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ON THE HOLOMORPHIC MAPS OF $\mathbf{C}^2$ TO $\mathbf{C}^2$ WHICH PRESERVES A GENERAL TYPE POLYNOMIAL OF TWO VARIABLES

YUKINOBU ADACHI

2-12-29 Kurakuen
Nishinomiya, Hyogo 662-0082, Japan
e-mail: fwjh5864@nifty.com

Abstract

Every holomorphic map of $\mathbf{C}^2$ to $\mathbf{C}^2$ which preserves general two level curves of a general type polynomial preserves its polynomial and such holomorphic maps make a finite subgroup of algebraic automorphisms of $\mathbf{C}^2$.

0. Introduction

It is well known that the rational function of a complex variable with same three values distributions is uniquely determined. Therefore, if $P(x)$ and $Q(x)$ are polynomials and $P(Q(x))$ is fixed two values distributions, $Q(x)$ is the identity. We study it in the two dimensional case.

Let $P(x, y)$ be a primitive general type polynomial (see Definitions 1.1 and 1.4 for its definition). Let A_i $(i = 1, 2)$ be a general level curve of $P(x, y)$ of (g, n) type such as $\{P = \alpha_i\}$, where α_i and α_2 are distinct complex numbers. If F is a holomorphic map of $\mathbf{C}^2$ to $\mathbf{C}^2$ such as $\{P \circ F = \alpha_i\} = A_i$ for $i = 1, 2,$ then $P \circ F$

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$= P$ and $F \in Aut_{alg}(\mathbf{C}^2)$. And such holomorphic maps F make a finite subgroup of $Aut_{alg}(\mathbf{C}^2)$ such as $\#\{F\} \leqq 12(g-1)+6n$ (Theorem 2.6).

The condition that $P(x, y)$ is a primitive polynomial is not essential (see Theorem 3.1) but the condition that $P(x, y)$ is a general type polynomial is essential (see Theorem 3.2).

1. Preliminaries

Let $P(x, y)$ be a nonconstant polynomial and S be an irreducible component of a level curve of $P(x, y)$. If the normalization of S is holomorphically isomorphic to a finite Riemann surface of (g, n) type, that is, its genus $g < \infty$ and its boundary consists of n punctured points $(n < \infty)$, then we call S is an *algebraic curve* of (g, n) type.

It is well known that every irreducible components of every level curve of $P(x, y)$ are same type except for a finite number of them. So, if general irreducible components of level curves of $P(x, y)$ are of (g, n) type, then we call that $P(x, y)$ is a *polynomial* of (g, n) type.

Definition 1.1. If except for a finite number of level curves of $P(x, y)$ every level curve is irreducible, then we call that $P(x, y)$ is a *primitive polynomial*.

Proposition 1.2 (cf. [3, Proposition 1])**.** *For every nonconstant polynomial* $P(x, y)$, *there are a primitive polynomial* $P_0(x, y)$ *and a polynomial of one complex variable* π *such that* $P = \pi \circ P_0$.

Following proposition is well known owing to M. Suzuki and T. Yoshioka.

Proposition 1.3. *Let* $P(x, y)$ *be a primitive polynomial of* (g, n) *type*. *Then except for a finite number of level curves every level curve is irreducible*, *nonsingular and of* (g, n) *type*. *If an irreducible component of exceptional level curves is of* (g', n') *type*, *then* $g' \leq g$ *and* $g' + n' \leq g + n$.

Definition 1.4. When $P(x, y)$ is a polynomial of (g, n) type, we call that $P(x, y)$ is *general type* if $2g - 2 + n > 0$ and we call that it is *exceptional type* if else, that is, $g = 0$ and $n = 1$ or $n = 2$.

Theorem 1.5 (Oikawa [4]). *If R is a finite open Riemann surface of* (g, n) *type, where* $2g - 2 + n > 0$, *then* $\#\{Aut(R)\} \leqq 12(g-1) + 6n = l$, *where* $Aut(R)$ *is the automorphisms of R.*

Proposition 1.6. *Let R be the same as of Theorem* 1.5. *Then every nonconstant self holomorphic map* φ *of R is an automorphism of R.*

Proof. Let $\widetilde{R}$ be a compactification of *R*. Since *R* is hyperbolically imbedded in $\widetilde{R}$, φ is extended holomorphically to $\widetilde{\varphi} : \widetilde{R} \to \widetilde{R}$. The image $\widetilde{\varphi}(\widetilde{R})$ is a compact and open set because φ is a nonconstant holomorphic map. Therefore, $\widetilde{\varphi}(\widetilde{R}) = \widetilde{R}$. Set $E = \widetilde{R} - R$. Since $\widetilde{\varphi}^{-1}(p) \in E$ for a point $p \in E$, $\widetilde{\varphi}$ permutes the points of *E*. So $\varphi : R \to R$ is an onto map and it is regarded as the covering surface of *R* without relative boundaries. Let *m* be a degree of φ. From the Hurwitz formula $2 - n - 2g \leqq m(2 - n - 2g)$. Then we conclude $m = 1$ because $2 - n - 2g < 0$.

2. Main Results

We denote that $F \in (E)$ if *F* is a nondegenerate holomorphic map of $\mathbf{C}^2$ to $\mathbf{C}^2$ and $F \in (PA)$ if $F \in (E)$ and *F* is a polynomial map. And we denote the automorphism group of $\mathbf{C}^2$ by $Aut(\mathbf{C}^2)$ and the algebraic one by $Aut_{alg}(\mathbf{C}^2)$.

Following proposition is easy to see from Theorem 4.9 in [1].

