JP Journal of Geometry and Topology
Volume 10, Number 1, 2010, Pages 41-62
Published Online: July 3, 2010
This paper is available online at http://pphmj.com/journals/jpgt.htm

CHARACTERIZATION OF MINIMAL 3-MANIFOLDS BY EDGE-COLOURED GRAPHS

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Abstract

We characterize combinatorial representations of minimal 3-manifolds by means of edge-coloured graphs. This enables their recognition among existing crystallization catalogues, and contemporarily enables the automatic construction of efficient and exhaustive catalogues representing all minimal 3-manifolds up to a fixed genus.

1. Introduction

Edge-coloured graphs (and crystallizations, in particular) are a combinatorial tool for representing compact PL-manifolds of arbitrary dimension (see [1, 9] and their bibliography, or simply the second paragraph for a brief account about this theory). One of the main features of crystallization theory relies on the purely

2010 Mathematics Subject Classification: 57Q15, 57M15, 57N10.

Keywords and phrases: edge-coloured graph, minimal 3-manifold, Heegaard genus, regular genus.

Work performed under the auspices of the G.N.S.A.G.A. of the I.N.D.A.M. (Istituto Nazionale di Alta Matematica “F. Severi” of Italy) and financially supported by MiUR of Italy (project “Proprietà geometriche delle varietà reali e complesse”) and by the University of Modena and Reggio Emilia (project “Modelli discreti di strutture algebriche e geometriche”).

Communicated by Yasuo Matsushita

Received March 10, 2010; Revised April 15, 2010

combinatorial nature of the representing objects, which makes them particularly suitable for computer manipulation; this enables - among other things - the automatic production and analysis of complete crystallization catalogues satisfying suitable combinatorial conditions, which represent all PL manifolds with given topological features (see, for example, [4, 6, 14] and [2], devoted to classes of orientable or non-orientable 3-manifolds admitting triangulations with a fixed number of tetrahedra).

Especially in dimension three, where many well-known manifold representation theories have been developed, different notions of "complexity" have been defined, in order to "measure" how complicated a manifold M is, by minimizing suitable features of the combinatorial objects representing M: this happens - for instance - with Matveev's notion of "complexity" $c(M^3)$, which computes the minimum number of true vertices of any spine of M^3 (see [15]), such as with the well-known Heegaard genus $\mathcal{H}(M^3)$ of M^3, which computes the minimum genus of a surface splitting M^3 as the union of two handlebodies (see [11]).

Within crystallization theory, a quite obvious "parameter" showing how complicated a manifold M^n of arbitrary dimension n is, consists of the minimum order of an edge-coloured graph (Γ, γ) representing M^n; more precisely, the so-called gem-complexity of M^n is the non-negative integer defined as follows (see [4]):

$$k(M^n) = \min\left\{\frac{\# V(\Gamma)}{2} - 1 / (\Gamma, \gamma) \text{ represents } M^n\right\}.$$

Naturally, to analyze the existing relationships between different notions of complexity, related to different representation theories, is an interesting and not trivial matter, to which many efforts have been devoted (see, for example, [3, 10, 15, 16].

The present paper takes into account a specific problem inserted in this context, by dealing about the so-called minimal 3-manifolds (originally introduced in [7]), i.e., closed connected 3-manifolds admitting a strong relation between gem-complexity and Heegaard genus[1]:

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[1]Note that, as it will be pointed out at the beginning of the third paragraph, $k(M^3) \geq 3\mathcal{H}(M^3)$ holds for each 3-manifold M^3.

$$M^3 \text{ minimal} \Leftrightarrow k(M^3) = 3\mathcal{H}(M^3).$$

It is worth noting that each edge-coloured graph Γ representing M^3 has an associated "genus" (actually, an associated genus $\rho_\varepsilon(\Gamma)$ for each chosen permutation ε of the colour set: see the second paragraph for details), and that the Heegaard genus $H(M^3)$ of M^3 coincides with the minimum among all genera $\rho_\varepsilon(\Gamma)$ associated to edge-coloured graphs representing M^3. As a matter of fact, a minimal 3-manifold M^3 admits a crystallization $\overline{\Gamma}$ realizing both the minimum order and the minimum genus among all edge-coloured graphs representing M^3; moreover, the order of $\overline{\Gamma}$ is the least possible associated to edge-coloured graphs representing manifolds with that genus (see the third paragraph, and in particular, Proposition 5).

Now, it is very easy to check that the 3-sphere $\mathbb{S}^3$ is a genus zero (actually, the only one) minimal 3-manifold: in fact, the standard crystallization of $\mathbb{S}^3$ with two vertices has null associated genus.

Moreover, as pointed out in [7], if M^3 is minimal with $\mathcal{G}(M^3) = 1$, then M^3 is PL-homeomorphic to either the real projective space $L(2, 1) = \mathbb{RP}^2$ or $\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1$ (i.e., either the orientable or non-orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$, according to the orientability of the manifold).

Hence, from the point of view of the comparison between different complexity notions, all minimal 3-manifolds, up to Heegaard genus one, turn out to have Matveev's complexity zero, but at least one non-minimal 3-manifold exists with Matveev's complexity zero (i.e., the lenticular space $L(3, 1)$). This simple fact shows that relationships between different representation theories is not trivial, and that suitable investigation is necessary.

With this aim, the third paragraph of the present paper is devoted to characterize combinatorial representations of minimal 3-manifolds by means of edge-coloured graphs (Proposition 7 and Proposition 8), and to detect conditions yielding topological information about the represented manifolds (Proposition 11, Proposition 12 and Corollary 13).

Note that, in dimension four, a slightly different definition of minimality exists (originally due to [7], too, but successively simplified in [5]). It involves a combinatorial invariant, called *regular genus* (defined for every PL-manifold M^n and denoted by $\mathcal{G}(M^n)$), which extends to arbitrary dimension the notion of Heegaard genus:

A closed connected 4-manifold M^4 is said to be *minimal* if $\mu_1(M^4) = \mathcal{G}(M^4)$, where $\mu_1(M^4) = \min\{g_{\hat{i}\hat{j}} - 1/i,\ j \in \Delta_4,\ (\Gamma, \gamma) \text{ represents } M^4\}$, $g_{\hat{i}\hat{j}}$ being the number of connected components of the subgraph obtained from Γ by deleting all i-coloured and j-coloured edges.

In [5], it is proved that a closed connected 4-manifold M^4 is minimal if and only if $M^4 \cong \#_m(\mathbb{S}^3 \underset{|\sim|}{\times} \mathbb{S}^1)$, where $\mathbb{S}^3 \underset{|\sim|}{\times} \mathbb{S}^1$ denotes either the orientable or non-orientable $\mathbb{S}^3$-bundle over $\mathbb{S}^1$, according to the orientability of the manifold. This fact improves the results obtained in [7; Theorem 6.12], and gives positive answer to some conjectures (actually, to a stronger version of one of them) settled in [7; page 133].

