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REFINEMENTS OF HERMITE-HADAMARD-TYPE INEQUALITIES FOR α-STAR *s*-CONVEX FUNCTIONS

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Abstract

In this article, some inequalities of Hermite-Hadamard-type for quasi-convex, convex, concave, s-convex, s-concave and α-star s-convex functions are given.

1. Preliminaries

For a convex mapping $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ defined on the interval $\mathbb{I}$ of real numbers and a, b in $\mathbb{I}$ with $a < b$, define the integral mean $\frac{1}{b-a}\int_a^b f(x)dx$ of f and the arithmetic mean $\frac{a+b}{2}$ on an interior subinterval (a, b) of $\mathbb{I}$, respectively.

The classical Hermite-Hadamard's inequality [1, 2, 5-8] asserts that:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

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A function $f : \mathbb{I} \subseteq \mathbb{R}^+ \to \mathbb{R}$ is said to be *s-convex* and α-*star s-convex* on $\mathbb{I}$ if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

and

$$f(tx + (1-t)y) \leq t^s f(x) + \{(1-t)\alpha\}^s f(y)$$

for any $x, y \in \mathbb{I}$ and $t, \alpha \in [0, 1]$, respectively. For the definitions of *s*-concave and α-star *s*-concave functions on $\mathbb{I}$, the inequalities in (2) are reversed.

For the simplicities of notations, define $R_f(a, b)$ and $L_f(a, b)$ by

$$R_f(a, b) = \frac{1}{b-a}\left[\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right]$$

and

$$L_f(a, b) = \frac{1}{b-a}\left[\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right].$$

In recent years, many authors established several inequalities connected to Hadamard's inequality. For recent results, refinements, counterparts, generalizations and new Hadamard's-type inequalities, see [1-13].

Abramovich and Pečarić [1], Pearce and Pečarić [12], Dragomir et al. [4, 5], Alomari et al. [2, 3], Dragomir and Agarwal [4] and Kirmaci et al. [8-10] obtained inequalities for differentiable convex mapping which are connected with Hermite-Hadamard's inequality, and they used the following lemma to prove them:

Lemma 1. *Let* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping defined on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *and* a, b *in* $\mathbb{I}$ *with* $a < b$. *If* $f' \in L([a, b])$, *then the following equalities hold*:

(a) $$R_f(a, b) = \frac{1}{2}\int_0^1 (1-2t)\, f'(ta + (1-t)b)\,dt, \tag{2}$$

(b) $$L_f(a, b) = \left[\int_0^{\frac{1}{2}} t\, f'(ta + (1-t)b)\,dt + \int_{\frac{1}{2}}^1 (t-1)\, f'(ta + (1-t)b)\,dt\right]. \tag{3}$$

The main inequality in [4] pointed out as follows:

Theorem 1.1. *Let* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping defined on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *and* a, b *in* $\mathbb{I}$ *with* $a < b$. *If* $f' \in L([a, b])$ *and* $|f'|$ *is convex on* $\mathbb{I}$, *then the following inequality holds*: $|R_f(a, b)| \leq \frac{1}{8}(|f'(a)| + |f'(b)|)$.

In [12], Pearce and Pečarić proved the following theorem by using the equalities (a) and (b) in Lemma 1.

Theorem 1.2. *Let* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping defined on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *and* a, b *in* $\mathbb{I}$ *with* $a < b$ *and* $p \geq 1$. *Then the following inequalities hold*:

(a) *If* $|f'|^p$ *is convex on* $\mathbb{I}$, *then* $|R_f(a, b)| \leq \frac{1}{4}\left[\frac{|f(a)|^p + |f(b)|^p}{2}\right]^{\frac{1}{p}}$.

(b) *If* $|f'|^p$ *is concave on* $\mathbb{I}$, *then* $|R_f(a, b)| \leq \frac{1}{4}\left|f'\left(\frac{a+b}{2}\right)\right|$.

A function $f : \mathbb{I} \to \mathbb{R}$ is said to be *quasi-convex* on $\mathbb{I}$ if

$$f(tx + (1-t)y) \leq \sup\{f(x), f(x)\}$$

for any $x, y \in \mathbb{I}$ and $t \in [0, 1]$.

Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex [7].

Recently, Ion [7] introduced two inequalities of the right hand side of Hadamard's type for quasi-convex functions, as follows:

Theorem 1.3. *Let* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping defined on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *and* a, b *in* $\mathbb{I}$ *with* $a < b$ *and* $p \geq 1$. *Then the following inequalities hold*:

(a) $|R_f(a, b)| \leq \frac{1}{4}\sup\{|f'(a)|, |f'(b)|\}$ *if* $|f'|$ *is quasi-convex on* $\mathbb{I}$.

(b) $|R_f(a, b)| \leq \frac{1}{2}\left(\frac{p-1}{2p-2}\right)^{\frac{p-1}{p}} \sup\{|f(a)|^p, |f(b)|^p\}^{1/p}$ *if* $|f'|^p$ *is quasi-convex* $\mathbb{I}$.

By the equality (b) in Lemma 1, Alomari et al. [2, 3] and Kirmaci and Özdemir [8, 10] established refinements inequalities of the right hand side of Hadamard's type for quasi-convex functions:

Theorem 1.4 [8-11]. *Let $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping defined on the interior $\mathbb{I}^0$ of $\mathbb{I}$ and a, b in $\mathbb{I}$ with $a < b$. Then the following inequalities hold*:

(a) *If $|f'|$ is quasi-convex on $\mathbb{I}$, then*

$$|R_f(a, b)| \leq \frac{1}{8}\left[\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(a)|\right\} + \sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(b)|\right\}\right].$$

(b) *If $|f'|^{p/(p-1)}$ is quasi-convex on $\mathbb{I}$ for $p > 1$, then*

$$|R_f(a, b)| \leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p, |f'(a)|^p\right\}\right)^{\frac{1}{p}}\right.$$
$$\left. + \left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p, |f'(b)|^p\right\}\right)^{\frac{1}{p}}\right].$$

Theorem 1.5 [2]. *Let $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping defined on the interior $\mathbb{I}^0$ of $\mathbb{I}$ and a, b in $\mathbb{I}$ with $a < b$. If $|f'|^p$ is quasi-convex on $\mathbb{I}$ for $p \geq 1$, then*

$$|R_f(a, b)|$$
$$\leq \frac{1}{8}\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p, |f'(a)|^p\right\}\right)^{\frac{1}{p}} + \left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p, |f'(b)|^p\right\}\right)^{\frac{1}{p}}\right].$$

Theorem 1.6 [6]. *Suppose that $f : \mathbb{I} \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping defined on the interior $\mathbb{I}^0$ of $\mathbb{I}$, and a, b in $\mathbb{I}$ with $a < b$, $s \in (0, 1)$, and $f \in L([a, b])$.*

(a) *If f is a convex mapping on $\mathbb{I}$, then*

$$\left|2^{s-1} f\left(\frac{a+b}{2}\right)\right| \leq \left|\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{2}{s+1}\left|\frac{f(a)+f(b)}{2}\right|. \tag{4}$$

(b) *If f is a concave mapping on* $\mathbb{I}$, *then*

$$\frac{2}{s+1}\left|\frac{f(a)+f(b)}{2}\right| \leq \left|\frac{1}{b-a}\int_a^b f(x)dx\right| \leq 2^{s-1}\left|f\left(\frac{a+b}{2}\right)\right|. \tag{5}$$

2. Hermite-Hadamard's Inequalities for α-star s-convex Functions

In this article, we will use the following new equalities not used in other articles:

Lemma 2. *Let* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping defined on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *and* a, b *in* $\mathbb{I}$ *with* $a < b$. *If* $f' \in L([a, b])$, *then the following equalities hold*:

(a)

$$R_f(a, b) = \frac{1}{4}\left[\int_0^1 (-t) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt + \int_0^1 tf'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)dt\right], \tag{6}$$

$$= \frac{1}{2}\left[\int_0^1 (1-t) f'(ta + (1-t)b)dt + \int_0^1 (t-1) f'(tb + (1-t)a)dt\right], \tag{7}$$

(b)

$$L_f(a, b) = \frac{1}{4}\left[\int_0^1 (1-t) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt\right.$$

$$\left. + \int_0^1 (t-1) f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)dt\right], \tag{8}$$

$$= -\left[\int_0^{\frac{1}{2}} tf'(tb + (1-t)a)dt + \int_{\frac{1}{2}}^1 tf'(ta + (1-t)b)dt\right]. \tag{9}$$

In the following theorem, we shall propose some new upper and lower bound for the left-hand and right-hand sides of Hermite-Hadamard's inequality for quasi-convex, convex, and concave mapping, which is better than the inequality had done in other articles.

