}$$

and we get

$$u_t = a_{0t} + a_1 A(f_t x + g_t) + [2a_2 A(f_t x + g_t) + a_{1t}]F(\xi)$$
$$+ [a_1 B(f_t x + g_t) + a_{2t}]F^2(\xi) + 2a_2 B(f_t x + g_t)F^3(\xi), \tag{8}$$

$$u_x = a_1 Af + 2a_2 AfF(\xi) + a_1 BfF^2(\xi) + 2a_2 BfF^3(\xi), \tag{9}$$

$$uu_x = a_0 a_1 Af + (2a_0 a_2 + a_1^2) AfF(\xi) + a_1(a_0 B + 3a_2 A) fF^2(\xi)$$
$$+ [(2a_0 a_2 + a_1^2)B + 2a_2^2 A] fF^3(\xi) + 3a_1 a_2 BfF^4(\xi) + 2a_2^2 BfF^5(\xi), \tag{10}$$

$$u_{xxx} = 2a_1 A^2 Bf^3 + 16a_2 A^2 Bf^3 F(\xi) + 8a_1 AB^2 f^3 F^2(\xi) + 40a_2 AB^2 f^3 F^3(\xi)$$
$$+ 6a_1 B^3 f^3 F^4(\xi) + 24a_2 B^3 f^3 F^5(\xi). \tag{11}$$

By substituting (7)-(11) into the vcKdV (1) yields a system of PDEs with respect to $F(\xi)$. Solving this system of equations for $a_0(t), a_1(t), ..., a_n(t)$, $f(t)$ and $g(t)$, we find that

$$f(t) = c, \quad g(t) = \int [-c\mu_1(t)a_0(t) - 8c^3\mu_2(t)AB - c^2\mu_3(t)B]dt + c_0, \quad a_1(t) = 0,$$

$$a_0(t) = \left[\int \mu_5(t)e^{\left(\int \mu_4(t)dt\right)}dt + c_1\right]e^{\left(\int \mu_4(t)dt\right)}, \quad a_2(t) = -\frac{12c^2\mu_2(t)B^2}{\mu_1(t)}, \tag{12}$$

with constraint condition

$$\mu_{1t}(t)\mu_2(t) - \mu_{2t}(t)\mu_1(t) - \mu_1(t)\mu_2(t)\mu_5(t) = 0, \tag{13}$$

where c, c_0, c_1 are constants of integration. Thus, we obtain the general formulae of the solutions of the generalized vcKdV equation (1):

$$u = \left[\int \mu_5(t)e^{\left(\int \mu_4(t)dt\right)}dt + c_1\right]e^{\left(\int \mu_4(t)dt\right)} - \frac{12c^2\mu_2(t)B^2}{\mu_1(t)}F^2(cx + g(t)), \tag{14}$$

where $g(t)$ is given in (12) and μ_i $(i = 1, 2, 5)$ satisfies the constraint condition (13). By selecting the special values of the A, B and the corresponding function $F(\xi)$, we have the following solutions of the vcKdV equation (1):

$$u_1 = a_0(t) - \frac{12c^2\mu_2(t)k^2}{\mu_1(t)}\tan^2_{pqk}(cx + g(t)), \tag{15}$$

with $g(t) = \int [-c\mu_1(t)a_0(t) - 8c^3\mu_2(t)k^2 - c^2\mu_3(t)k]dt + c_0$ and

$$u_2 = a_0(t) - \frac{12c^2\mu_2(t)k^2}{\mu_1(t)} \cot^2_{pqk}(cx + g(t)), \tag{16}$$

$$u_3 = a_0(t) - \frac{12c^2\mu_2(t)k^2}{\mu_1(t)} \tanh^2_{pqk}(cx + g(t)), \tag{17}$$

$$u_4 = a_0(t) - \frac{12c^2\mu_2(t)k^2}{\mu_1(t)} \coth^2_{pqk}(cx + g(t)), \tag{18}$$

with $g(t) = \int [-c\mu_1(t)a_0(t) + 8c^3\mu_2(t)k^2 + c^2\mu_3(t)k]dt + c_0$ and

$$u_5 = a_0(t) - \frac{3c^2\mu_2(t)k^2}{\mu_1(t)} \left[\frac{\tanh_{pqk}(cx + g(t))}{1 \pm \operatorname{sech}_{pqk}(cx + g(t))}\right]^2, \tag{19}$$

with $g(t) = \int \left[-c\mu_1(t)a_0(t) + 2c^3\mu_2(t)k^2 + \frac{c^2\mu_3(t)k}{2}\right]dt + c_0$ and

$$u_6 = a_0(t) - \frac{3c^2\mu_2(t)k^2}{\mu_1(t)} [\tan_{pqk}(cx + g(t)) \pm \sec_{pqk}(cx + g(t))]^2, \tag{20}$$

