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# A THEORETICAL STUDY ON FLOW OF HERCHEL-BULKLEY FLUID IN A SLIGHTLY CURVED TUBE

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## Abstract

The present paper investigates the steady laminar flow of a Herchel-Bulkley (H.B.) fluid in a curved tube of circular cross-section for the case of high Dean number region by using boundary layer approximation. The flow domain has been considered to be consisting of a central plug core formation region surrounded by a viscous dominated boundary layer region near the wall. The momentum integral approach has been used which reduces the governing equations to a system of nonlinear ordinary differential equations. These equations have been solved by Runge-Kutta techniques followed by an iterative procedure.

## 1. Introduction

The study of flow characteristics of a viscous fluid in a curved tube constitutes a problem of fundamental interest in the field of internal fluid mechanics. Owing to the frequent occurrence of the curved tube geometry in industries, heat engines, heat exchanges, chemical reactors and biophysical problems, an understanding of the complex flow field and secondary flow is very essential.

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The flow of a Newtonian fluid in a curved tube has been studied extensively in the past and brief reviews have been given by Ghia and Sokhey [3] and Ito [4]. The flow of a Power – Law fluid in a curved tube was investigated by Raju and Rathna [7].

The purpose of the present study is to investigate the flow of a Herchel-Bulkley fluid through a curved tube of circular cross-section for large value of Dean number. Since for a H.B. fluid there is always a plug core formation region away from the walls, in the present analysis a central plug core formation region and viscous dominated flow region near the walls have been considered.

The modified Navier-Stokes equations governing the flow of a H.B. fluid have been reduced to a system of ordinary differential equations under boundary layer approximations. These equations have been solved numerically to obtain velocity distributions, and boundary layer thickness for various values of yield number.

## 2. Formulation of the Problem

The geometry of the problem is shown in Figure 1. We consider the steady, laminar and fully developed flow of H.B. fluid in the curved tube of circular cross-section under a constant pressure gradient along the axis of the tube.

For convenience we use the co-ordinate system introduced by Dean [1] as shown in Figure 1.

NS is the axis of the anchor ring formed by the tube wall, O is the center of the section of the tube by a plane through NS making an angle $\phi$ with the fixed axis plane NM. ON is drawn perpendicular to NS and is of length $R$. The position of any point Q at a distance $r$ from O in the cross section is specified by the orthogonal co-ordinates $(r, \theta, \phi)$, where $\theta$ is the angle made by OQ with the extended line ON. The circumference of the cross-section of the tube is given by $r = a$, where $a$ is the radius of cross-section. The element of arc length $ds$ in this co-ordinate system can be written as

$$(ds)^2 = (dr)^2 + (rd\theta)^2 + ((R + r\sin\theta)d\phi)^2. \tag{1}$$

The motion of the fluid is due to a fall in pressure $p$ along the tube. The problem has been solved for tubes of small curvature, so that the curvature ratio $L = a/R$ is small.

Under these conditions, the equations governing the flow can be written as:

Equations of continuity:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = 0. \tag{2}$$

Equations of motion:

$$\rho\left[u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} - \frac{w^2\cos\theta}{R}\right] = -\frac{\partial p}{\partial r} + \frac{\partial \tau_{rr}}{\partial r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}, \tag{3}$$

$$\rho\left[u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} + \frac{w^2\sin\theta}{R}\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r}, \tag{4}$$

$$\rho\left[u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta}\right] = -\frac{1}{R}\frac{\partial p}{\partial \phi} + \frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta}, \tag{5}$$

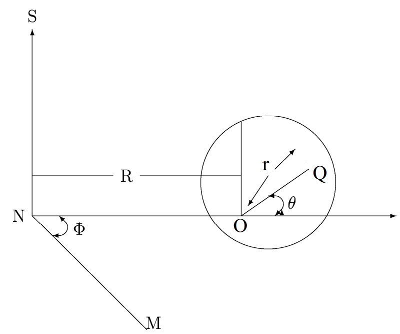


**Figure 1.** Geometric of the problem.

where ρ is the density of the fluid, $\tau_{ij}$, $i$, $j = r, \theta, \phi$ represents the stress components and for a H.B. fluid is given by the relation

$$\tau_{ij} = 2\mu(I)e_{ij}, \tag{6}$$

where $\mu(I) = [\tau_0 I^{-1} + KI^{n-1}]$ and $I = \sum(2e_{ij}e_{ij})^{1/2}$ is the second invariant of rate of strain tensor.