Theorem 2.1. *Let* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *such that* $f' \in L([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$ *and* $s \in (0, 1]$.

(a) *If* $|f'|$ *is an s-convex mapping on* $\mathbb{I} = [a, b]$, *then*

(i)

$$|R_f(a, b)| \leq \frac{1}{4}\left(\frac{2s + 2^{1-s}}{(s+1)(s+2)}\right)(|f'(a)| + |f'(b)|) \leq \frac{1}{2}(|f'(a)| + |f'(b)|)$$

or

$$|R_f(a, b)| \leq \frac{1}{2}\frac{1}{s+1}[|f'(a)| + |f'(b)|] \leq \frac{1}{4}[|f'(a)| + |f'(b)|].$$

(ii)

$$|L_f(a, b)| \leq \frac{1}{2}\left(\frac{2 - 2^{-s}}{(s+1)(s+2)}\right)(|f'(a)| + |f'(b)|) \leq \frac{1}{4}(|f'(a)| + |f'(b)|).$$

(b) *If* $|f'|^p$ *is an s-convex mapping on* $\mathbb{I} = [a, b]$ *for* $p > 1$, *then*

(i)

$$|R_f(a, b)| \leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{p}}\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'(a)|^p\right\}^{\frac{1}{p}} + \left\{|f'(b)|^p + \left|f'\left(\frac{a+b}{2}\right)\right|^p\right\}^{\frac{1}{p}}\right]$$

or

$$|R_f(a, b)| \leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} 2^{\frac{s-1}{p}}\left[\left|f'\left(\frac{a+3b}{4}\right)\right|^{\frac{1}{p}} + \left|f'\left(\frac{3a+b}{4}\right)\right|^{\frac{1}{p}}\right].$$

(ii)

$$|L_f(a, b)| \leq \frac{1}{4}\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'(a)|^p\right\}^{\frac{1}{p}} + \left\{|f'(b)|^p + \left|f'\left(\frac{a+b}{2}\right)\right|^p\right\}^{\frac{1}{p}}\right].$$

Proof. (a) By (6), (7) and (8) in Lemma 2, we have

(i)

$$
\begin{aligned}
&| R_f(a, b) | \\
&\le \frac{1}{4}\left[| f'(a)| \int_0^1 t\left(\frac{1+t}{2}\right)^s dt + | f'(b)| \int_0^1 t\left(\frac{1-t}{2}\right)^s dt \right. \\
&\left. + | f'(a)| \int_0^1 t\left(\frac{1-t}{2}\right)^s dt + | f'(b)| \int_0^1 t\left(\frac{1+t}{2}\right)^s dt \right] \\
&\le \frac{1}{4}\left(\frac{2s + 2^{1-s}}{(s+1)(s+2)}\right)(| f'(a)| + | f'(b)|) \le \frac{1}{4}(| f'(a)| + | f'(b)|)
\end{aligned}
$$

or

$$
\begin{aligned}
&| R_f(a, b) | \\
&\le \frac{1}{2}\left[\int_0^1 |1-t||\, f'(ta + (1-t)b)| dt + \int_0^1 |t-1||\, f'(tb + (1-t)a)| dt\right] \\
&= \frac{1}{2}\frac{1}{s+1}[| f'(a)| + | f'(b)|] \le \frac{1}{4}(| f'(a)| + | f'(b)|)
\end{aligned}
$$

by the fact that $\frac{1}{2} < \frac{2s + 2^{1-s}}{(s+1)(s+2)} \le 1$ for $0 < s \le 1$.

