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NON-COMMUTATIVITY OF SELF-HOMOTOPY GROUPS OF SOME SIMPLE *p*-COMPACT GROUPS

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Abstract

The non-commutativity of the self-homotopy groups $\mathcal{H}(X) = [X, X]_*$ of some simple p-compact groups X is studied, by extending the method due to Kono and Ōshima for compact Lie groups.

1. Introduction

Kono and Ōshima [7] studied the non-commutativity of the self-homotopy groups $\mathcal{H}(G)$ for compact Lie groups G. As an extension of their work, we consider the case of simple p-compact groups X.

Let p be an odd prime. The mod p cohomology $H^*(X; \mathbb{Z}/p)$ of a simply connected p-torsion free p-compact group X is an exterior algebra with primitive generators of odd degree:

$$H^*(X; \mathbb{Z}/p) = \Lambda(x_1, ..., x_k),$$

$$\deg x_i = 2n_i - 1 \quad (n_1 \leq \cdots \leq n_k), \quad x_i : \text{primitive}.$$

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We call the set of odd numbers $(2n_1 - 1, ..., 2n_k - 1)$ the type of *X*. Here, a *p*-compact group is a space *X* with finite mod *p* cohomology $H^*(X; \mathbb{Z}/p)$ together with a homotopy equivalence $X \to \Omega BX$ to the loop space of a path connected *p*-complete space *BX*. The space *BX* is called the *classifying space* of *X*. Then $H^*(BX; \mathbb{Z}/p)$ is a finitely generated polynomial algebra:

$$H^*(BX; \mathbb{Z}/p) = \mathbb{Z}/p[y_1, ..., y_k], \quad \deg y_i = 2n_i.$$

We call the set of even numbers $(2n_1, ..., 2n_k)$ the type of the above polynomial algebra.

Let *X* be a simply connected *p*-torsion free *p*-compact group. By Dwyer et al. [3, Theorem 1.1], there exist a *p*-adic pseudoreflection group $W \subset GL(k, \mathbb{Z}_p^\wedge)$ and a map $f : BT \to BX$ equivariant up to homotopy to the projection with respect to the natural action of *W* on *BT* such that *f* induces an isomorphism $H^*(BX; \mathbb{Z}/p) \cong H^*(BT; \mathbb{Z}/p)^W$, where $\mathbb{Z}_p^\wedge$ is the ring of *p*-adic integers, and *BT* is the classifying space of the *p*-completion of the *k*-dimensional torus T^k. The space *BX* is called a *realization* of *W*.

All irreducible *p*-adic pseudoreflection groups are classified by Clark and Ewing [2], using the classification of irreducible pseudoreflection groups over $\mathbb{C}$ by Shephard and Todd [12]. By a simple *p*-compact group, we mean a simply connected *p*-compact group *X* such that *BX* realizes one of the irreducible *p*-adic pseudoreflection groups in Clark-Ewing list.

Which simple *p*-compact group is homotopy commutative is determined by McGibbon and Saumell.

Theorem 1.1 ([9, Theorem 4], [11, Theorem 1.1])**.** *The multiplication of a simple p-compact group X of type* $(2n_1 - 1, ..., 2n_k - 1)$ *is homotopy commutative if and only if*

(i) $p > 2n_k$, *or*

(ii) *X is p-equivalent to* $B_1(p)$ *for p odd*, $B_7(17)$, $B_5(19)$, $B_{19}(41)$, $B_{11}(19)$, $B_3(11) \times S^{11}$ *or* $B_1(19) \times B_{11}(19)$.

If the multiplication of X is homotopy commutative, then $\mathcal{H}(X)$ is commutative. Thus, if X satisfies (i) or (ii) in the above theorem, then $\mathcal{H}(X)$ is commutative.