The flow conditions are given by

$$e_{ij} = \begin{cases} 0, & \text{if } I < \tau_y^2; \\ \dfrac{1}{2\mu(I)}\tau_{ij}, & \text{if } I \geq \tau_y^2. \end{cases} \tag{7}$$

The flow is induced by a pressure gradient $-\dfrac{\partial p}{\partial \phi} = C$ in the axial direction. The above equations have to be solved with the boundary conditions.

$u = v = w = 0$ at $r = a$, for all θ (no slip conditions at the wall),

$$\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial \theta} = \frac{\partial w}{\partial \theta} = 0 \text{ at } \theta = 0 \text{ and } \pi \text{ (symmetry condition about the central axis).} \tag{8}$$

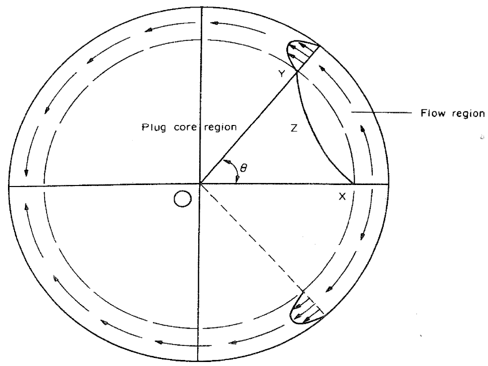


**Figure 2.** Schematic representation of the flow model.

## 3. Flow Model

Experimental measurements show that for large values of the Dean number $K = \mathrm{Re}(a/R)^{1/2}$. There is a complete change in the velocity distribution in the tube due to the secondary flow caused by the centrifugal force, the maximum axial velocity being near the outside wall. In the region near the wall of the tube, the secondary flow appears in the form of a boundary layer. This relation is shown in Figure 2. The fluid enters the boundary layer in the outer part of the wall and is driven by a pressure gradient towards the inner part where it leaves the boundary layer. Since for a H.B. fluid, there is always a plug core formation region away from the walls, we have a central plug core formation region and viscosity dominated boundary layer region near the wall.

## 4. Equations of Motion for the Plug Core Region

In the plug core region, the axial velocity distribution is more uniform than in the boundary layer and the transverse velocity components may be taken small in comparison with the axial velocity [16, 17].

Under the assumption that $u, v \ll W,$ the equations governing the flow in the plug core region are obtained from the equations (3), (4), (5) with condition (6) and are given by

$$-\frac{w^2 \cos\theta}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{9}$$

$$-\frac{w^2 \sin\theta}{R} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}, \tag{10}$$

$$u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} = \frac{C}{\rho R}. \tag{11}$$

Eliminating the pressure terms from the equations (9) and (10), we get

$$\sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta} = 0. \tag{12}$$

Introducing the transformation $x = r\cos\theta,\ \ y = r\sin\theta;$ The above equation becomes

$$\frac{\partial w}{\partial y} = 0 \quad \text{or} \quad w = F(x), \tag{13}$$

where $F$ is an arbitrary function of $x$. We define the stream function $\psi$ by

$$u = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v = -\frac{\partial \psi}{\partial r}. \tag{14}$$

The continuity equation (2) is identically satisfied and the equation (11) takes form

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial w}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial w}{\partial \theta} = \frac{C}{\rho R}. \tag{15}$$

Using the equation (13) in (15) and integrating the resulting expression, we get

$$\psi = \frac{Cy}{\rho R F'(x)} + \psi_0, \tag{16}$$

where

$$F'(x) = -\frac{1}{r\sin\theta}\frac{\partial w}{\partial \theta} \tag{17}$$

and $\psi_0$ is a constant of integration. The function $F(x),$ which depends on the flow of in and out of the boundary layer, will be determined from the continuity of the secondary flow together with the momentum equations of the boundary layer.