The present paper may be considered as a first step toward a similar, possible characterization of minimal 3-manifolds:

Conjecture. A closed connected 3-manifold M^3 $(M^3 \neq \mathbb{S}^3)$ is minimal if and only if $M^3 = N_1 \# N_2 \# \cdots \# N_h$, where $h = \mathcal{G}(M^3)$ and, for each $i = 1, ..., h$, N_i is homeomorphic to either $L(2, 1) = \mathbb{RP}^3$ or $\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1$.

In fact, results obtained in the third paragraph prove the above conjecture to be true up to regular genus four and for each existing crystallization catalogue, and enable to construct efficient catalogues of rigid crystallizations, containing - if any - all prime minimal 3-manifolds up to a fixed genus.

2. A Quick Trip through Crystallization Theory

As already pointed out, representation theory via edge-coloured graphs is useful to deal with the whole class of piecewise linear (PL) manifolds, without assumptions about the dimension, the connectedness, the orientability or the boundary properties. In the present work, however, we restrict our attention to closed and connected PL-manifolds of dimension $n = 3$; hence, we will briefly review only basic notions and results of the theory concerning this particular case.

A 4-*coloured graph* (*without boundary*) is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a regular multigraph (i.e., it may include multiple edges, but no loop) of degree four and $\gamma : E(\Gamma) \to \Delta_3 = \{0, 1, 2, 3\}$ is a proper edge-colouration (i.e., it is injective when restricted to edges incident to any vertex $v \in V(\Gamma)$).

The elements of the set $\Delta_3 = \{0, 1, 2, 3\}$ are said to be *colours* of Γ; thus, for every $i \in \Delta_3$, an *i-coloured edge* is an element $e \in E(\Gamma)$ such that $\gamma(e) = i$. For every $i, j \in \Delta_3$ let $\Gamma_{\hat{i}}$ (resp. $\Gamma_{i, j}$) be the subgraph obtained from (Γ, γ) by deleting all the edges of colour i (resp. by deleting all the edges of colour $c \in \Delta_3 - \{i, j\}$). The connected components of $\Gamma_{i, j}$ (resp. $\Gamma_{\hat{i}}$) are said to be $\{i, j\}$-*coloured cycles* (resp. $\hat{i}$-*residues*) of Γ, and their number is denoted by $g_{i, j}$ (resp. $g_{\hat{i}}$). A 4-coloured graph (Γ, γ) is called *contracted* iff for each $i \in \Delta_3$, the subgraph $\Gamma_{\hat{i}}$ is connected (i.e., iff $g_{\hat{i}} = 1, \forall i \in \Delta_3$).

Every 4-coloured graph (Γ, γ) may be thought of as the combinatorial visualization of a 3-dimensional labelled pseudocomplex (see [13]) $K(\Gamma)$, which is constructed according to the following instructions:

- for each vertex $v \in V(\Gamma)$, take a 3-simplex $\sigma(v)$, with its vertices labelled 0, 1, 2, 3;
- for each j-coloured edge between v and w $(v, w \in V(\Gamma))$, identify the bidimensional faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertex labelled j, so that equally labelled vertices coincide.

In case $K(\Gamma)$ triangulates a (closed) PL 3-manifold M^3, then (Γ, γ) is said to *represent* M^3, or to be a *gem* (gem = graph encoded manifold) of M^3.[2]

Finally, a 4-coloured graph representing a (closed) 3-manifold M^3 is said to be

[2]Construction of $K(\Gamma)$ directly ensures that, if (Γ, γ) is a gem of M^3, then M^3 turns out to be orientable (resp. non-orientable) iff Γ is bipartite (resp. non-bipartite).

a *crystallization* of M^3 if it is also a contracted graph; by construction, it is not difficult to check that this is equivalent to require that the associated pseudocomplex $K(\Gamma)$ contains exactly one *i*-labelled vertex, for every $i \in \Delta_3$. The representation theory of PL-manifolds by edge-coloured graphs is often called *crystallization theory*, since every PL-manifold M^n is proved to admit a crystallization: see Pezzana theorem and its subsequent improvements ([9] or [1]).

Unlike greater dimensions, crystallizations of 3-manifolds may be easily recognized by means of convenient combinatorial conditions:

Proposition 1. *Let* (Γ, γ) *be a contracted* 4-*coloured graph*, *with* $\# V(\Gamma) = p$. *Then* (Γ, γ) *is a crystallization of a* 3-*manifold* M^3 *iff*:

(a) $g_{01} + g_{02} + g_{03} = 2 + p/2$;

(b) *for every permutation* $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ *of* Δ_3, $g_{\varepsilon_0\varepsilon_1} = g_{\varepsilon_2\varepsilon_3}$.

Another important advantage of the assumption $n = 3$ consists of the possibility of representing all closed connected 3-manifolds by a restricted class of crystallizations, i.e., the so-called *rigid* ones.

Definition 1. A pair (e, f) of distinct *i*-coloured edges in a 4-coloured graph (Γ, γ) is said to be a ρ_m *-pair* $(m = 2, 3)$ if and only if *e* and *f* both belong to exactly *m* bicoloured cycles of Γ. By a ρ-pair, we mean a ρ_m -pair, for $m \in \{2, 3\}$.

A crystallization (Γ, γ) of a 3-manifold M^3 is said to be *rigid* if it contains no ρ-pair.

Definition 2. We will say that a 3-manifold M^3 contains a *handle* if a decomposition $M^3 = J \# (\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1)$ holds, where $\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1$ denotes either the orientable or non-orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$ and *J* is a suitable non-empty 3-manifold (possibly homeomorphic to $\mathbb{S}^3$). In the opposite case, M^3 is said to be *handle-free*.

Proposition 2 [4, 14]**.** *Every closed connected* 3-*manifold* M^3 *admits a rigid* *crystallization. Moreover, if* M^3 *is handle-free and* (Γ, γ) *is any gem of* M^3, *with* $\# V(\Gamma) = p$, *then a rigid crystallization* $(\overline{\Gamma}, \overline{\gamma})$ *of* M^3 *exists, with* $\# V(\overline{\Gamma}) \leq p$.

The following result explains the topological meaning of a special configuration involving an edge for each colour, called *quartet*.