On the other hand, even if the multiplication of X is not homotopy commutative, then $\mathcal{H}(X)$ can be commutative. Lie group S^3 is such an example: S^3 is not homotopy commutative, but $\mathcal{H}(S^3) = \pi_3(S^3) \cong \mathbb{Z}$ is commutative. Thus, our problem is to determine simple p-compact groups X such that the groups $\mathcal{H}(X)$ are not commutative.

To study this problem we first show the following theorem:

Theorem 1.2. *Suppose that a loop space X is homotopy equivalent to a product space $X_1 \times X_2$. Let $f : S^n \to X_1$ and $g : S^m \to X_2$ be continuous maps. We consider $\alpha_1 \in \pi_n(X_1 \times X_2)$ and $\alpha_2 \in \pi_m(X_1 \times X_2)$ defined by*

$$\alpha_1(x) = (f(x), *) \text{ and } \alpha_2(y) = (*, g(y)).$$

If the Samelson product $\langle \alpha_1, \alpha_2 \rangle$ is non-trivial, then the self-homotopy group $\mathcal{H}(X)$ is not commutative.

Kono and Ōshima [7, Lemma 2.4] showed the above theorem in the case that the space X has S^n and S^m as product factors.

By using Theorem 1.2, we get the following theorem which is a partial answer to the problem.

Theorem 1.3. *If X is a simple p-compact group of the following type for a prime p, then the self-homotopy group $\mathcal{H}(X)$ of X is not commutative.*

(1) $(7, 11), (3, 9, 11, 15, 17, 23)$ *for* $p = 7$.

(2) $(15, 23), (11, 17, 23), (3, 11, 15, 23)$,

$(7, 11, 19, 23, 35), (3, 9, 11, 15, 17, 23)$ *for* $p = 13$.

(3) $(15, 23, 39, 47)$ *for* $p = 17$.

(4) $(3, 11, 15, 23), (3, 9, 11, 15, 17, 23)$,

$(3, 11, 15, 19, 23, 27, 35), (3, 15, 23, 27, 35, 39, 47, 59)$ *for* $p = 19$.

(5) $(39, 59)$, $(23, 59)$, $(3, 23, 39, 59)$,

$(23, 35, 47, 59)$, $(3, 15, 23, 27, 35, 39, 47, 59)$ *for* $p = 31$.

(6) $(3, 23, 39, 59)$ *for* $p = 41$.

2. Proof of Theorem 1.2

From now on we assume that p is a fixed odd prime.

Proof of Theorem 1.2. Let $p : S^n \times S^m \to S^n \wedge S^m \approx S^{n+m}$ be the quotient map and $p_1 : S^n \times S^m \to S^n$ and $p_2 : S^n \times S^m \to S^m$ the projection maps. Then we consider the following diagram:

$$[S^{n+m}, X_1 \times X_2]_* \xrightarrow{p^*} [S^n \times S^m, X_1 \times X_2]_* \xleftarrow{(f \times g)^*} [X_1 \times X_2, X_1 \times X_2]_*.$$

By the definition of the Samelson product, we have

$$p^*(\langle \alpha_1, \alpha_2 \rangle) = (p_1^* \alpha_1) \cdot (p_2^* \alpha_2) \cdot (p_1^* \alpha_1)^{-1} \cdot (p_2^* \alpha_2)^{-1}.$$

Here, we consider the commutator $[i_1] \cdot [i_2] \cdot [i_1]^{-1} \cdot [i_2]^{-1} \in [X_1 \times X_2, X_1 \times X_2]_*$, where $i_1, i_2 : X_1 \times X_2 \to X_1 \times X_2$ are defined by $i_1(x, y) = (x, *)$, $i_2(x, y) = (*, y)$. Then it is clear that

$$p^*(\langle \alpha_1, \alpha_2 \rangle) = (f \times g)^*([i_1] \cdot [i_2] \cdot [i_1]^{-1} \cdot [i_2]^{-1}).$$