### 5. Boundary Layer Momentum Equations

Within the boundary layer, the axial velocity falls sharply to zero, therefore the magnitude of the transverse velocity $v$ becomes comparable to that of $w$. Applying the boundary layer approximation by taking $u$ of the order of boundary layer thickness $\delta$; $v$, $w \sim o(1)$, $\frac{\partial}{\partial r} = \sim o(\delta^{-1})$, $\frac{\partial}{\partial \psi} = \sim o(1)$ and $\frac{\partial}{\partial \phi} = \sim o(1)$ and by neglecting the variation of $r$ in thin boundary layer, we get

$$\frac{\partial u}{\partial r} + \frac{1}{a}\frac{\partial v}{\partial \theta} = 0, \tag{18}$$

$$\frac{v^2}{a} + \frac{w^2 \cos\theta}{R} = \frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{19}$$

$$u\frac{\partial v}{\partial r} + \frac{v}{a}\frac{\partial v}{\partial \theta} + \frac{w^2 \sin\theta}{R} = -\frac{1}{\rho a}\frac{\partial p}{\partial \theta} + \frac{1}{\rho}\frac{\partial \tau_{r\theta}}{\partial r}, \tag{20}$$

$$u\frac{\partial w}{\partial r} + \frac{v}{a}\frac{\partial w}{\partial \theta} = -\frac{1}{\rho R}\frac{\partial p}{\partial \phi} + \frac{1}{\rho}\frac{\partial \tau_{r\phi}}{\partial r}. \tag{21}$$

From the equation (19), we observe that the integrated value of $\frac{\partial p}{\partial r}$ across the boundary layer is of small magnitude and hence the variation of pressure across the boundary layer can be neglected.

Let $W_1(x)$ be the value of $W(x)$ at the edge of the boundary layer. Then taking into account the continuity of the pressure gradient, we have from the equation (10)

$$-\frac{\partial p}{\partial \theta} = -\frac{\rho a}{R} W_1^2 \sin\theta. \tag{22}$$

Substituting this value of $\frac{\partial p}{\partial \theta}$ in (20), we get

$$u\frac{\partial v}{\partial r} + \frac{v}{a}\frac{\partial v}{\partial \theta} = \frac{W_1^2 - W^2}{R}\sin\theta + \frac{1}{\rho}\frac{\partial \tau_{r\theta}}{\partial r}. \tag{23}$$

Balancing the constant axial pressure gradient with the shear stress at the wall, we obtain

$$C = -\frac{\partial p}{\partial \phi} = \frac{2R}{\pi a}\int_0^{\pi} \tau_{r\phi}\, d\theta. \tag{24}$$

Expressing $v$ and $w$ in terms of the mean axial velocity $v_m$, we can write

$$v \sim \varepsilon v_m, \quad w \sim v_m, \tag{25}$$

where $\varepsilon$ is an unknown parameter. With this definition, making an order of magnitude analysis of the equation (23), we can get

$$\varepsilon \sim o\left(\sqrt{\frac{a}{R}}\right). \tag{26}$$

Using the equations (25) and (26) and the constitutive relation (6), the stress components can be written as

$$\tau_{r\theta} = \left[\tau_y \left|\frac{\partial w}{\partial r}\right|^{-1} + K_H \left|\frac{\partial w}{\partial r}\right|^{n-1}\right] \frac{\partial v}{\partial r}, \tag{27}$$

$$\tau_{r\phi} = \left[\tau_y \left|\frac{\partial w}{\partial r}\right|^{-1} + K_H \left|\frac{\partial w}{\partial r}\right|^{n-1}\right] \frac{\partial w}{\partial r}. \tag{28}$$