$$u_7 = a_0(t) - \frac{3c^2\mu_2(t)k^2}{\mu_1(t)} \left[\frac{\tan_{pqk}(cx + g(t))}{1 \pm \sec_{pqk}(cx + g(t))}\right]^2, \tag{21}$$

with $g(t) = \int \left[-c\mu_1(t)a_0(t) - 2c^3\mu_2(t)k^2 - \frac{c^2\mu_3(t)k}{2}\right]dt + c_0$ and

$$u_8 = a_0(t) - \frac{192c^2\mu_2(t)k^2}{\mu_1(t)} \left[\frac{\tan_{pqk}(cx + g(t))}{1 + \tan^2_{pqk}(cx + g(t))}\right]^2, \tag{22}$$

with $g(t) = \int [-c\mu_1(t)a_0(t) - 32c^3\mu_2(t)k^2 - 4c^2\mu_3(t)k]dt + c_0$ and

$$u_9 = a_0(t) - \frac{192c^2\mu_2(t)k^2}{\mu_1(t)} \left[\frac{\tanh_{pqk}(cx + g(t))}{1 + \tanh^2_{pqk}(cx + g(t))}\right]^2, \tag{23}$$

with $g(t) = \int [-c\mu_1(t)a_0(t) + 2c^3\mu_2(t)k^2 + 4c^2\nu_3(t)k]dt + c_0,\ \ p = l,\ \left(\text{or } p = \frac{1}{l}\right),$

$q = \frac{1}{l}$ (or $q = l$) and $a_0(t) = \left[\int \mu_5(t)e^{\left(\int \mu_4(t)dt\right)}dt + c_1\right]e^{\left(\int \mu_4(t)dt\right)}.$

4. Exact Analytic Solutions of the Generalized vcKdV Equation (2)

In order to obtain the exact solution of the generalized vcKdV equation (2), we first assume that the form of solution to equation (2) is the same as equation (7). By substituting (7)-(11) into the vcKdV (1) yields a system of PDEs with respect to $F(\xi)$. Solving this system of equations for $a_0(t), a_1(t), \ldots, a_n(t),$ $f(t)$ and $g(t)$, we find that

$$a_1(t) = 0, \quad a_0(t) = c_0e^{\left(-2\int \beta(t)dt\right)}, \quad a_2(t) = -\frac{4f^2(t)B^2}{c},$$

$$g(t) = \int [-\alpha(t)f(t) - 8f^3(t)\alpha(t)AB + 3c\alpha(t)a_0(t)f(t)]dt + c_1,$$

$$f(t) = c_2e^{\left(-\int \beta(t)dt\right)}, \tag{24}$$

where c_0, c_1 and c_2 are constants of integration. Thus, we obtain the general formulae of the solutions of the generalized vcKdV equation (1):

$$u = c_0e^{\left(-2\int \beta(t)dt\right)} - \frac{4f^2(t)B^2}{2}F^2(f(t)x + g(t)), \tag{25}$$

where $f(t)$ and $g(t)$ are given in (24). By selecting the special values of the A, B and the corresponding function $F(\xi)$, we have the following solutions of the generalized vcKdV equation (2):

$$u_1 = c_0e^{\left(-2\int \beta(t)dt\right)} - \frac{4f^2(t)k^2}{c}\tan^2_{pqk}(f(t)x + g(t)), \tag{26}$$

with

$$g(t) = \int [-\alpha(t)f(t) - 8f^3(t)\alpha(t)k^2 + 3c\alpha(t)a_0(t)f(t)]dt + c_1,$$

$$f(t) = c_2e^{\left(-\int \beta(t)dt\right)}$$

and

$$u_2 = c_0 e^{\left(-2\int \beta(t)\,dt\right)} - \frac{4f^2(t)k^2}{2}\tanh^2_{pqk}(f(t)x + g(t)), \tag{27}$$

with

$$g(t) = \int [-\alpha(t) f(t) + 8f^3(t)\alpha(t)k^2 + 3c\alpha(t)a_0(t) f(t)]dt + c_1,$$

$$f(t) = c_2 e^{\left(-\int \beta(t)\,dt\right)}.$$

We omitted the remaining solutions for simplicity.