Proposition 2.1. *Let* $P(x, y)$ *be a primitive general type polynomial of* (g, n) *type and* A_i $(i = 1, 2)$ *be a general level curve of* $P(x, y)$ *such as* $\{P = \alpha_i\}$ *with distinct* α_1, α_2. *If a map* $F \in (E)$ *satisfies* $\{P \circ F = \alpha_i\} = A_i$ *for* $i = 1, 2$, *then* $F \in (PA)$.

Proposition 2.2. *Let F be the same as of Proposition* 2.1. *Then* $P \circ F = P$.

Proof. Let *L* be an arbitrary complex line which is not contained in a level curve of *P*. As $P \circ F|_L$ is a polynomial on *L* and value distributions of α_1 and α_2 are the same of P_L, where $P \circ F|_L = P_L$. Since almost *L* is not contained in a level of *P*, $P \circ F = P$.

Lemma 2.3. *Let X and Y be Hausdorff spaces. A set* $\mathcal{F}$ *is included in* $C(X, Y)$, *where* $C(X, Y)$ *is a space of the continuous maps from X onto Y. Assume that* $N = \sup_{x \in X} \#\{f(x);\ f \in \mathcal{F}\} < \infty$. *Then there is a domain U in X and* $f_1, \ldots, f_N \in \mathcal{F}$ *such that for every* $f \in \mathcal{F}$, $f|_U = f_i|_U$, *where* $i \in \{1, \ldots, N\}$.

Proof. From the assumption there is a point $x_0 \in X$ and $f_1, \ldots, f_N \in \mathcal{F}$ such that $\{f(x_0);\ f \in \mathcal{F}\} = \{f_1(x_0), \ldots, f_N(x_0)\}$, where $f_1(x_0), \ldots, f_N(x_0)$ are mutually different points of *Y*. Let *U* be a sufficiently small neighborhood of x_0 such that for every $x \in U$, $\{f(x);\ f \in \mathcal{F}\} \subset \{f_1(U), \ldots, f_N(U)\}$ and $f_i(U)$ $(i = 1, \ldots, N)$ are mutually disjoint. If we fix $f \in \mathcal{F}$ arbitrary and we set $E_i = \{x \in U;\ f(x) = f_i(x)\}$, then the set E_i is open one. Since $U = E_1 \cup \cdots \cup E_N$, $E_i \cap E_j = \varnothing$ for $i \neq j$ and *U* is connected, there is an integer $i \in \{1, \ldots, N\}$ such as $U = E_i$.

Proposition 2.4. *Let F be a map such as the same in Proposition* 2.1. *Then* $\#\{F\} \leqq l$.

Proof. For every point $p \in \mathbf{C}^2$, we shall prove $\sup_{p \in \mathbf{C}^2} \#\{F(p)\} \leqq l = 12(g-1) + 6n$. Let *p* be a point of general level curve of $P(x, y)$ such as $\{P = \alpha\}$. Since $P \circ F = P$ from Proposition 2.2 and *F* is nonconstant on $\{P = \alpha\}$, $\#\{F(p)\} \leqq l$ by Theorem 1.5. Let *E* be the set of exceptional level curves of $P(x, y)$. Since $\mathbf{C}^2 - E$ is dense in $\mathbf{C}^2$, $\sup_{p \in \mathbf{C}^2} \#\{F(p)\} \leqq l$. Then we have the conclusion by Lemma 2.3.

Proposition 2.5. *Let F be a map such as the same in Proposition* 2.1. *Then* $F \in Aut_{alg}(\mathbf{C}^2)$.

Proof. Since F^k satisfies the condition of Proposition 2.1, where F^k is a *k*-ply iteration of *F*, there is an integer $k_0 \leqq l$ such as $F^{k_0} = id$, by Proposition 2.4. Then we have the conclusion easily.

From the above discussion, following theorem is proved.

Theorem 2.6. *Let* $P(x, y)$ *be a primitive general type polynomial of* (g, n) *type and* A_i $(i = 1, 2)$ *be a general level curve of* $P(x, y)$ *such as* $\{P = \alpha_i\}$ *with* *distinct* α_1, α_2. *If a map* $F \in (E)$ *satisfies* $\{P \circ F = \alpha_i\} = A_i$ *for* $i = 1, 2,$ *then* $F \in Aut_{alg}(\mathbf{C}^2)$ *and* $P \circ F = P$. *And such F makes a subgroup of* $Aut_{alg}(\mathbf{C}^2)$ *with* $\#\{F\} \leqq l = 12(g-1) + 6n.$

3. Some Remarks

Let $P(x, y)$ be a general type polynomial of (g, n) type. From Proposition 1.2, there is a primitive polynomial $P_0(x, y)$ and a polynomial π of one complex variable such that $P(x, y) = \pi \circ P_0(x, y)$. Let A_1 and A_2 be general level curves of $P_0(x, y)$ such as $A_i = \{P_0(x, y) = \alpha_i\}$ with distinct α_1, α_2.

Theorem 3.1. *Let* $P(x, y)$ *and* A_i *be the same above and a map* $F \in (E)$ *satisfy* $\{P_0 \circ F = \alpha_i\} = A_i$ $(i = 1, 2)$. *Then such F makes a subgroup of* $Aut_{alg}(\mathbf{C}^2)$ *such as* $P \circ F = P$ *with* $\#\{F\} \leqq l$, *where* $l = 12(g-1) + 6n.$

Proof. It is the direct consequence of Theorem 2.6 and $\pi \circ P_0 \circ F = \pi \circ P_0$.

In the case of a polynomial of exceptional type, there is a following:

Theorem 3.2 ([5] and [2])**.** *Let* $P(x, y)$ *be a polynomial of exceptional type and* $T \in Aut(\mathbf{C}^2)$ *satisfy* $P \circ T = P$. *Then such T makes a transcendental infinite subgroup of* $Aut(\mathbf{C}^2)$.

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