Proposition 3 [14]. *Let* (Γ, γ) *be a gem of the closed connected* 3*-manifold* M^3. *Suppose there exist four edges* $\{e_0, e_1, e_2, e_3\}$ *in* (Γ, γ) *such that* $\gamma(e_i) = i$, *for each* $i \in \Delta_3$, *and* $\{e_i, e_j\}$ *belongs to the same* $\{i, j\}$*-coloured cycle, for each* $i, j \in \Delta_3$. *If* $\{u_i, v_i\}$ *denote the end-points of* e_i, *for each* $i \in \Delta_3$, *with the assumption that an even number of edges connects* u_i *to* u_{i+1} *along the* $\{i, i+1\}$*-coloured cycle, for* $i \in \mathbb{Z}_3$, *let H be the graph obtained from* (Γ, γ) *by deleting* $\{e_0, e_1, e_2, e_3\}$ *and by connecting vertices* u_0, u_1, u_2, u_3 $($*resp.* $v_0, v_1, v_2, v_3)$ *to a new vertex x* (*resp. y*). *Then*:

• *if H is connected, then it is a gem of a* 3*-manifold* N^3 *so that* $M^3 = N^3 \# (\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1)$;

• *if H has two connected components* H_1 *and* H_2, *then* H_1 $($*resp.* $H_2)$ *is a gem of a* 3*-manifold* N_1^3 $($*resp.* $N_2^3)$ *so that* $M^3 = N_1^3 \# N_2^3$.

Conversely, the topological notion of connected sum of PL-manifolds has an easy combinatorial realization on edge-coloured graphs (actually, in arbitrary dimension):

Definition 3. Let (Γ', γ') and (Γ'', γ'') be two 4-coloured graphs. Let us consider two vertices $v' \in V(\Gamma')$ and $v'' \in V(\Gamma'')$ and let $\Gamma' \#_{\{v', v''\}} \Gamma''$ be the 4-coloured graph obtained from Γ' and Γ'' by deleting $\{v', v''\}$ and welding the "hanging" edges of the same colour $c \in \Delta_3$. Then the process leading from Γ', Γ'' to $\Gamma' \#_{\{v', v''\}} \Gamma''$ is said to be a *graph connected sum*.

Proposition 4 [9]. *If* Γ' $($*resp.* $\Gamma'')$ *is a gem of the* 3*-manifold* M_1^3 $($*resp.* $M_2^3)$, *then* $\Gamma' \#_{\{v', v''\}} \Gamma''$ *is a gem of the* 3*-manifold* $M_1^3 \# M_2^3$, *obtained by connected sum of* 3*-manifolds* M_1^3 *and* M_2^3.

Among essential constructions and results of crystallization theory, let us recall the existence of a complete (finite) set of graph-moves - called *dipole moves* - which allow to translate the (PL)-homeomorphism problem for closed *n*-manifolds into an equivalence problem for edge-coloured graphs: for any dimension *n*, it has been proved that two edge-coloured graphs do represent the same PL *n*-manifold if and only if they can be obtained one each other by a finite sequence of dipole moves: see [8].

In dimension $n = 3$, in particular, another combinatorial move on gems exists, not affecting the homeomorphism class of the represented manifold.

Definition 4. Let (Γ, γ) be a gem of a closed connected 3-manifold M^3. If there exists an $\{i, j\}$-coloured cycle of length $m+1$ and a $\{k, l\}$-coloured cycle of length $n+1$ in Γ (with $\{i, j, k, l\} = \Delta_3$) having exactly one common vertex $\bar{v}$, then (Γ, γ) is said to contain a (m, n)*-generalized dipole* of type $\{i, j\}$ at vertex $\bar{v}$.

To *cancel* a (m, n)-generalized dipole from a gem (Γ, γ) means to perform on (Γ, γ) the operation visualized in Figure 1 (in case $m = 3$; $n = 5$).

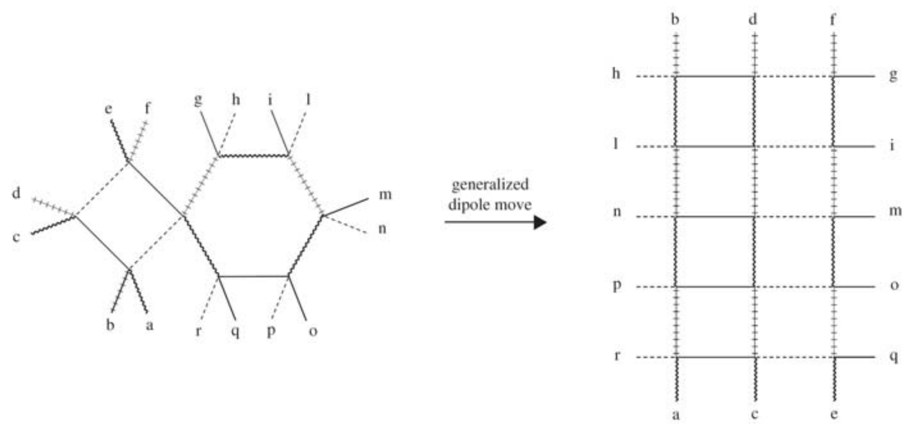

Figure 1

We refer to the cancellation of a (m, n)-generalized dipole and to its inverse procedure as *generalized dipole moves*. They both connect different gems of the same 3-manifold, as proved in [8], too.

Finally, we resume in dimension $n = 3$ the notion of *regular genus*.[3]

If Γ is a bipartite (resp. non-bipartite) crystallization of a 3-manifold M^3, then for any permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ of the colour set Δ_3 a regular embedding[4] $i_\varepsilon : \Gamma \to F_\varepsilon$ is proved to exist (see [9] or [1], together with their references), where F_ε is the orientable (resp. non-orientable) surface of genus $\rho_\varepsilon(\Gamma) = g_{\varepsilon_0, \varepsilon_2} - 1 = g_{\varepsilon_1, \varepsilon_3} - 1$ (resp. of genus $2 \cdot \rho_\varepsilon(\Gamma)$).

The regular genus $\rho(\Gamma)$ of (Γ, γ) is, by definition, the minimum genus $\rho_\varepsilon(\Gamma)$, among all permutations ε of Δ_3, while the *regular genus* of a 3-manifold M^3 is defined as:

$$\mathcal{G}(M^3) = \min\{\rho(\Gamma)/(\Gamma, \gamma) \text{ represents } M^3\}.$$

Note that, as already pointed out in the first paragraph, regular genus extends to arbitrary dimension, the well-known notion of Heegaard genus of a 3-manifold M^3: in fact the equality $\mathcal{G}(M^3) = \mathcal{H}(M^3)$ holds for any M^3: see again [9] or [1].

3. Representing Minimal 3-manifolds

By means of the notions summarized in the second paragraph, we are now able to state the original definition of *minimal 3-manifold*:

Definition 5 [7]**.** A closed connected 3-manifold M^3 is said to be *minimal* if $p^*(M^3) - 2 = 6\mathcal{G}(M^3)$, where $p^*(M^3) = \min\{\# V(\Gamma)/(\Gamma, \gamma) \text{ represents } M^3\}$.

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[3]Actually, *regular genus* subsists for PL-manifolds of arbitrary dimension and plays a central role within crystallization theory, since it enables to obtain important classification results on PL-manifolds: see [1; Section 5].