(ii)

$$
\begin{aligned}
&| L_f(a, b) | \\
&\le \frac{1}{4}\left[\left\{| f'(a)| \int_0^1 (1-t)\left(\frac{1+t}{2}\right)^s dt + | f'(b)| \int_0^1 (1-t)\left(\frac{1-t}{2}\right)^s dt\right\} \right. \\
&\left. + \left\{| f'(a)| \int_0^1 (1-t)\left(\frac{1-t}{2}\right)^s dt + | f'(b)| \int_0^1 (1-t)\left(\frac{1+t}{2}\right)^s dt\right\}\right] \\
&\le \frac{1}{4}(| f'(a)| + | f'(b)|).
\end{aligned}
$$

(b) By (6) and (8) in Lemma 2, we have

(i)

$$|R_f(a, b)|$$

$$\leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left\{\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'(a)|^p}{s+1}\right\}^{\frac{1}{p}} + \left\{\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'(b)|^p}{s+1}\right\}^{\frac{1}{p}}\right]$$

$$\leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{p}}\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'(a)|^p\right\}^{\frac{1}{p}}\right.$$

$$\left. + \left\{|f'(b)|^p + \left|f'\left(\frac{a+b}{2}\right)\right|^p\right\}^{\frac{1}{p}}\right]$$

or

$$|R_f(a, b)|$$

$$\leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left\{\frac{2}{b-a}\int_a^{\frac{a+b}{2}} |(f')^p| dt\right\}^{\frac{1}{p}} + \left\{\frac{2}{b-a}\int_{\frac{a+b}{2}}^b |(f')^p| dt\right\}^{\frac{1}{p}}\right]$$

$$\leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left(2^{s-1}\left|f'\left(\frac{a+3b}{4}\right)\right|\right)^{\frac{1}{p}} + \left(2^{s-1}\left|f'\left(\frac{3a+b}{4}\right)\right|\right)^{\frac{1}{p}}\right].$$

(ii)

$$|L_f(a, b)|$$

$$= \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left\{\int_a^{\frac{a+b}{2}} |f'(t)|^p dt\right\}^{\frac{1}{p}} + \left\{\int_{\frac{a+b}{2}}^b |f'(t)|^p dt\right\}^{\frac{1}{p}}\right]$$

$$\leq \frac{1}{4}\left[\left\{\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'(a)|^p}{s+1}\right\}^{\frac{1}{p}} + \left\{\frac{|f'(b)|^p + \left|f'\left(\frac{a+b}{2}\right)\right|^p}{s+1}\right\}^{\frac{1}{p}}\right]$$

$$\leq \frac{1}{4}\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'(a)|^p\right\}^{\frac{1}{p}} + \left\{|f'(b)|^p + \left|f'\left(\frac{a+b}{2}\right)\right|^p\right\}^{\frac{1}{p}}\right]$$

by the fact that $\frac{1}{2} \leq \left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \leq 1$ and $\left(\frac{1}{s+1}\right)^{\frac{1}{p}} \leq 1$ for $0 < s \leq 1$ and $p > 1$ and by Theorem 1.6(a).

Theorem 2.2. *Let* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *such that* $f' \in L([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$, $s \in (0, 1]$ *and* $p \geq 1$. *If* $|f'|^p$ *is an s-concave mapping on* $\mathbb{I} = [a, b]$, *then the following inequalities hold*:

(i)

$$|R_f(a, b)| \leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left(\frac{1}{2}\right)^{\frac{1-s}{p}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right|\right]$$

or

$$|R_f(a, b)| \leq \left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} 2^{\frac{s-1}{p}}\left[\left|f'\left(\frac{a+b}{2}\right)\right|\right].$$

(ii)

$$|L_f(a, b)| \leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left(\frac{1}{2}\right)^{\frac{1-s}{p}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right|\right].$$

Proof. By Theorem 1.6(b), we have the following inequalities:

$$\int_0^1 \left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^p dt$$

$$\leq 2^{s-1}\left|f'\left(\frac{\frac{a+b}{2}+a}{2}\right)\right|^p = 2^{s-1}\left|f'\left(\frac{3a+b}{4}\right)\right|^p,$$

$$\int_0^1 \left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|^p dt$$

$$\leq 2^{s-1}\left| f'\left(\frac{b+\frac{a+b}{2}}{2}\right)\right|^p = 2^{s-1}\left| f'\left(A\left(\frac{a+3b}{4}\right)\right)\right|^p.$$