We will show that p^* is an injection. In fact, the inclusion $\Sigma i : \Sigma(S^n \vee S^m) \to \Sigma(S^n \times S^m)$ has a homotopy left inverse $r : \Sigma(S^n \times S^m) \to \Sigma(S^n \vee S^m) : r \circ \Sigma i \simeq id$. Thus, the map $S^n \wedge S^m \to \Sigma(S^n \vee S^m)$ in the cofibers sequence $S^n \times S^m \xrightarrow{p} S^n \wedge S^m \to \Sigma(S^n \vee S^m) \to \Sigma(S^n \times S^m)$ is null-homotopic, and so p^* is an injection. Thus, we have

$$[i_1] \cdot [i_2] \cdot [i_1]^{-1} \cdot [i_2]^{-1} \neq 0,$$

since $\langle \alpha_1, \alpha_2 \rangle \neq 0$. Therefore, $\mathcal{H}(X) \cong \mathcal{H}(X_1 \times X_2)$ is not commutative. □

Now to show the non-triviality of Samelson product $\langle \alpha_1, \alpha_2 \rangle$ in Theorem 1.2, we study the cohomology of the classifying space of X as follows:

Lemma 2.1. *Let* $X \simeq \Omega BX$ *be a loop space. Suppose that* $X \simeq X_1 \times X_2$ *for some* X_1 *and* X_2, *and*

$$H^*(BX; \mathbb{Z}/p) = \mathbb{Z}/p[y_1, \ldots, y_k].$$

We assume that there are a, i, j *and* l *with* $a > 0$ *and* $i < j$ *such that*

$$\mathcal{P}^a(y_l) \equiv \sum_{s \le t} c_{s,t} y_s y_t \mod D^3H^*(BX; \mathbb{Z}/p) \quad \text{with} \quad c_{i,j} \neq 0,$$

where $D^3H^*(BX; \mathbb{Z}/p)$ *is the module of* 3*-fold decomposables in* $H^*(BX; \mathbb{Z}/p)$. *Furthermore, we suppose also that there are maps* $f : S^n \to X_1$ *and* $g : S^m \to X_2$ *such that the adjoint maps*

$$\beta_1 : S^{n+1} \to BX, \quad \beta_2 : S^{m+1} \to BX$$

of α_1 *and* α_2, *respectively, satisfy*

$$\beta_1^*(y_i) \neq 0, \ \beta_1^*(y_s) = 0 \ (\forall s \neq i), \ \beta_2^*(y_j) \neq 0, \ \beta_2^*(y_t) = 0 \ (\forall t \neq j),$$

for the maps $\alpha_1 : S^n \xrightarrow{f} X_1 \subset X_1 \times X_2 \simeq \Omega BX$ *and* $\alpha_2 : S^m \xrightarrow{g} X_2 \subset X_1 \times X_2 \simeq \Omega BX$ *are defined in Theorem* 1.2. *Then Samelson product* $\langle \alpha_1, \alpha_2 \rangle$ *is non-trivial.*

Proof. Suppose that Whitehead product $[\beta_1, \beta_2]$ is trivial. Then there is a continuous map

$$\mu : S^{n+1} \times S^{m+1} \to BX$$

such that $\mu(x, *) = \beta_1(x)$, $\mu(*, y) = \beta_2(y)$. We consider the homomorphism

$$\mu^* : H^*(BX; \mathbb{Z}/p) \to H^*(S^{n+1} \times S^{m+1}; \mathbb{Z}/p) \cong H^*(S^{n+1}; \mathbb{Z}/p) \otimes H^*(S^{m+1}; \mathbb{Z}/p).$$

Taking i and j $(i < j)$ in the assumption, we have

$$\mu^*(y_i) = \beta_1^*(y_i) \otimes 1, \quad \mu^*(y_j) = 1 \otimes \beta_2^*(y_j),$$

$$\mu^*(y_i y_j) = \mu^*(y_i)\mu^*(y_j) = \beta_1^*(y_i) \otimes \beta_2^*(y_j) \neq 0.$$