[4]A cellular embedding *i* of a 4-coloured graph Γ into a surface is said to be *regular* if there exists a permutation ε of Δ_3 such that the regions of *i* are bounded by the images of $\{\varepsilon_j, \varepsilon_{j+1}\}$-residues of Γ $(j \in \mathbb{Z}_4)$.

Note that relation $p^*(M^3) - 2 \geq 6\mathcal{G}(M^3)$ (or, equivalently, $k(M^3) \geq 3\mathcal{H}(M^3)$) holds for each 3-manifold M^3, in virtue of the equality $\rho_\varepsilon(\Gamma) = g_{\varepsilon_0, \varepsilon_2} - 1$ and of Proposition 1(a) and (b), together with the definition itself of regular genus of M^3.

The first result we are going to prove (Proposition 5) states that - as pointed out in the Introduction - a minimal 3-manifold M^3 admits a crystallization $\overline{\Gamma}$ realizing both the minimum order and the minimum genus among all edge-coloured graphs representing M^3; moreover, the order of $\overline{\Gamma}$ is the least possible associated to edge-coloured graphs representing manifolds with that genus and $\overline{\Gamma}$ exactly realizes the minimum genus with respect to any permutation of the colour set.

On the other hand, it is not difficult to check that, if M^3 is assumed to be a lenticular space $L(p, q)$, then the following facts occur:

• In case $q = 1$ and $p \neq 2$, the standard genus one crystallization $\Lambda_{p,q}$ of $L(p, q)$ (with $4p$ vertices: see [5; Figure 1]) is minimal with respect to the number of vertices, but $\Lambda_{p,q}$ contains at least one $(3, 3)$-generalized dipole of type $\{\varepsilon_0, \varepsilon_1\}$, where $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is the permutation of Δ_3 so that $\rho(\Lambda_{p,q}) = \rho_\varepsilon(\Lambda_{p,q}) = 1$. Hence, by elimination of the generalized dipole, a gem $\Lambda'_{p,q}$ of $L(p, q)$ is obtained, with $\rho_{\varepsilon'}(\Lambda'_{p,q}) < \rho_{\varepsilon'}(\Lambda_{p,q}) \neq \rho(\Lambda_{p,q})$, where $\varepsilon' = (\varepsilon_0, \varepsilon_2, \varepsilon_1, \varepsilon_3)$.

• In case $q \neq 1$, the standard genus one crystallization $\Lambda_{p,q}$ of $L(p, q)$ (with $4p$ vertices, too) is minimal with respect to genus; however, $\Lambda_{p,q}$ contains at least one *cluster*, i.e., a nine-vertex structure consisting in four bicoloured cycles of length four with a common vertex, which can be easily removed giving rise to a gem of $L(p, q)$ with two less vertices (see Figure 2, or [14; paragraph 4.1.4] and [6; Proposition 6.1] for details).

This is a confirmation of the known fact that $L(2, 1)$ is the only lenticular space which turns out to be a minimal 3-manifold.

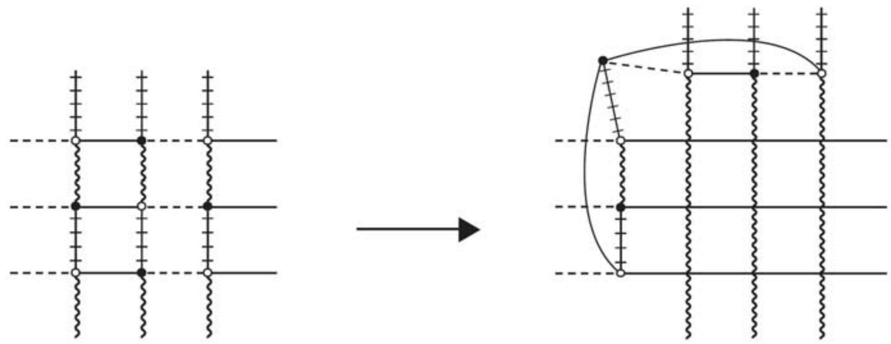

Figure 2

Proposition 5. *Let* M^3 *be a minimal* 3-*manifold and let* $(\overline{\Gamma}, \overline{\gamma})$ *be a* 4-*coloured graph representing* M^3 *with* $\# V(\overline{\Gamma}) = \overline{p} = p^*(M^3)$. *Then*:

(a) $(\overline{\Gamma}, \overline{\gamma})$ *is a crystallization of* M^3;

(b) *for every permutation* ε *of* Δ_3, $\rho_\varepsilon(\overline{\Gamma}) = \mathcal{G}(M^3)$;

(c) *for every* $i, j \in \Delta_3$, $g_{ij} = \dfrac{\overline{p}+4}{6}$.

Proof. Property (a) is a direct consequence of the fact that $\overline{\Gamma}$ realizes the minimal order among all edge-coloured graphs representing M^3: in fact, given any (non-contracted) edge-coloured graph Γ representing M^3, a (non-void) finite sequence of dipole moves exists (more precisely: a finite sequence of 1-dipole eliminations exists: see [8] or [9] for details) giving rise to a crystallization $\overline{\Gamma}$ of M^3 with $\# V(\overline{\Gamma}) < \# V(\Gamma)$.

Let now $\overline{\varepsilon} = (\overline{\varepsilon}_0, \overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3)$ be a permutation of Δ_3 such that $\rho_{\overline{\varepsilon}}(\overline{\Gamma}) = \rho(\overline{\Gamma})$. Then from Proposition 1(a) and (b), the equality $2g_{\overline{\varepsilon}_0\overline{\varepsilon}_1} + 2g_{\overline{\varepsilon}_1\overline{\varepsilon}_2} + 2g_{\overline{\varepsilon}_0\overline{\varepsilon}_2} = 4 + \overline{p}$ follows. On the other hand, $\rho(\overline{\Gamma}) = \rho_{\overline{\varepsilon}}(\overline{\Gamma}) = g_{\overline{\varepsilon}_0\overline{\varepsilon}_2} - 1$ implies $g_{ij} \geq g_{\overline{\varepsilon}_0\overline{\varepsilon}_2}$, $\forall i, j \in \Delta_3$, while the minimality of M^3 implies $\overline{p} = p^*(M^3) = 6\mathcal{G}(M^3) + 2$. Hence,

$$4 + \overline{p} = 2g_{\overline{\varepsilon}_0\overline{\varepsilon}_1} + 2g_{\overline{\varepsilon}_1\overline{\varepsilon}_2} + 2g_{\overline{\varepsilon}_0\overline{\varepsilon}_2} \geq 6g_{\overline{\varepsilon}_0\overline{\varepsilon}_2}$$

$$= 6(\rho_{\overline{\varepsilon}}(\overline{\Gamma}) + 1) \geq 6(\mathcal{G}(M^3) + 1) = 4 + \overline{p}.$$

As a consequence, $g_{i,j} = g_{\bar{\varepsilon}_0\bar{\varepsilon}_2}$, $\forall i, j \in \Delta_3$ and $\rho_{\bar{\varepsilon}}(\overline{\Gamma}) = \mathcal{G}(M^3)$ follow. This proves that $\overline{\Gamma}$ also realizes the minimal genus among all 4-coloured graphs representing M^3 (i.e., the Heegaard genus of M^3), for any permutation ε of Δ_3 (property (b)), and that the equality $g_{ij} = \dfrac{\bar{p}+4}{6}$ is satisfied $\forall i, j \in \Delta_3$ (property (c)). $\square$

Proposition 6. *Let* M_1^3 *and* M_2^3 *be two minimal* 3-*manifolds*. *Then* $M_1^3 \# M_2^3$ *is minimal*, *too*.