By using (6), (7) and (8) in Lemma 2 and the fact that $\frac{1}{2} \leq \left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \leq 1$ and $\frac{1}{2} \leq \left(\frac{1}{2}\right)^{\frac{1-s}{p}} \leq 1,$ we have

(i)

$$| R_f(a, b) |$$

$$\leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left(\int_0^1 \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^p dt\right)^{\frac{1}{p}}\right.$$

$$\left. + \left(\int_0^1 \left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|^p dt\right)^{\frac{1}{p}}\right]$$

$$\leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left(\frac{1}{2}\right)^{\frac{1-s}{p}}\left[\left| f'\left(\frac{3a+b}{4}\right)\right| + \left| f'\left(\frac{a+3b}{4}\right)\right|\right]$$

or

$$| R_f(a, b) |$$

$$\leq \frac{1}{2}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left(2(2^{s-1})\left| f'\left(\frac{a+b}{2}\right)\right|^p\right)^{\frac{1}{p}}\right]$$

$$\leq \left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} 2^{\frac{s-1}{p}}\left[\left| f'\left(\frac{a+b}{2}\right)\right|\right]$$

by the fact that

$$\int_0^1 | f'(ta + (1-t)b)|^p \, dt \le 2^{s-1} \left| f'\left(\frac{a+b}{2}\right)\right|^p,$$

$$\int_0^1 | f'(tb + (1-t)a)|^p \, dt \le 2^{s-1} \left| f'\left(\frac{a+b}{2}\right)\right|^p$$

by Theorem 1.2(b).

(ii)

$$\begin{aligned} &| L_f(a, b)| \\ &\le \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \left[\left(\int_0^1 \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^p dt\right)^{\frac{1}{p}}\right. \\ &\left. + \left(\int_0^1 \left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|^p dt\right)^{\frac{1}{p}}\right] \\ &\le \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \left[\left(2^{s-1}\left| f'\left(\frac{3a+b}{4}\right)\right|^p\right)^{\frac{1}{p}} + \left(2^{s-1}\left| f'\left(\frac{a+3b}{4}\right)\right|^p\right)^{\frac{1}{p}}\right] \\ &\le \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \left(\frac{1}{2}\right)^{\frac{1-s}{p}} \left[\left| f'\left(\frac{3a+b}{4}\right)\right| + \left| f'\left(\frac{a+3b}{4}\right)\right|\right]. \end{aligned}$$

Theorem 2.3. *Let* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *such that* $f' \in L([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$ *and* $\alpha, s \in (0, 1]$.

(a) *If* $| f' |$ *is an* α*-star s-convex mapping on* $\mathbb{I} = [a, b]$, *then*

(i)

$$| R_f(a, b)| \le \frac{1}{4}\left\{\frac{1 + 2^{s+1}s + \alpha^s}{2^{-s}(s+1)(s+2)}\right\}(| f'(a)| + | f'(b)|)$$

or

$$| R_f(a, b)| \le \frac{1}{2}\,\frac{\alpha^s(s+1)+1}{(s+1)(s+2)}[| f'(a)| + | f'(b)|].$$

(ii)

$$| L_f(a, b) | \leq \frac{1}{8}\left\{\frac{2^{s+2} - s + 3 + (s+1)\alpha^s}{(s+1)(s+2)2^s}\right\}(| f'(a) | + | f'(b) |).$$

(b) *If* $| f' |^p$ *is an* α*-star s-convex mapping on* $\mathbb{I} = [a, b]$ *for* $p > 1$, *then*

(i)

$$| R_f(a, b) | \leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left\{\left(\frac{2-2^{-s}}{s+1}\right)^{\frac{1}{p}} + \left(\frac{2^{-s}}{s+1}\right)^{\frac{1}{p}}\right\}(| f'(a) | + | f'(b) |).$$

(ii)

$$| L_f(a, b) | \leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left\{\left(\frac{2-2^{-s}}{s+1}\right)^{\frac{1}{p}} + \left(\frac{2^{-s}}{s+1}\right)^{\frac{1}{p}}\right\}(| f'(a) | + | f'(b) |).$$