5. Exact Analytic Solutions of the Generalized vcmKdV Equation (3)

In this section, we apply the EGHF method to the vcmKdV equation (1), balancing the highest derivative term u_{xxx} with the nonlinear term $u^2 u_x$, gives $n = 1$. Therefore, the solution of equation (3) can be expressed as

$$u(x, t) = a_0(t) + a_1(t)F(\xi). \tag{28}$$

Also, we get

$$u_t = a_{0t} + a_1 A(f_t x + g_t) + a_{1t}F(\xi) + a_1 B(f_t x + g_t)F^2(\xi), \tag{29}$$

$$u_x = a_1 Af + a_1 BfF^2(\xi), \tag{30}$$

$$u^2 u_x = a_0^2 a_1 Af + 2a_0 a_1^2 AfF(\xi) + a_1^2(2a_0 B + a_1 A)\,fF^2(\xi)$$

$$+ 2a_0 a_1^3 BfF^3(\xi) + a_1^3 BfF^4(\xi), \tag{31}$$

$$u_{xxx} = 2a_1 A^2 Bf^3 + 8a_1 AB^2 f^3 F^2(\xi) + 6a_1 B^3 f^3 F^4(\xi). \tag{32}$$

By substituting (28)-(31) into the generalized vcmKdV (3) yields a system of PDEs with respect to $F(\xi)$. Solving this system of equations for $a_0(t), a_1(t), ..., a_n(t)$, $f(t)$ and $g(t)$, we find that

$$a_0(t) = 0, \quad a_1(t) = \pm Bf(t), \quad f(t) = c_0 e^{\left(-\int \beta(t)\,dt\right)},$$

$$g(t) = \int [4\alpha(t) f(t) + 2f^3(t)\gamma(t)AB]dt + c_1, \tag{33}$$

where c_0 and c_1 are constants of integration. Thus, we obtain the general formulae of the solutions of the generalized vcmKdV equation (2):

$$u = \pm Bc_0 e^{\left(-\int \beta(t)\,dt\right)} F\left(c_0 e^{\left(-\int \beta(t)\,dt\right)} x + \int [4\alpha(t)\, f(t) + 2 f^3(t)\gamma(t)\, AB]\,dt + c_1\right). \quad (34)$$

By selecting the special values of the A, B and the corresponding function $F(\xi)$, we have the following solutions of the generalized vcmKdV equation (3):

$$u = \pm kc_0 e^{\left(-\int \beta(t)\,dt\right)} \times \tan_{pqk}\left(c_0 e^{\left(-\int \beta(t)\,dt\right)} x + \int [4\alpha(t)\, f(t) + 2 f^3(t)\gamma(t) k^2]\,dt + c_1\right), \quad (35)$$

$$u_2 = \mp kc_0 e^{\left(-\int \beta(t)\,dt\right)} \times \tanh_{pqk}\left(c_0 e^{\left(-\int \beta(t)\,dt\right)} x + \int [4\alpha(t)\, f(t) - 2 f^3(t)\gamma(t) k^2]\,dt + c_1\right). \quad (36)$$

We omitted the remaining solutions for simplicity.

Remark 1. The method proposed is an extension of the methods of [14, 20, 27, 34]. If setting $p = q = k = 1$ and $a_0(t), a_1(t), ..., a_n(t)$ to be real constants, the tanh method and the generalized hyperbolic function method can be recovered.

Remark 2. Because some arbitrary constants are included in the solutions (14), (25) and (34), we can obtain some special exact solutions of a class of vcKdV equations. We omitted here for simplicity.

6. Summary and Discussion

In this paper, using the variable-coefficient EGHF method, we present explicit solutions of a class of vcKdV equations. These solutions include solitary wave solution, soliton like solutions and trigonometric function solutions, among which some are found for the first time. The obtained solutions may be of important significance for the explanation of some practical physical problems. We can successfully recover the known solitary wave solutions that had been found by the tanh-function method and other methods. The EGHF method can be applied to other NLEEs.

7. Appendix A

7.1. The generalized hyperbolic functions

The generalized hyperbolic sine, the generalized hyperbolic cosine and the generalized hyperbolic tangent functions are

$$\sinh_{pqk}(\xi) = \frac{pe^{k\xi} - qe^{-k\xi}}{2}, \quad \cosh_{pqk}(\xi) = \frac{pe^{k\xi} + qe^{-k\xi}}{2},$$

$$\tanh_{pqk}(\xi) = \frac{pe^{k\xi} - qe^{-k\xi}}{pe^{k\xi} + qe^{-k\xi}}, \tag{37}$$

where ξ is an independent variable, p, q and k are arbitrary constants greater than zero. The generalized hyperbolic cotangent function is $\coth_{pqk}(\xi) = 1/\tanh_{pqk}(\xi)$, the generalized hyperbolic secant function is $\operatorname{sech}_{pqk}(\xi) = 1/\cosh_{pqk}(\xi)$, the generalized hyperbolic cosecant function is $\operatorname{csch}_{pqk}(\xi) = 1/\sinh_{pqk}(\xi)$, the above six kinds of functions are said *GHFs*. These functions satisfy the following relations [2, 28]:

$$\cosh^2_{pqk}(\xi) - \sinh^2_{pqk}(\xi) = pq, \quad \sinh_{pqk}(\xi) = \sqrt{pq}\sinh\left(k\xi - \frac{1}{2}\ln\frac{q}{p}\right). \tag{38}$$