Proof. In virtue of Proposition 5, let $(\overline{\Gamma}_1, \bar{\gamma}_1)$ (resp. $(\overline{\Gamma}_2, \bar{\gamma}_2)$) be a 4-coloured graph representing M_1^3 (resp. M_2^3) with $\#V(\overline{\Gamma}_1) = \bar{p}_1 = p^*(M_1^3)$, $\rho_\varepsilon(\overline{\Gamma}_1) = \mathcal{G}(M_1^3)$, $\forall \varepsilon$ and $p^*(M_1^3) - 2 = 6\mathcal{G}(M_1^3)$ (resp. $\#V(\overline{\Gamma}_2) = \bar{p}_2 = p^*(M_2^3)$, $\rho_\varepsilon(\overline{\Gamma}_2) = \mathcal{G}(M_2^3)$, $\forall \varepsilon$ and $p^*(M_2^3) - 2 = 6\mathcal{G}(M_2^3)$). By making use also of the additivity property of Heegaard genus (see [11]) and of the construction of graph connected sum (see Definition 3), it is easy to check that

$$\begin{aligned} p^*(M_1^3 \# M_2^3) \le \#V(\overline{\Gamma}_1 \# \overline{\Gamma}_2) &= \#V(\overline{\Gamma}_1) + \#V(\overline{\Gamma}_2) - 2 \\ &= p^*(M_1^3) + p^*(M_2^3) - 2 = 6\mathcal{G}(M_1^3) + 6\mathcal{G}(M_2^3) + 2 \\ &= 6\mathcal{G}(M_1^3 \# M_2^3) + 2. \end{aligned}$$

Since $6\mathcal{G}(N^3) \le p^*(N^3) - 2$ holds for any 3-manifold N^3, the equality $p^*(M_1^3 \# M_2^3) = 6\mathcal{G}(M_1^3 \# M_2^3) + 2$ follows; hence, $M_1^3 \# M_2^3$ is a minimal 3-manifold, too, and the edge-coloured graph $\overline{\Gamma}_1 \# \overline{\Gamma}_2$ (in fact: a crystallization, like $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$) realizes its minimality. $\square$

Proposition 7. *Let* M^3 *be a minimal* 3-*manifold*. *Then*:

(a) *if* M^3 *is handle-free*, *then a rigid crystallization exists realizing the minimality of* M^3;

(b) *if* $M^3 = J \# (\#_h(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1)),\ h \geq 1,$ *then a non-negative integer* $\bar{s},\ 0 \leq \bar{s} \leq h$ *exists*, *so that* $J \# (\#_{\bar{s}}(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1))$ *is minimal and a rigid crystallization exists realizing the minimality of* $J \# (\#_{\bar{s}}(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1))$.

Proof. As stated in Proposition 2, for each handle-free 3-manifold M^3, the minimal order of an edge-coloured graph representing M^3 is realized by a *rigid* crystallization. Hence, in virtue of Proposition 5, statement (a) directly follows.

Let us now assume $M^3 = J \# (\#_h(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1)),\ h \geq 1.$ Then the minimality of M^3 and the additivity of Heegaard genus easily yield:

$$p^*(J \# (\#_h(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1))) = 6\mathcal{G}(J \# (\#_h(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1))) + 2$$

$$= 6\mathcal{G}(J) + 6\mathcal{G}(\#_h(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1)) + 2 = 6\mathcal{G}(J) + 6h + 2.$$

If $p_r^*(N)$ denotes the minimum order of a rigid crystallization representing *N*, then [4; Proposition 8(b)] implies

$$p^*(M^3) = \min\{p_r^*(J \# (\#_s(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1))) + 6(h - s)/s \in \{0, 1, \ldots, h\}\}.$$

Let $\bar{s} \in \{0, 1, \ldots, h\}$ be the non-negative integer so that

$$p^*(M^3) = p_r^*(J \# (\#_{\bar{s}}(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1))) + 6(h - \bar{s}).$$

Hence, $p_r^*(J \# (\#_{\bar{s}}(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1))) + 6(h - \bar{s}) = 6\mathcal{G}(J) + 6h + 2,$ which yields

$$p_r^*(J \# (\#_{\bar{s}}(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1))) = 6\mathcal{G}(J) + 6\bar{s} + 2 = 6\mathcal{G}(J \# (\#_{\bar{s}}(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1))) + 2.$$

The minimality of $J \# (\#_{\bar{s}}(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1))$ is so proved, together with the existence of a rigid crystallization realizing it, as item (b) states. □

Remark. The standard order eight (non-rigid) crystallization of $\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1$ realizes the minimality of $\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1$: $p^*(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1) = 8 = 2 + 6 = 2 + 6 \cdot \mathcal{G}(\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1)$.

Proposition 8. *Let M^3 be a minimal handle-free* 3*-manifold and $(\overline{\Gamma}, \overline{\gamma})$ be a* 4*-coloured graph representing M^3 with $\#V(\overline{\Gamma}) = \overline{p} = p^*(M^3)$. Then*:

(a) $(\overline{\Gamma}, \overline{\gamma})$ *is a rigid crystallization of M^3*;

(b) $\overline{p} = p(\overline{\Gamma}) \equiv 2 \bmod 6$;

(c) *for every $i, j \in \Delta_3$, $g_{ij} = \dfrac{\overline{p} + 4}{6}$*;

(d) $\overline{\Gamma}$ *lacks in generalized dipoles*;

(e) $\overline{\Gamma}$ *is cluster-less*, *i.e.*, *no vertex $v \in V(\overline{\Gamma})$ exists*, *so that four bicoloured cycles containing v have length four and altogether involve exactly nine vertices*.