Since

$$\mathcal{P}^a(y_l) \equiv \sum_{s \le t} c_{s,t} y_s y_t \mod D^3 H^*(BX; \mathbb{Z}/p) \quad \text{with} \quad c_{i,j} \ne 0,$$

for l in the assumption, we have

$$c_{i,j}\mu^*(y_i y_j) = \mu^*(\mathcal{P}^a(y_l)) = \mathcal{P}^a \mu^*(y_l) = 0.$$

This is a contradiction. Therefore, Whitehead product $[\beta_1, \beta_2]$ is non-trivial. Thus, Samelson product $\langle \alpha_1, \alpha_2 \rangle$ is non-trivial. □

3. Proof of Theorem 1.3

First, we recall the following:

Theorem 3.1 ([4], [8]). *Let X be a simple p-compact group of type* $(2n_1 - 1, \ldots, 2n_k - 1)$.

(1) *If* $2n_k - 2n_1 < 2(p-1)$, *then X is p-regular*.

(2) *If* $2n_k - 2n_1 < 4(p-1)$, *then X is quasi p-regular*.

Here, a simply connected H-space is called *p-regular* if it is p-equivalent to a product of odd-dimensional spheres and is called *quasi p-regular* if it is p-equivalent to a product of odd-dimensional spheres and $B_n(p)$'s. Here, $B_n(p)$ is a space introduced in [10] and the mod p cohomology of $B_n(p)$ is as follows:

$$H^*(B_n(p); \mathbb{Z}/p) = \Lambda(x, y),$$

$$\deg x = 2n + 1, \quad \deg y = 2n + 2p - 1, \quad \mathcal{P}^1(x) = y.$$

Now, we show Theorem 1.3 by dividing the proof into several cases.

Lemma 3.2. *The self-homotopy groups of the simple p-compact groups of the following types are not commutative*:

$(7, 11)$	*for*	$p = 7$,
$(15, 23)$	*for*	$p = 13$, *and*
$(23, 59), (39, 59)$	*for*	$p = 31$.

Proof. We only give the proof for the type $(7, 11)$ with $p = 7$ since the proofs for the other types are similar.

Let X be a simple p-compact group of type $(7, 11)$ for $p = 7$. Then X is p-regular:

$$X \simeq_7 S^7 \times S^{11}.$$

In $H^*(BX; \mathbb{Z}/7) \cong \mathbb{Z}/7[y_8, y_{12}]$, we have

$$\mathcal{P}^1(y_8) = hy_8y_{12},$$

where $\deg y_i = i$. We see $h \neq 0$, since otherwise $y_8^7 = \mathcal{P}^4(y_8) = (4!)^{-1}(\mathcal{P}^1)^4(y_8) = 0$, which is a contradiction. So, the assumptions of Lemma 2.1 are satisfied by taking the identity maps $1 : S^7 \to S^7$ and $1 : S^{11} \to S^{11}$ for f and g. Thus, the self-homotopy group of X is not commutative by Theorem 1.2 and Lemma 2.1. □

Lemma 3.3. *The self-homotopy group of the simple p-compact group of type* $(11, 17, 23)$ *with* $p = 13$ *is not commutative.*

Proof. Let X be a simple p-compact group of type $(11, 17, 23)$ with $p = 13$. Then X is p-regular:

$$X \simeq_{13} S^{11} \times S^{17} \times S^{23}.$$

In $H^*(BX; \mathbb{Z}/13) \cong \mathbb{Z}/13[y_{12}, y_{18}, y_{24}]$, we have

$$\mathcal{P}^1(y_{12}) = h_1y_{12}^3 + h_2y_{12}y_{24} + h_3y_{18}^2,$$

$$\mathcal{P}^1(y_{18}) = l_1y_{12}^2y_{18} + l_2y_{18}y_{24},$$

where $\deg y_i = i$. We see $(h_2, l_2) \neq (0, 0)$. In fact, if $(h_2, l_2) = (0, 0)$, then the subalgebra $\mathbb{Z}/13[y_{12}, y_{18}]$ is a non-modular $\mathcal{A}_{(p)}$-algebra, where $\mathcal{A}_{(p)}$ is the mod p Steenrod algebra. According to Adams and Wilkerson [1, Theorem 1.2], the type of any non-modular polynomial $\mathcal{A}_{(p)}$-algebra must be a union of types in Clark-Ewing list, which is not the case. Thus, we have $(h_2, l_2) \neq (0, 0)$.