Also, from the above definition, we give the derivative formulas of GHFs as follows:

$$(\sinh_{pqk}(\xi))' = k\cosh_{pqk}(\xi), \quad (\cosh_{pqk}(\xi))' = k\sinh_{pqk}(\xi),$$

$$(\tanh_{pqk}(\xi))' = kpq\operatorname{sech}^2_{pqk}(\xi), \quad (\coth_{pqk}(\xi))' = -kpq\operatorname{csch}^2_{pqk}(\xi). \tag{39}$$

We see that when $p = q = k = 1$ in (37), the GHF $\sinh_{pqk}(\xi)$, $\cosh_{pqk}(\xi)$ and $\tanh_{pqk}(\xi)$ degenerate as hyperbolic function $\sinh(\xi)$, $\cosh(\xi)$ and $\tanh(\xi)$, respectively. Also, when $p = k = 1$ and q is an arbitrary parameter in (37), the GHF $\sinh_{pqk}(\xi)$, $\cosh_{pqk}(\xi)$ and $\tanh_{pqk}(\xi)$ degenerate as q-deformed hyperbolic function $\sinh_q(\xi)$, $\cosh_q(\xi)$ and $\tanh_q(\xi)$, respectively. If $q = k = 1$ and p is an arbitrary parameter in (37), the GHF $\sinh_{pqk}(\xi)$, $\cosh_{pqk}(\xi)$ and $\tanh_{pqk}(\xi)$ degenerate as p-deformed hyperbolic function $\sinh_p(\xi)$, $\cosh_p(\xi)$ and $\tanh_p(\xi)$, respectively, [2, 7, 28, 33, 36].

7.2. The generalized triangular functions

The generalized triangular sine, the generalized triangular cosine and the generalized triangular tangent functions are

$$\sin_{pqk}(\xi) = \frac{pe^{ik\xi} - qe^{-ik\xi}}{2}, \quad \cos_{pqk}(\xi) = \frac{pe^{ik\xi} + qe^{-ik\xi}}{2},$$

$$\tan_{pqk}(\xi) = \frac{pe^{ik\xi} - qe^{-ik\xi}}{pe^{ik\xi} + qe^{-ik\xi}}, \tag{40}$$

where ξ is an independent variable, p, q and k are arbitrary constants greater than zero. The generalized triangular cotangent function is $\cot_{pqk}(\xi) = 1/\tan_{pqk}(\xi)$, the generalized triangular secant function is $\sec_{pqk}(\xi) = 1/\cos_{pqk}(\xi)$, the generalized triangular cosecant function is $\csc_{pqk}(\xi) = 1/\sin_{pqk}(\xi)$, the above six kinds of functions are said *GTFs*. These functions satisfy the following relations:

$$\cos^2_{pqk}(\xi) + \sin^2_{pqk}(\xi) = pq, \quad \sin_{pqk}(\xi) = \sqrt{pq}\,\sin\left(k\xi + \frac{i}{2}\ln\frac{q}{p}\right). \tag{41}$$

Also, from the above definition, we give the derivative formulas of GTFs as follows:

$$(\sin_{pqk}(\xi))' = k\cos_{pqk}(\xi), \quad (\cos_{pqk}(\xi))' = -k\sin_{pqk}(\xi),$$

$$(\tan_{pqk}(\xi))' = kpq\sec^2_{pqk}(\xi), \quad (\coth_{pqk}(\xi))' = -kpq\csc^2_{pqk}(\xi). \tag{42}$$

We see that when $p = q = k = 1$ in (40), the GTF $\sin_{pqk}(\xi)$, $\cos_{pqk}(\xi)$ and $\tan_{pqk}(\xi)$ degenerate as triangular function $\sin(\xi)$, $\cos(\xi)$ and $\tan(\xi)$, respectively. Also, when $p = k = 1$ and q is an arbitrary parameter in (40), the GTF $\sin_{pqk}(\xi)$, $\cos_{pqk}(\xi)$ and $\tan_{pqk}(\xi)$ degenerate as q-deformed triangular function $\sin_q(\xi)$, $\cos_q(\xi)$ and $\tan_q(\xi)$, respectively. If $q = k = 1$ and p is an arbitrary parameter in (40), the GTF $\sin_{pqk}(\xi)$, $\cos_{pqk}(\xi)$ and $\tan_{pqk}(\xi)$ degenerate as p-deformed triangular function $\sin_p(\xi)$, $\cos_p(\xi)$ and $\tan_p(\xi)$, respectively.

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