Proof. Item (a) directly follows from item (a) of Proposition 7; item (b) holds in virtue of the definition itself of minimal 3-manifold; item (c) directly follows from item (c) of Proposition 5. On the other hand, the existence of a generalized dipole in $\overline{\Gamma}$ would imply the existence of a 4-coloured graph representing M^3 with lower genus than $\overline{\Gamma}$, with respect to a suitable permutation, against the hypothesis $\rho_\varepsilon(\overline{\Gamma}) = \mathcal{G}(M^3)$ for any permutation ε (see Figure 1, or [8] for details): this proves item (d). Finally, the existence in $\overline{\Gamma}$ of a so called *cluster-type vertex* (i.e., a vertex $v \in V(\overline{\Gamma})$ so that $(\overline{\Gamma}, \overline{\gamma})$ admits four bicoloured cycles containing *v* with length four, which altogether involve exactly nine vertices) would imply the existence of a 4-coloured graph representing M^3 with less vertices than $\overline{\Gamma}$, against the hypothesis $\#V(\overline{\Gamma}) = p^*(M^3)$ (see Figure 2, or [14; Paragraph 4.1.4] and [6; Proposition 6.1] for details): this proves item (e). □

Corollary 9. *Let M^3 be a minimal handle-free* 3*-manifold and $(\overline{\Gamma}, \overline{\gamma})$ be a* 4*-coloured graph realizing its minimality*, *with $\#V(\overline{\Gamma}) = \overline{p}$ ($\overline{p} > 2$). Then*, *for every*

$i, j \in \Delta_3$, *a non-negative integer* h_{ij} *exists, so that* $\overline{\Gamma}$ *admits*:

- $2 + h_{ij} \geq 2$ $\{i, j\}$*-cycles of length four*;

- r_{ij} ($r_{ij} = 0$ *if* $h_{ij} = 0$; $1 \leq r_{ij} \leq h_{ij}$ *otherwise*) $\{i, j\}$*-cycles of length* $l_{ij}^{1}, \ldots, l_{ij}^{r_{ij}}$, *with* $l_{ij}^{s} \geq 8$, $\forall s = 1, \ldots, r_{ij}$ *and* $\sum_{s=1}^{r_{ij}} l_{ij}^{s} = 6r_{ij} + 2h_{ij}$;

- *all other* $\dfrac{\overline{p} - 8 - 6(h_{ij} + r_{ij})}{6}$ $\{i, j\}$*-cycles of length six.*

Proof. In virtue of Proposition 5(c), $g_{ij} = \dfrac{\overline{p} + 4}{6}$ for every $i, j \in \Delta_3$. On the other hand, the hypothesis of M^3 being handle-free implies $\overline{\Gamma}$ to be a rigid crystallization (Proposition 7(a)), and this - under the hypothesis $\overline{p} > 2$ -prevents the existence of length two $\{i, j\}$-cycles. The statement now easily follows by direct calculation. □

Proposition 10. *Let* M^3 *be a minimal* 3*-manifold and* $(\overline{\Gamma}, \overline{\gamma})$ *be a* 4*-coloured graph realizing its minimality. If two* 4*-coloured graphs* $(\overline{\Gamma}_1, \overline{\gamma}_1)$ *and* $(\overline{\Gamma}_2, \overline{\gamma}_2)$ *exist, so that* $\overline{\Gamma} = \overline{\Gamma}_1 \# \overline{\Gamma}_2$, *then* $M_1^3 = K(\overline{\Gamma}_1)$ (*resp.* $M_2^3 = K(\overline{\Gamma}_2)$) *is a minimal* 3*-manifold, and* $(\overline{\Gamma}_1, \overline{\gamma}_1)$ (*resp.* $\overline{\Gamma}_2, \overline{\gamma}_2$)) *realizes its minimality.*

Proof. By the hypothesis on $\overline{\Gamma}$, $g_{ij} = \dfrac{\overline{p} + 4}{6}$ for every $i, j \in \Delta_3$ (Proposition 5(c)). On the other hand, if $\overline{\Gamma} = \overline{\Gamma}_1 \# \overline{\Gamma}_2$, then $\#V(\overline{\Gamma}) = \#V(\overline{\Gamma}_1) + \#V(\overline{\Gamma}_2) - 2$ and $g_{ij} = g'_{ij} + g''_{ij} - 1$ (where, for every $i, j \in \Delta_3$, g_{ij}, g'_{ij} and g''_{ij}, respectively, denote the number of $\{i, j\}$-cycles in $\overline{\Gamma}$, $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$). Let us now assume $g'_{ij} < \dfrac{\#V(\overline{\Gamma}_1) + 4}{6}$ $\left(\text{or, equivalently, } g''_{ij} < \dfrac{\#V(\overline{\Gamma}_2) + 4}{6}\right)$. As a consequence, we have

$$g_{ij} = g'_{ij} + g''_{ij} - 1 < \frac{\#V(\overline{\Gamma}_1) + \#V(\overline{\Gamma}_2) + 8}{6} - 1 = \frac{\#V(\overline{\Gamma}) + 10}{6} - 1 = \frac{\overline{p} + 4}{6},$$

which contradicts the hypothesis on $\overline{\Gamma}$. Hence, $g'_{ij} = \dfrac{\#V(\overline{\Gamma}_1) + 4}{6}$ and $g''_{ij} =$

$\dfrac{\#V(\overline{\Gamma}_2)+4}{6}$ hold for every $i, j \in \Delta_3$, which, respectively, imply $6\rho_\varepsilon(\overline{\Gamma}_1) = \#V(\overline{\Gamma}_1) - 2$ and $6\rho_\varepsilon(\overline{\Gamma}_2) = \#V(\overline{\Gamma}_2) - 2$ for any permutation ε of Δ_3. On the other hand, it is easy to check that both M_1^3 and M_2^3 cannot be represented by a 4-coloured graph with less vertices (resp. with a strictly lower genus) than $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$, since otherwise $\overline{\Gamma}$ could not realize the minimality of M^3. This proves that $(\overline{\Gamma}_1, \overline{\gamma}_1)$ and $(\overline{\Gamma}_2, \overline{\gamma}_2)$, respectively, realize the minimality of M_1^3 and M_2^3. □

Proposition 11. *Let M^3 be a minimal (handle-free) 3-manifold with $\mathcal{G}(M^3) \geq 2$ and $(\overline{\Gamma}, \overline{\gamma})$ be a 4-coloured graph realizing its minimality. If $(\overline{\Gamma}, \overline{\gamma})$ admits a vertex $\overline{v}$ so that both the $\{i, j\}$-cycle and the $\{k, l\}$-cycle containing $\overline{v}$ have length four $((i, j, k, l)$ being a suitable permutation of $\Delta_3)$, then two minimal 3-manifolds M_1^3 and M_2^3 exist (with $M_i^3 \neq \mathbb{S}^3, \forall i = 1, 2)$, so that $M^3 = M_1^3 \# M_2^3$.*

Proof. Let $\{\overline{v} = v_0, v_1, v_2, v_3\}$ be the vertex set of the $\{i, j\}$-cycle of $(\overline{\Gamma}, \overline{\gamma})$ containing $\overline{v}$, with v_{2r} *i*-adjacent to v_{2r+1} and *j*-adjacent to v_{2r-1}, for $r = 0, 1$ (indices being assumed in $\mathbb{Z}_4$), and let $\{\overline{v} = u_0, u_1, u_2, u_3\}$ be the vertex set of the $\{k, l\}$-cycle of $(\overline{\Gamma}, \overline{\gamma})$ containing $\overline{v}$, with u_{2r} *k*-adjacent to u_{2r+1} and *l*-adjacent to u_{2r-1}, for $r = 0, 1$ (indices being assumed in $\mathbb{Z}_4$). In virtue of Proposition 8, $(\overline{\Gamma}, \overline{\gamma})$ is a rigid crystallization lacking in generalized dipoles. Hence, $v_2 = u_2$ necessarily follows, together with $\{v_1, v_3\} \cap \{u_1, u_3\} = \emptyset$; moreover, v_1 and v_3 (resp. u_1 and u_3) belong to the same $\{k. l\}$-cycle (resp. $\{i, j\}$-cycle).