Substituting (27) in the equation (20), we obtain

$$u\frac{\partial v}{\partial r} + \frac{v}{a}\frac{\partial v}{\partial \theta} = \frac{W_1^2 - W^2}{R}\sin\theta + \frac{1}{\rho}\frac{\partial}{\partial r}\left[\tau_y \left|\frac{\partial w}{\partial r}\right|^{-1} + K_H \left|\frac{\partial w}{\partial r}\right|^{n-1}\right] \frac{\partial v}{\partial r}. \tag{29}$$

On examining (29) in the view of (25) and (26), we obtain for the boundary layer thickness $\delta$,

$$\frac{\delta}{a} \sim \left[\mathrm{Re}\sqrt{\frac{a}{R}}\right]^{-1/2}, \text{ that is, } \frac{\delta}{a} \sim K^{1/2}, \tag{30}$$

since

$$\tau_{r\phi} \sim K_H \frac{v_m}{\delta}.$$

From (24) and (30), we get

$$C \sim \rho v_m^2 \left(\frac{a}{R}\right)^{-\frac{1}{2}} \left(\frac{\delta}{a}\right). \tag{31}$$

Using (28), the equation (21) becomes

$$u\frac{\partial w}{\partial r} + \frac{v}{a}\frac{\partial w}{\partial \theta} = \frac{1}{\rho}\frac{\partial}{\partial r}\left[\tau_y \left|\frac{\partial w}{\partial r}\right|^{-1} + K_H \left|\frac{\partial w}{\partial r}\right|^{n-1}\right] \frac{\partial w}{\partial r}. \tag{32}$$

## 6. Derivation of Boundary Conditions

Outside the boundary layer $\frac{\partial}{\partial r} \sim o\left(\frac{1}{a}\right)$. Hence by using the velocity expressions (13) and (14) for $v$ and $W$, and the relations (16) and (31) for $\psi$ and $C$, applying the condition that $\delta \leq 1$, the boundary conditions at the edge of the boundary layer $r = a - \delta$ can be written as [8, 16, 17]

$$v = 0, \quad \frac{\partial v}{\partial r} = 0, \quad W = W_1, \quad \frac{\partial w}{\partial r} = 0. \tag{33}$$

## 7. Momentum Integral Equations

Integrating the equations (29) and (32) through boundary layer and making use of the relations (18) and (33), we obtain

$$\frac{1}{a}\int_{a-\delta}^{a} \frac{\partial}{\partial \theta}(v^2)\,dr + [uv]_{a-\delta}^{a}$$

$$= \frac{\sin\theta}{R}\int_{a-\delta}^{a}(w_1^2 - w^2)\,dr + \frac{1}{\rho}\left[\tau_y\left|\frac{\partial w}{\partial r}\right|^{-1} + K_H\left|\frac{\partial w}{\partial r}\right|^{n-1}\right]\frac{\partial v}{\partial r}\Bigg|_{r=a}, \tag{34}$$

$$\frac{1}{a}\int_{a-\delta}^{a} \frac{\partial}{\partial \theta}(vw)\,dr + [uw]_{a-\delta}^{a} = \frac{1}{\rho}\left[\tau_y\left|\frac{\partial w}{\partial r}\right|^{-1} + K_H\left|\frac{\partial w}{\partial r}\right|^{n-1}\right]\frac{\partial w}{\partial r}\Bigg|_{r=a}. \tag{35}$$

Applying the no-slip condition $u = v = 0$ at the tube wall and converting the variable $\xi = a - r$, we get

$$\frac{\tau_\theta a}{\rho} = -\frac{\partial}{\partial \theta}\int_0^{\delta} v^2\,d\xi + \frac{a}{R}\sin\theta\int_0^{\delta}(w_1^2 - w^2)\,d\xi, \tag{36}$$

and

$$\frac{\tau_\phi a}{\rho} = w_1\frac{\partial}{\partial \theta}\int_0^{\delta} v\,d\xi - \frac{\partial}{\partial \theta}\int_0^{\delta} vw\,d\xi, \tag{37}$$

where

$$\tau_\theta = \left[\left(\tau_y\left|\frac{\partial w}{\partial}\right|^{-1} + K_H\left|\frac{\partial w}{\partial \xi}\right|^{n-1}\right)\frac{\partial v}{\partial \xi}\right]_{\xi=0}$$

and

$$\tau_\phi = \left[\left(\tau_y\left|\frac{\partial w}{\partial \xi}\right|^{-1} + K_H\left|\frac{\partial w}{\partial \xi}\right|^{n-1}\right)\frac{\partial w}{\partial \xi}\right]_{\xi=0}.$$