If $h_2 \neq 0$, we can see that the assumptions of Lemma 2.1 are satisfied by taking the identity map $1 : S^{11} \to S^{11}$ and the inclusion map $i : S^{23} \to S^{17} \times S^{23}$ for f and g with $X_1 = S^{11}$ and $X_2 = S^{17} \times S^{23}$.

On the other hand, if $l_2 \neq 0,$ then we can see that the assumptions of Lemma 2.1 are satisfied by taking the identity map $1 : S^{17} \to S^{17}$ and the inclusion map $i : S^{23} \to S^{11} \times S^{23}$ for f and g with $X_1 = S^{17}$ and $X_2 = S^{11} \times S^{23}$. Thus the self-homotopy group of X is not commutative by Theorem 1.2 and Lemma 2.1. □

Lemma 3.4. *The self-homotopy groups of the simple p-compact groups of the following types are not commutative*:

$$(3, 23, 39, 59) \quad \textit{for} \quad p = 31 \textit{ or } 41,$$

$$(23, 35, 47, 59) \quad \textit{for} \quad p = 31, \textit{ and}$$

$$(3, 9, 11, 15, 17, 23) \quad \textit{for} \quad p = 13.$$

Proof. Let X be a simple p-compact group of type $(3, 23, 39, 59)$ with $p = 31$. Then X is p-regular:

$$X \simeq_{31} S^3 \times S^{23} \times S^{39} \times S^{59}.$$

In $H^*(BX; \mathbb{Z}/31) \cong \mathbb{Z}/31[y_4, y_{24}, y_{40}, y_{60}],$ we have

$$\mathcal{P}^1(y_4) \equiv h_1 y_4 y_{60} + h_2 y_{24} y_{40} \mod D^3 H^*(BX; \mathbb{Z}/31),$$

where $\deg y_i = i$. We see $h_2 \neq 0$. In fact, if $h_2 = 0,$ then the ideal (y_4) generated by y_4 is closed under the action of $\mathcal{A}_{(p)}$. Therefore,

$$\mathbb{Z}/31[y_4, y_{24}, y_{40}, y_{60}]/(y_4) \cong \mathbb{Z}/31[y_{24}, y_{40}, y_{60}]$$

is a non-modular $\mathcal{A}_{(p)}$-algebra. Hence, as in the proof of Lemma 3.3, we have a contradiction, and so $h_2 \neq 0$.

Then we can see that the assumptions of Lemma 2.1 are satisfied by taking the inclusion maps $i_1 : S^{23} \to S^3 \times S^{23}$ and $i_2 : S^{39} \to S^{39} \times S^{59}$ for f and g. Thus, the self-homotopy group of X is not commutative by Theorem 1.2 and Lemma 2.1.

For the other spaces, we use the same method. In fact, for the type (3, 23, 39, 59) with $p = 41$ the coefficient of $y_{24} y_{60}$ in $\mathcal{P}^1(y_4)$ is non-zero, while for the type (23, 35, 47, 59) with $p = 31$ the coefficient of $y_{36} y_{48}$ in $\mathcal{P}^1(y_{24})$ is non-zero. On the other hand, for the last one for $p = 13,$ we can show that at least one of the coefficients of $y_{10}y_{18}$ and $y_{12}y_{16}$ in $\mathcal{P}^1(y_4)$ is non-zero. □

Lemma 3.5. *The self-homotopy groups of the simple p-compact groups of the following types are not commutative*:

$$(15, 23, 39, 47) \quad \textit{for} \quad p = 17, \textit{ and}$$
$$(7, 11, 19, 23, 35) \quad \textit{for} \quad p = 13.$$