Note now that the *i*-coloured edge having an end-point in u_3 (resp. in u_1), the *j*-coloured edge having an end-point in u_1 (resp. in u_3), the *k*-coloured edge having an end-point in v_3 (resp. in v_1) and the *l*-coloured edge having an end-point in v_1 (resp. in v_3) form a *quartet* Q (resp. Q'), since they pairwise belong to the same bicoloured cycle of $(\overline{\Gamma}, \overline{\gamma})$. Moreover, note that if both the quartets Q and Q' consist in four incident edges, then $(\overline{\Gamma}, \overline{\gamma})$ is the (order eight) standard crystallization of $L(2, 1)$, against the hypothesis $\mathcal{G}(M^3) \geq 2$.

The thesis is now a direct consequence of Proposition 3, applied to a quartet chosen between the above ones so that its edges are not incident: in fact, since M^3 is assumed to be handle-free, $(\overline{\Gamma}, \overline{\gamma})$ splits as a connected sum and the fact that $(\overline{\Gamma}, \overline{\gamma})$ realizes the minimality of M^3 prevents the case that one of the two components represents $\mathbb{S}^3$. □

Proposition 12. *Let* M^3 *be a minimal (handle-free) 3-manifold and* $(\overline{\Gamma}, \overline{\gamma})$ *be a 4-coloured graph realizing its minimality. Then for every pair* $i, j \in \Delta_3$, $h_{ij} \leq \mathcal{G}(M^3) - 1$.

Moreover, in case $\mathcal{G}(M^3) \geq 2$:

- *If a pair* $i, j \in \Delta_3$ *exists, so that* $h_{ij} \geq \mathcal{G}(M^3) - 2$, *then two minimal 3-manifolds* M_1^3 *and* M_2^3 *exist, so that* $M^3 = M_1^3 \# M_2^3$.
- *If a pair* $i, j \in \Delta_3$ *exists, so that* $h_{ij} = \mathcal{G}(M^3) - 3$ *and* $r_{ij} \leq 1$, *then two minimal 3-manifolds* M_1^3 *and* M_2^3 *exist, so that* $M^3 = M_1^3 \# M_2^3$.

Proof. In virtue of Corollary 9, $\overline{\Gamma}$ admits $\dfrac{\overline{p} - 8 - 6 \cdot h_{ij}}{6}$ $\{i, j\}$-cycles of length greater or equal to six. On the other hand, by Proposition 8(c), $\dfrac{\overline{p} - 8 - 6 \cdot h_{ij}}{6} = g_{ij} - 2 - h_{ij} = \mathcal{G}(M^3) - 1 - h_{ij}$. Hence, $h_{ij} \leq \mathcal{G}(M^3) - 1$ directly follows, for each pair $i, j \in \Delta_3$.

Let us now assume the existence of a pair $i, j \in \Delta_3$ so that $h_{ij} \geq \mathcal{G}(M^3) - 2$. Then the above calculation implies that $\overline{\Gamma}$ contains at most one $\{i, j\}$-cycle of length greater or equal to six. Since $\overline{\Gamma}$ contains at least two $\{k, l\}$-cycle of length four (where $\{k, l\} = \Delta_3 - \{i, j\}$ is assumed), the rigidity of $\overline{\Gamma}$ implies the existence of at least one $\{i, j\}$-cycle of length four and one $\{k, l\}$-cycle of length four having common vertices. Hence, the second point of the statement follows by Proposition 11.

Let us now assume the existence of a pair $i,\ j \in \Delta_3$ so that $h_{ij} = \mathcal{G}(M^3) - 3$ and $r_{ij} \leq 1$. Then the above calculation (by means of the hypothesis $h_{ij} = \mathcal{G}(M^3) - 3$) implies that $\overline{\Gamma}$ contains exactly two $\{i,\ j\}$-cycle of length greater or equal to six; moreover, at least one of them turns out to have length six (since also $r_{ij} \leq 1$ is assumed). Note that, if $\overline{\Gamma}$ contains an $\{i,\ j\}$-cycle of length four and a $\{k,\ l\}$-cycle of length four having common vertices, then the thesis follows by Proposition 11. So, we can restrict our attention to the case that each $\{k,\ l\}$-cycle of length four of $\overline{\Gamma}$ has vertices in common only with $\{i,\ j\}$-cycles of length greater or equal to six. The fact that $\overline{\Gamma}$ lacks in ρ-pairs and generalized dipoles implies that each $\{i,\ j\}$-cycle of length six has exactly two pairs of vertices (each pair consisting of vertices of the same bipartition class, if $\overline{\Gamma}$ is a bipartite graph) belonging to a $\{k,\ l\}$-cycle of length four.

Now, let us recall a basic result of crystallization theory, which allows to directly obtain a presentation for the fundamental group of a PL n-manifold from any edge-coloured graph representing it (see [9] for details): in particular, if $(\Gamma,\ \gamma)$ is a crystallization of a 3-manifold M^3 and if $i,\ j \in \Delta_3$ are two arbitrarily chosen colours, then the generator set for $\pi_1(M^3)$ is in bijection with the set of all $\{i,\ j\}$-coloured cycles of Γ but one, while relators follow from all $(\Delta_3 - \{i,\ j\})$-coloured cycles but one.