Proof. By (6), (7) and (8) in Lemma 2, we have

(a)(i)

$$| R_f(a, b) | \leq \frac{1}{4}\left\{\int_0^1 t\left(\left|\frac{1+t}{2}\right|\right)^s dt + \int_0^1 t\left(\alpha\left|\frac{1-t}{2}\right|\right)^s dt\right\}(| f'(a) | + | f'(b) |)$$

$$\leq \frac{1}{4}\left(\frac{1 + 2^{s+1}s + \alpha^s}{2^{-s}(s+1)(s+2)}\right)(| f'(a) | + | f'(b) |)$$

or

$$| R_f(a, b) | \leq \frac{1}{2}\left[\int_0^1 (1-t)\{(t^s | f'(a) | + ((1-t)\alpha)^s | f'(b) |)\} dt\right.$$

$$\left. + \int_0^1 (1-t)\{(t^s | f'(b) | + ((1-t)\alpha)^s | f'(a) |)\} dt\right]$$

$$\leq \frac{1}{2}\frac{\alpha^s(s+1)+1}{(s+1)(s+2)}[| f'(a) | + | f'(b) |].$$

(ii)

$$| L_f(a, b) | \le \frac{1}{4}\left[| f'(a) |\left\{\int_0^1 (1-t)\left\{\left(\frac{1+t}{2}\right)^s dt\right\}\right\} + \int_0^1 (1-t)\left\{\left(\alpha\frac{1-t}{2}\right)^s dt\right\}\right.$$

$$\left. + | f'(b) |\left\{\int_0^1 (1-t)\left\{\left(\alpha\frac{1+t}{2}\right)^s dt\right\}\right\} + \int_0^1 (1-t)\left\{\left(\frac{1+t}{2}\right)^s dt\right\}\right]$$

$$= \frac{1}{8}\left\{\frac{2^{s+2} - s + 3 + (s+1)\alpha^s}{(s+1)(s+2)2^s}\right\}(| f'(a) | + | f'(b) |).$$

(b)(i)

$$| R_f(a, b) | \le \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left\{| f'(a) |^p\left(\frac{2-2^{-s}}{s+1}\right) + | f'(b) |^p \alpha^s\left(\frac{2^{-s}}{s+1}\right)\right\}^{\frac{1}{p}}\right.$$

$$\left. + \left\{| f'(a) |^p \alpha^s\left(\frac{2^{-s}}{s+1}\right) + | f'(b) |^p\left(\frac{2-2^{-s}}{s+1}\right)\right\}^{\frac{1}{p}}\right]$$

$$\le \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left\{\left(\frac{2-2^{-s}}{s+1}\right)^{\frac{1}{p}} + \left(\frac{2^{-s}}{s+1}\right)^{\frac{1}{p}}\right\}(| f'(a) | + | f'(b) |)$$

or

$$| R_f(a, b) | \le \frac{1}{2}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left(\int_0^1 \{(t^s | f'(a) |^p + ((1-t)\alpha)^s | f'(b) |^p)\} dt\right)^{\frac{1}{p}}\right.$$

$$\left. + \left(\int_0^1 \{(t^s | f'(b) |^p + ((1-t)\alpha)^s | f'(a) |^p)\} dt\right)^{\frac{1}{p}}\right]$$

$$\le \frac{1}{2}\left(\frac{1}{s+1}\right)^{\frac{1}{p}}[(| f'(a) |^p + \alpha^s | f'(b) |^p)^{\frac{1}{p}} + (| f'(b) |^p + \alpha^s | f'(a) |^p)^{\frac{1}{p}}].$$

(ii)