Proof. Let X be a simple p-compact group of type $(15, 23, 39, 47)$ with $p = 17.$ Then X is quasi p-regular and we have

$$X \simeq_{17} B_7(17) \times S^{23} \times S^{39},$$

since X is not p-regular by [6]. In $H^*(BX; \mathbb{Z}/17) \cong \mathbb{Z}/17\,[y_{16}, y_{24}, y_{40}, y_{48}],$ we have

$$\mathcal{P}^1(y_{24}) \equiv hy_{16}y_{40} \bmod D^3H^*(BX; \mathbb{Z}/17),$$

where $\deg y_i = i.$ Furthermore, we can show $h \neq 0$ by using the same method as in the proof of Lemma 3.4.

So, we can see that the assumptions of Lemma 2.1 are satisfied by taking the inclusion maps $i_1 : S^{15} \to B_7(17)$ and $i_2 : S^{39} \to S^{23} \times S^{39}$ for f and g. Thus, the self-homotopy group of X is not commutative by Theorem 1.2 and Lemma 2.1.

We can show the non-commutativity for the type $(7, 11, 19, 23, 35)$ with $p = 13$ by using the same method. In fact, we can show that the coefficient of $y_{12}y_{20}$ in $\mathcal{P}^1(y_8)$ is non-zero. □

Lemma 3.6. *The self-homotopy groups of the simple p-compact groups of the following types are not commutative*:

$$(3, 11, 15, 23) \quad \textit{for} \quad p = 13,$$
$$(3, 9, 11, 15, 17, 23) \quad \textit{for} \quad p = 7, \quad \textit{and}$$
$$(3, 15, 23, 27, 35, 39, 47, 59) \quad \textit{for} \quad p = 19.$$

Proof. Let X be a simple p-compact group of type $(3, 11, 15, 23)$ with $p = 13.$

Then X is p-regular:

$$X \simeq_{13} S^3 \times S^{11} \times S^{15} \times S^{23}.$$

In $H^*(BX; \mathbb{Z}/13) \cong \mathbb{Z}/13[y_4, y_{12}, y_{16}, y_{24}]$, we have

$$\mathcal{P}^1(y_{16}) \equiv h y_{16} y_{24} \mod (y_4, y_{12})$$

for some h, where $\deg y_i = i$ and (y_4, y_{12}) is the ideal generated by y_4 and y_{12}. We see $h \neq 0$. In fact, if $h = 0$, then we have $\mathcal{P}^1(y_{16}) \in (y_4, y_{12})$. Furthermore, we have $\mathcal{P}^1((y_4, y_{12})) \subset (y_4, y_{12})$. Therefore

$$y_{16}^{13} = \mathcal{P}^8(y_{16}) = (8!)^{-1}(\mathcal{P}^1)^8(y_{16}) \in (y_4, y_{12}).$$

This is a contradiction and we have $h \neq 0$.

So, we see that the assumptions of Lemma 2.1 are satisfied by taking the inclusion maps $i_1 : S^{15} \to S^3 \times S^{15}$ and $i_2 : S^{23} \to S^{11} \times S^{23}$ for f and g. Thus, the self-homotopy group of X is not commutative by Theorem 1.2 and Lemma 2.1.

Let X be a simple p-compact group of type $(3, 9, 11, 15, 17, 23)$ with $p = 7$. Then X is quasi p-regular. Furthermore, by [5, Theorem 1.2], we can choose primitive generators x_i of $H^*(X; \mathbb{Z}/7)$ with $\deg x_i = i$ such that $\mathcal{P}^1(x_3) = x_{15}$ and $\mathcal{P}^1(x_{11}) = x_{23}$. Thus,

$$X \simeq_7 B_1(7) \times B_5(7) \times S^9 \times S^{17}.$$

In $H^*(BX; \mathbb{Z}/7) \cong \mathbb{Z}/7[y_4, y_{10}, y_{12}, y_{16}, y_{18}, y_{24}]$, we have