So, if A (resp. B) denotes the generator of the $\{i,\ j\}$-presentation $\langle X/R \rangle$ of the fundamental group $\pi_1(M^3)$ of M^3 corresponding to the $\{i,\ j\}$-cycle of $\overline{\Gamma}$ of length six (resp. to the other $\{i,\ j\}$-cycle of $\overline{\Gamma}$ of length greater or equal to six), then R surely contains relation $(AB^{-1})^2 = 1$ (twice), and generator A appears only in one other relation $\bar{r} \in R$. By setting $B = 1$ and by deleting relation $\bar{r}$, then a new presentation $\langle X'/R' \rangle$ of $\pi_1(M^3)$ is obtained, where R' contains relation $A^2 = 1$, and no other relation of R' contains generator A. Hence, $\pi_1(M^3)$ splits into the free product between $\mathbb{Z}_2$ and a suitable group G, and so the last point of the statement follows by Kneser's theorem (see [12; Theorem 7.1]). □

Corollary 13. *Let* M^3 *be a minimal* 3-*manifold*, $M^3 \neq \mathbb{S}^3$. *If* $\mathcal{G}(M^3) \leq 3$, *then* $M^3 = N_1 \# \cdots \# N_h$, *where* $h = \mathcal{G}(M^3)$ *and*, *for each* $i = 1, \ldots, h$, N_i *is homeomorphic to either* $L(2, 1) = \mathbb{RP}^3$ *or* $\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1$.

Proof. First of all note that, if M^3 is not handle-free, then $M^3 = \overline{M} \#_r H$, where $r \in \{1, \ldots, \mathcal{G}(M^3)\}$, $\#_r H$ denotes the connected sum of r copies of $\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1$ and $\overline{M}$ is an handle-free minimal 3-manifold of genus $\mathcal{G}(M^3) - r$ ($\overline{M} = \mathbb{S}^3$ in case $r = \mathcal{G}(M^3)$). Hence, the attention may be restricted to the case of M^3 being a minimal handle-free 3-manifold.

Now, apart from the well-known case of genus one minimal 3-manifolds, the assumption $\mathcal{G}(M^3) \leq 3$ easily implies that each 4-coloured graph realizing the minimality of M^3 satisfies the hypothesis of one of the statements of Proposition 12 concerning manifolds with $\mathcal{G}(M^3) \geq 2$; the thesis now directly follows. □

4. Cataloguing Minimal 3-manifolds

Proposition 12, together with Proposition 7 and Proposition 8, yield a list of conditions which have to be satisfied by any gem realizing the minimality of a minimal prime handle-free 3-manifold.

These conditions - which are collected and simplified in the following statement - may be used either to examine existing crystallization catalogues, in order to recognize minimal 3-manifolds possibly represented in them, or to produce new reduced catalogues of gems with increasing order, representing all minimal prime handle-free 3-manifolds up to a given gem-complexity.

Proposition 14. *Let* M^3 *be a minimal* 3-*manifold with gem-complexity* $3k$, $k \in \mathbb{Z}$ (*i.e.*, *with* $p^*(M^3) = 6k + 2$ *and* $\mathcal{G}(M^3) = k$). *Then* M^3 *is represented* (*possibly by means of connected sums and*/*or adding of handles*) *by the catalogue of edge-coloured graphs* (Γ, γ) *satisfying the following conditions*:

- $\#V(\overline{\Gamma}) = 6h + 2$ *with* $h \leq k$;

- $g_{ij} = h + 1, \forall i, j \in \Delta_3$;

- (Γ, γ) *is a rigid crystallization*;

- *if* $h \geq 2$, *then* (Γ, γ) *contains at least two* $\{i, j\}$*-cycles with length greater or equal to six*, $\forall i, j \in \Delta_3$; *in particular, if* (Γ, γ) *contains exactly two* $\{i, j\}$*-cycles with length greater or equal to six* $(i, j \in \Delta_3)$, *then they both have length greater or equal to eight*;

- *if* $h \geq 2$, *then for any vertex* v *of* (Γ, γ) *and for any permutation* $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ *of* Δ_3, *at least one between the* $\{\varepsilon_0, \varepsilon_1\}$*-cycle of* (Γ, γ) *containing* v *and the* $\{\varepsilon_2, \varepsilon_3\}$*-cycle of* (Γ, γ) *containing* v *have length greater or equal to six*;

- (Γ, γ) *lacks in generalized dipoles*.

Proof. First of all note that, if $h \leq 1$, then the first three conditions are satisfied only by the standard order two crystallization of $\mathbb{S}^3$ and by the standard order eight crystallization of $L(2, 1) = \mathbb{RP}^3$, for which also the last condition holds. If $h \geq 2$ is assumed, then the statement follows almost directly - as already pointed out - from Proposition 12, together with Proposition 7 and Proposition 8. Note, however, that the condition of (Γ, γ) being cluster-less is not included in the above list: in fact, it is easy to check that the condition about bicoloured cycles of opposite colours containing the same vertex is actually stronger than cluster-less one (for details, see [6; Definitions 6.1-6.2 and Proposition 6.3] together with [14; Proposition 24]). □

Remark. Note that - apart from the well-known case of minimal 3-manifolds of genus zero and one - a gem (Γ, γ) satisfying all conditions of Proposition 14 must have at least 26 vertices. In fact, if (Γ, γ) contains no $\{i, j\}$-cycle with length six and exactly two $\{i, j\}$-cycles with length greater or equal to eight, $r_{ij} = 2$ holds, and hence $h_{ij} \geq 2$ follows, i.e., (Γ, γ) contains at least four $\{i, j\}$-cycles with length four. In this case, $\#V(\overline{\Gamma}) \geq 2 \cdot 8 + 4 \cdot 4 = 32$ follows. On the other hand, if (Γ, γ) contains at least three $\{i, j\}$-cycle with length greater or equal to six, then $\#V(\overline{\Gamma}) \geq 3 \cdot 6 + 2 \cdot 4 = 26$ follows.

A direct analysis of the existing crystallization catalogue $\mathcal{C}^{(30)}$ (resp. $\tilde{\mathcal{C}}^{(30)}$) representing all orientable (resp. non-orientable) 3-manifolds admitting coloured triangulations up to 30 tetrahedra (see [6], together with previous work [14], for the orientable case and [2], together with previous work [4], for the non-orientable case) enables to check that the standard order two crystallization of $\mathbb{S}^3$ and the standard order eight crystallization of $L(2, 1) = \mathbb{RP}^3$ are the only elements of the above catalogues satisfying all conditions of Proposition 14. This proves - without making use of the topological recognition of the manifolds involved in catalogues $\mathcal{C}^{(30)}$ and $\tilde{\mathcal{C}}^{(30)}$ -the validity of the conjecture stated in paragraph 1 up to regular genus four:

Corollary 15. *Let* M^3 *be a closed connected minimal* 3-*manifold*, $M^3 \neq \mathbb{S}^3$. *If* $\mathcal{G}(M^3) = h \leq 4$, *then* $M^3 = N_1 \# N_2 \# \cdots \# N_h$, *where* N_i *is homeomorphic to either* $L(2, 1) = \mathbb{RP}^3$ *or* $\mathbb{S}^2 \underset{|\sim|}{\times} \mathbb{S}^1$, *for each* $i = 1, ..., h$.

In a forthcoming paper, conditions listed in Proposition 14 will be used to implement a computer program generating a catalogue of all prime handle-free minimal 3-manifolds (if any) with fixed gem-complexity.

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