$$
\begin{aligned}
&| L_f(a, b) | \\
&\leq \frac{1}{4}\left[\int_0^1 (1-t)\left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right| dt + \int_0^1 (1-t)\left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right| dt\right] \\
&\leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left(\int_0^1 \left| f'\left(\left|\frac{1+t}{2}a + \frac{1-t}{2}b\right|\right)\right|^p dt\right)^{\frac{1}{p}}\right. \\
&\quad \left. + \left(\int_0^1 \left| f'\left(\left|\frac{1+t}{2}b + \frac{1-t}{2}a\right|\right)\right|^p dt\right)^{\frac{1}{p}}\right] \\
&\leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left\{| f'(a) |^p \int_0^1 \left(\left|\frac{1+t}{2}\right|\right)^s dt + | f'(b) |^p \int_0^1 \left(\alpha\left|\frac{1-t}{2}\right|\right)^s dt\right\}^{\frac{1}{p}}\right. \\
&\quad \left. + \left\{| f'(a) |^p \int_0^1 \left(\alpha\left|\frac{1-t}{2}\right|\right)^s dt + | f'(b) |^p \int_0^1 \left(\left|\frac{1+t}{2}\right|\right)^s dt\right\}^{\frac{1}{p}}\right] \\
&\leq \frac{1}{2}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}(| f'(a) | + | f'(b) |) \leq \frac{1}{2}(| f'(a) | + | f'(b) |).
\end{aligned}
$$

By using Theorem 1.6, we have the following inequalities:

Theorem 2.4. *Let* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *such that* $f' \in L([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$ *and* $\alpha, s \in (0, 1]$. *If* $| f' |^p$ *is an* α*-star s-convex mapping on* $\mathbb{I} = [a, b]$, *then*

(i)

$$
\begin{aligned}
| R_f(a, b) | \leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{p}}&\left[\left\{\left| f'\left(\frac{a+b}{2}\right)\right|^p + | f'(a) |^p\right\}^{\frac{1}{p}}\right. \\
&\left. + \left\{| f'(b) |^p + \left| f'\left(\frac{a+b}{2}\right)\right|^p\right\}^{\frac{1}{p}}\right]
\end{aligned}
$$

or

$$|R_f(a, b)| \leq \left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{p}}\{|f'(a)|^p + |f'(b)|^p\}^{\frac{1}{p}}.$$

(ii)

$$|L_f(a, b)| \leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{p}}\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'(a)|^p\right\}^{\frac{1}{p}}\right.$$

$$\left. + \left\{|f'(b)|^p + \left|f'\left(\frac{a+b}{2}\right)\right|^p\right\}^{\frac{1}{p}}\right].$$

Proof. By (6), (7) and (8) in Lemma 2, we have

(i)

$|R_f(a, b)|$

$$\leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left[\left\{\frac{2}{b-a}\int_a^{\frac{a+b}{2}} |(f')^p| dt\right\}^{\frac{1}{p}} + \left\{\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} |(f')^p| dt\right\}^{\frac{1}{p}}\right]$$

$$\leq \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{p}}\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'(a)|^p\right\}^{\frac{1}{p}}\right.$$

$$\left. + \left\{|f'(b)|^p + \left|f'\left(\frac{a+b}{2}\right)\right|^p\right\}^{\frac{1}{p}}\right]$$

$$\leq \frac{1}{4}\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'(a)|^p\right\}^{\frac{1}{p}} + \left\{|f'(b)|^p + \left|f'\left(\frac{a+b}{2}\right)\right|^p\right\}^{\frac{1}{p}}\right]$$

or

$$|R_f(a,b)| \le \frac{1}{2}\left[\left(\int_0^1 (1-t)^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \left(\int_0^1 |f'(ta+(1-t)b)|^p dt\right)^{\frac{1}{p}}\right.$$

$$\left. + \left(\int_0^1 (1-t)^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \left(\int_0^1 |f'(tb+(1-t)a)|^p dt\right)^{\frac{1}{p}}\right]$$

$$\le \left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \left(\frac{1}{s+1}\right)^{\frac{1}{p}} [|f'(a)|^p + |f'(b)|^p]^{\frac{1}{p}}.$$

(ii)

$$|L_f(a,b)|$$

$$\le \frac{1}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \left[\left\{\frac{2}{b-a}\int_a^{\frac{a+b}{2}} |(f')^p| dt\right\}^{\frac{1}{p}} + \left\{\frac{2}{b-a}\int_{\frac{a+b}{2}}^b |(f')^p| dt\right\}^{\frac{1}{p}}\right]$$

$$\le \frac{1}{4}\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'(a)|^p\right\}^{\frac{1}{p}} + \left\{|f'(b)|^p + \left|f'\left(\frac{a+b}{2}\right)\right|^p\right\}^{\frac{1}{p}}\right].$$

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