## 8. Continuity of Secondary Flow

In Figure 2, we consider $X$ and $Y$ as the points on the outer edge of the boundary layer specified by $\theta = 0$ and $\theta = \theta$, respectively. Neglecting the variation of $r$ in the thin boundary layer and using the equation (16), the flux from left to right across the curve $XZY$ drawn outside the boundary layer is given by

$$\int_X^Y d\psi = [\psi_Y] - [\psi_X] = \frac{aC\sin\theta}{\rho RF^1[x]}. \tag{38}$$

The flux through the boundary layer at $\theta$ is given by $\left[\int_{a-r}^{a} vdr\right]_0^\theta$ since $v = 0$ at $\theta = 0$, the above flux is equal to

$$\int_0^\delta vd\xi \text{ at } \theta = 0, \text{ where } \xi = a - r. \tag{39}$$

The flux expressed by (38) and (39) are equal and so, we get

$$\frac{dF}{dx} = \frac{aC\sin\theta}{\rho R\int_0^\delta vd\xi}. \tag{40}$$

Furthermore using the equation (17) for $\frac{dF}{dx}$ in (40), we obtain

$$\frac{dw_1}{d\theta} = -\frac{a^2C\sin^2\theta}{\rho R\int_0^\delta vd\xi}. \tag{41}$$

## 9. Solution of the Problem

We use Phlhausen's method of solution [17]. The boundary conditions for $v$ and $w$ reduce to

$$\text{At } \xi = 0, \quad v = 0, \quad w = 0, \quad \frac{d^2 w}{d\xi^2} = 0, \tag{42}$$

$$\text{At } \xi = 0, \quad -\frac{w_1^2 \sin\theta}{R} = \frac{1}{\rho}\frac{\partial}{\partial r}\left[\tau_y \left|\frac{\partial w}{\partial r}\right|^{-1} + K_H \left|\frac{\partial w}{\partial r}\right|^{n-1}\right]\frac{\partial v}{\partial r} \tag{43}$$

and

$$\text{At } \xi = \delta, \quad v = 0, \quad w = w_1, \quad \frac{\partial w}{\partial \xi} = 0, \quad \frac{\partial v}{\partial \xi} = 0. \tag{44}$$

The boundary conditions (42) and (44) are satisfied by the velocity expressions given by,

$$v = Ag(\eta) + \Lambda h(\eta), \tag{45}$$

$$w = w_1 K(\eta), \tag{46}$$

where

$$g(\eta) = \frac{1}{6}[2\eta - 3\eta^2 + 4\eta^4],$$

$$h(\eta) = \frac{1}{6}[\eta - 3\eta^3 + 2\eta^4] \text{ and}$$

$$k(\eta) = \frac{1}{2}[3\eta - \eta^3]. \tag{47}$$

Here $\eta = \frac{\xi}{\delta}$ and $A$ and $\Lambda$ are unknowns to be determined. Using the condition (43), $A$ can be written in terms of $w_1$ and $\delta$ as

$$A = \frac{3\rho w_1^3 2^{n-1}\delta^{n+1}\sin\theta}{R}[2^n \tau_y \delta^n + K_H w_1^n 3^n]^{-1}. \tag{48}$$