$$\mathcal{P}^1(y_{10}) \equiv h y_{10} y_{12} \mod (y_4, y_{16})$$

for some h, where $\deg y_i = i$ and (y_4, y_{16}) is the ideal generated by y_4 and y_{16}. We see $h \neq 0$. In fact, if $h = 0$, then we have $\mathcal{P}^1(y_{10}) \in (y_4, y_{16})$. Furthermore, the generators y_i can be chosen to satisfy $\mathcal{P}^1(y_4) = y_{16}$, and so $\mathcal{P}^1(y_{16}) = \mathcal{P}^1\mathcal{P}^1(y_4)$ $= 2y_4^7$. Thus, we have $\mathcal{P}^1((y_4, y_{16})) \subset (y_4, y_{16})$. Therefore,

$$y_{10}^7 = \mathcal{P}^5(y_{10}) = (5!)^{-1}(\mathcal{P}^1)^5(y_{10}) \in (y_4, y_{16}).$$

This is a contradiction and we have $h \neq 0$. Thus, we can apply Theorem 1.2 and Lemma 2.1 to prove the desired fact.

Let X be a simple p-compact group of type $(3, 15, 23, 27, 35, 39, 47, 59)$ with $p = 19$. Then X is quasi p-regular. Furthermore, by [5, Theorem 1.2], we can choose primitive generators x_i of $H^*(X; \mathbb{Z}/19)$ with $\deg x_i = i$ such that $\mathcal{P}^1(x_3) = x_{39}$ and $\mathcal{P}^1(x_{23}) = x_{59}$. Thus,

$$X \simeq_{19} B_1(19) \times B_{11}(19) \times S^{15} \times S^{27} \times S^{35} \times S^{47}.$$

In $H^*(BX; \mathbb{Z}/19) \cong \mathbb{Z}/19[y_4, y_{16}, y_{24}, y_{28}, y_{36}, y_{40}, y_{48}, y_{60}]$, we have

$$\mathcal{P}^1(y_{16}) \equiv h_1 y_{16} y_{36} + h_2 y_{24} y_{28} \mod (y_4, y_{40})$$

for some h_1 and h_2, where $\deg y_i = i$ and (y_4, y_{40}) is the ideal generated by y_4 and y_{40}. We see $(h_1, h_2) \neq (0, 0)$. In fact, if $(h_1, h_2) = (0, 0)$, then we have $\mathcal{P}^1(y_{16}) \in (y_4, y_{40})$. Furthermore, the generators y_i can be chosen to satisfy $\mathcal{P}^1(y_4) = y_{40}$, and so $\mathcal{P}^1(y_{40}) = \mathcal{P}^1\mathcal{P}^1(y_4) = 2y_4^{19}$. Thus, we have $\mathcal{P}^1((y_4, y_{40})) \subset (y_4, y_{40})$. Therefore,

$$y_{16}^{19} = \mathcal{P}^8(y_{16}) = (8!)^{-1}(\mathcal{P}^1)^8(y_{16}) \in (y_4, y_{40}).$$

This is a contradiction and we have $(h_1, h_2) \neq (0, 0)$. Thus, we can apply Theorem 1.2 and Lemma 2.1 to prove the desired result. □

Lemma 3.7. *The self-homotopy groups of the simple p-compact groups of the following types are not commutative*:

$(3, 11, 15, 23)$	*for*	$p = 19$,
$(3, 9, 11, 15, 17, 23)$	*for*	$p = 19$,
$(3, 11, 15, 19, 23, 27, 35)$	*for*	$p = 19$, *and*
$(3, 15, 23, 27, 35, 39, 47, 59)$	*for*	$p = 31$.

Proof. This lemma is proved by Kono and Ōshima [7]. In fact, these spaces are the p-completion of the Lie groups F_4, E_6, E_7 and E_8. □

Proof of Theorem 1.3 is completed through Lemmas 3.2-3.7.

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