Substituting the value of $A$ in the velocity expression (45), and (47) in the equations (36), (37) and (41), evaluating the integrals, we get

$$-\frac{d}{d\theta}\left[\delta\left\{\frac{17}{11340}A^2 + \frac{19}{22680}\Lambda^2 + \frac{101}{45360}A\Lambda\right\}\right] + \frac{18\delta a}{35R}w_1^2 \sin\theta$$

$$= \frac{a}{\rho\delta}\frac{(2A + \Lambda)}{6}\left[K_H + \frac{2^n \delta^n \tau_y}{3^n w_1^n}\right], \tag{49}$$

$$w_1 \frac{d}{d\theta}\left[\delta\left(\frac{A}{30}+\frac{\Lambda}{40}\right)\right]-\frac{d}{d\theta}\left[\delta w_1\left(\frac{3A}{160}+\frac{5\Lambda}{336}\right)\right]=\frac{a}{\rho}\left[\tau_y+\frac{3^n w_1^n K_H}{2^n \delta^n}\right] \tag{50}$$

and

$$\frac{dw_1}{d\theta}=-\frac{a^2 C \sin\theta}{\rho R\delta\left[\frac{A}{30}+\frac{\Lambda}{40}\right]}, \tag{51}$$

where

$$C=\frac{2R}{\pi a}\int_0^{\pi}\left[\tau_y+\frac{3^n w_1^n K_H}{2^n \delta^n}\right]d\theta. \tag{52}$$

The unknown quantities $\delta$, $w_1$ and $\Lambda$ are to be determined by solving the above equations (49), (50) and (51). Taking into account the relation (30), the equations are reduced to the non-dimensional form

$$\delta=\frac{K_H\left(\frac{a}{R}\right)^{1/2}}{w_{10}}\delta_0,$$

$$A=w_{10}\left(\frac{a}{R}\right)^{1/2}A_0,$$

$$\Lambda=w_{10}\left(\frac{a}{R}\right)^{1/2}\Lambda_0,$$

$$w_1=w_{10}w_0, \tag{53}$$

where $w_{10}$ is the characteristic axial velocity represents the value of $w_1$ at $\theta = 0$ and from (48), we can write

$$A_0=3w_0^3 2^{n-1}\delta_0^{n+1}\sin\theta[2^n\delta_0^n H_D+w_0^n 3^n]^{-1}, \tag{54}$$

where $H_D$ is the yield number expressed by

$$H_D=\left[\frac{\tau_y a}{K_H w_{10}}\frac{\rho w_{10} a}{K_H}\left(\frac{a}{R}\right)^{1/2}\right]^{-\frac{1}{2}} \text{ or } H_D=H\left(\frac{k}{2}\right)^{-\frac{1}{2}}\left(\frac{v_m}{w_{10}}\right)^{\frac{3}{2}}. \tag{55}$$

The dimensionless form of the equations (49), (50) and (51) then becomes

$$\frac{d}{d\theta}\left[\delta_0\left\{\frac{17}{11340}A_0^2+\frac{19}{22680}\Lambda_0^2+\frac{101}{45360}A_0\Lambda_0\right\}\right]$$

$$=\frac{18}{35}\delta_0 w_0^2\sin\theta-\frac{(2A_0+\Lambda_0)}{6\delta_0}\left[1+\frac{2^n\delta_0^n H_D}{3^n w_0^n}\right], \tag{56}$$

$$w_0\frac{d}{d\theta}\left[\delta_0\left(\frac{A_0}{30}+\frac{\Lambda_0}{40}\right)\right]-\frac{d}{d\theta}\left[\delta_0 w_0\left(\frac{3A_0}{160}+\frac{5\Lambda_0}{336}\right)\right]=H_D+\frac{3^n w_0^n}{2^n\delta_0^n} \tag{57}$$

and

$$\frac{dw_0}{d\theta}=-\frac{120\alpha\sin^2\theta}{(4A_0+3\Lambda_0)\delta_0}, \tag{58}$$

where

$$\alpha=\frac{2}{\pi}\int_0^\pi\left(H_D+\frac{3^n w_0^n}{2^n\delta_0^n}\right)d\theta. \tag{59}$$

By carrying out the necessary differentiation and by substituting the derivative of $A_0$ obtained from the relation (54), the equations (56) and (57) take the form

$$A_1\frac{d\delta_0}{d\theta}+A_2\frac{dw_0}{d\theta}+A_3\frac{d\Lambda_0}{d\theta}=A_4, \tag{60}$$

$$B_1\frac{d\delta_0}{d\theta}+B_2\frac{dw_0}{d\theta}+B_3\frac{d\Lambda_0}{d\theta}=B_4, \tag{61}$$

where

$$A_1=\delta_0 c_2[136A_0+101\Lambda_0]+[68A_0^2+38\Lambda_0^2+101A_0\Lambda_0],$$

$$A_2=\delta_0 c_3[136A_0+101\Lambda_0],$$

$$A_3=\delta_0[76\Lambda_0+101A_0],$$

$$A_4=8208\delta_0 w_0^2\sin\theta-\frac{2520\Lambda_0 f}{\delta_0 w_0}-\delta_0 c_1(136A_0+101\Lambda_0),$$

$$B_1=w_0[49A_0+34\Lambda_0]+49c_2\delta_0 w_0,$$

$$B_2 = 49c_3\delta_0 w_0 + \delta_0[63A_0 + 50\Lambda_0],$$

$$B_3 = 34\delta_0 w_0, \quad B_4 = \frac{1680}{\delta_0} f - 49c_1\delta_0 w_0,$$

$$c_1 = \frac{3w_0^3 2^{n-1}\delta_0^{n+1}}{f}\cos\theta,$$

$$c_2 = \frac{3.2^{n-1}w_0^3\delta_0^n \sin\theta}{f}\left[(n+1) - 2^n \frac{H_D\delta_0^n}{f}\right],$$

$$c_3 = \frac{9.2^{n-1}w_0^2\delta_0^{n+1}\sin\theta}{f}\left[1 - 3^{n-1}\frac{nw_0^n}{f}\right] \text{ and } f = 2^n H_D\delta_0^n + 3^n w_0^n.$$

In order to start the numerical integration of the system of nonlinear ordinary differential equations (60), (61), and (62), the values of $\delta_0$, $w_0$ and $\Lambda_0$ as well as the derivatives $\frac{d\delta_0}{d\theta}$, $\frac{dw_0}{d\theta}$ and $\frac{d\Lambda_0}{d\theta}$ are required at $\theta = 0$. Following the scheme developed by Ito, in the neighborhood of $\theta = 0$, we express $\delta_0$, $w_0$ and $\Lambda_0$ in the form of infinite series given by

$$\delta_0 = \delta_1[1 + \delta_2\theta^2 + \cdots],$$

$$\Lambda_0 = \Lambda_1[1 + \Lambda_2\theta^2 + \cdots],$$

$$w_0 = 1 + w_2\theta^2 + \cdots. \tag{62}$$

Substituting these expansions into (60) and (61), and retaining the terms independent of $\theta$, we get the following algebraic equations to determine the new unknowns $\delta_1$ and $\Lambda_1$ as

$$612\delta_1^6 + 303\delta_1^4\Lambda_1 f_1 + 385\delta_1^2\Lambda_1^2 f_1^2 - 4104\delta_1^2 f_1^2 + 1260\Lambda_1 f_1^3 = 0 \tag{63}$$

and

$$34\delta_1^2\Lambda_1 f_1 + 147\delta_1^4 - 1680 f_1^2 = 0, \tag{64}$$

where $f_1 = 2^n H_D\delta_1^n + 3^n$.

These nonlinear equations have been solved by using Newton-Raphson method to determine $\delta_1$ and $\Lambda_1$. Making use of these values, the derivatives $\frac{d\delta_0}{d\theta}$, $\frac{dw_0}{d\theta}$ and $\frac{d\Lambda_0}{d\theta}$ are obtained from (62). The systems of equations (58), (60) and (61) have been solved by Range-Kutta fifth order technique following an iterative procedure. An initial value of $\alpha$ is assumed to start the integration. The numerical results $\delta_0$, $w_0$ and $\Lambda_0$ thus obtained are substituted in (59) within an error less than $10^{-7}$. The above values of $\alpha$ and the corresponding values of $\delta_0$, $w_0$ and $\Lambda_0$ have been calculated for different values of the yield number.

Variation of dimensionless shape factor $\Lambda_0$, boundary layer thickness $\delta_0$ and the variation of axial velocity $(w_0)$ at $n = 1$ and $H_D = 0$.

**Table 1**

| $\theta/(\pi/50)$ | $\Lambda_0$ | $\delta_0$ | $w_0$ |
|---|---|---|---|
| 1 | –0.22807 | 3.364289 | 0.999168 |
| 2 | –0.4179383 | 3.374019 | 0.9964439 |
| 3 | –0.6238825 | 3.392045 | 0.9919806 |
| 4 | –0.8046964 | 3.398309 | 0.9777644 |
| 5 | –0.9495977 | 3.405105 | 0.9680737 |
| 10 | –0.9918481 | 3.441314 | 0.9130184 |
| 15 | 0.3519171 | 3.501525 | 0.8117503 |
| 20 | 2.846617 | 3.559993 | 0.6829094 |
| 25 | 5.578194 | 3.584061 | 0.537304 |
| 30 | 7.340095 | 3.534242 | 0.3865756 |
| 40 | 5.962785 | 3.25291 | 0.1201479 |
| 48 | 2.07689 | 6.651795 | 0.00845406 |
| 50 | 0.3477815 | 12.68648 | 0.00541737 |

Pressure drop and volumetric flow relationship. The volumetric flow rate

$$Q = \pi a^2\left[\frac{2}{\pi}\int_0^{\pi} w_1 \sin^2\theta - \frac{3}{4\pi a}\int_0^{\pi} w_1\theta d\theta\right].$$

From the above relation the mean velocity $V_m$ $\left(V_m = \frac{Q}{\pi a^2}\right)$ is expressed in the non-dimensional form as

$$\frac{V_m}{w_{10}} = \beta - \gamma k^{-1/2}\left(\frac{V_m}{w_{10}}\right)^{1/2}, \tag{65}$$

where $\beta$ and $\gamma$ are constants and are expressed by

$$\beta = \frac{2}{\pi}\int_0^{\pi} w_0 \sin^2\theta d\theta,$$

$$\gamma = \frac{3}{\pi}2^{-3/2}\int_0^{\pi} w_0\delta_0 d\theta.$$

The values of $\beta$ and $\gamma$ have been calculated for $n = 1$ and $H_D = 0$. The equation (65) is a quadratic equation in $\left(\frac{V_m}{w_{10}}\right)^{1/2}$ and the only physically feasible solution is given by

$$\frac{V_m}{w_{10}} = \beta\left[\left(1 + \frac{\gamma^2}{4\beta K}\right)^{1/2} - \left(\frac{\gamma^2}{4\beta K}\right)^{1/2}\right]^2 \tag{66}$$

value of dimensionless mean velocity $\frac{V_m}{w_{10}}$ for different values of Dean number for $n = 1$ and $H_D = 0$.

**Table 2**

| | $\frac{V_m}{w_{10}}$ | | |
|---|---|---|---|
| $K$ | 100 | 500 | 1000 |
| | 0.407798006 | 0.471500722 | 0.48805766 |

## 10. Conclusion

The flow characteristic of a H.B. fluid through a curved tube of circular cross-section have been investigated in the high Dean number region for the dean number $K \geq 100$. Considering a central plug core formation region and viscous dominated boundary layer region near the wall, the corresponding equations for each region have been obtained from the modified Navier-Stokes equations for a H.B. fluids. The momentum integral equations together with the equations obtained by considering the continuity of the secondary flow at the edge of the boundary layer, give rise to a system of nonlinear ordinary differential equations. These equations have been solved numerically by using Runge-Kutta technique followed by iterative procedure.

The effect of H.B. number and the power - law index on the variation of the dimensionless boundary layer thickness along the angular co-ordinate has been analyzed.

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