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ONE-POINT EXTENSIONS OF λ-KOSZUL ALGEBRAS

WANG YUE-FENG[1], **PAN YUAN**[2], **LU JIA-FENG**[3] **and CHEN MIAO-SEN**[1,*]

[1,3]Department of Mathematics
Zhejiang Normal University
Jinhua, Zhejiang, 321004, P. R. China
e-mail: yfwlyr@qq.com
mschen@zjnu.edu.cn
jiafenglv@zjnu.edu.cn

[2]Yiwu Industrial and Commercial College
Yiwu, Zhejiang, 322000, P. R. China
e-mail: mathpanyuan@126.com

Abstract

In this paper, we mainly discuss the one-point extensions of λ-Koszul algebras. More precisely, we mainly discuss the "λ-Koszulity" of the extension algebra of the forms $E_M^A := \begin{pmatrix} A & M \\ 0 & \Bbbk \end{pmatrix}$, where $\Bbbk$ is a fixed ground field, *A* is a positively graded algebra and *M* is a finitely generated graded *A*-module.

1. Preliminaries

λ-*Koszul algebra*, another class of "Koszul-type" algebras possessing a lot of beautiful homological properties similar to Koszul algebras (see [5]), was first introduced by Lü in [3] in 2009. However, it is a pity that [3] does not provide enough examples of such algebras. Therefore, it is important, of course interesting to

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[*]Corresponding author

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construct new λ-Koszul algebras from the given ones. In this paper, by using the matrix-construction method: “One-point extension”, we mainly discuss how to construct new λ-Koszul algebras from the known ones. More precisely, we mainly discuss the conditions for the extension algebra

$$E_M^A := \begin{pmatrix} A & M \\ 0 & \mathbb{k} \end{pmatrix}$$

to be λ-Koszul, where A is a positively graded $\mathbb{k}$-algebra and M is a finitely generated graded A-module.

The name of *positively graded algebra* will stand for a graded $\mathbb{k}$-algebra $A = \oplus_{i \geq 0} A_i$ such that $A_0 = \mathbb{k} \times \cdots \times \mathbb{k}$ is a finite product of the fixed ground field $\mathbb{k}$ and A_1 is of finite dimension as a $\mathbb{k}$-space. A graded $\mathbb{k}$-algebra $A = \oplus_{i \geq 0} A_i$ is called *locally finite* if for all $i \geq 0$, $\dim_{\mathbb{k}} A_i < \infty$. If the graded $\mathbb{k}$-algebra $A = \oplus_{i \geq 0} A_i$ is positively, locally finite and generated in degrees 0 and 1, i.e., $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$, then A is called *standard*.

Let A be a $\mathbb{k}$-algebra and M be an A-module (not necessary graded). In general, the *one-point extension of A by M* is defined to be the upper triangular matrix algebra $\begin{pmatrix} A & M \\ 0 & \mathbb{k} \end{pmatrix}$, we call the upper triangular matrix algebra *one-point extension algebra*, denoted by E_M^A. It is obvious that the elements in the extension algebra E_M^A have the form: $\begin{pmatrix} a & m \\ 0 & k \end{pmatrix}$. Hence we have

$$E_M^A = \left\{ \begin{pmatrix} a & m \\ 0 & k \end{pmatrix} \mid a \in A, m \in M, k \in \mathbb{k} \right\}.$$

The addition and the multiplication in E_M^A are given by the matrix addition and matrix multiplication, that is,

$$\begin{pmatrix} a_1 & m_1 \\ 0 & k_1 \end{pmatrix} + \begin{pmatrix} a_2 & m_2 \\ 0 & k_2 \end{pmatrix} := \begin{pmatrix} a_1 + a_2 & m_1 + m_2 \\ 0 & k_1 + k_2 \end{pmatrix}$$

and

$$\begin{pmatrix} a_1 & m_1 \\ 0 & k_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & m_2 \\ 0 & k_2 \end{pmatrix} := \begin{pmatrix} a_1 a_2 & a_1 \cdot m_2 + k_2 \cdot m_1 \\ 0 & k_1 k_2 \end{pmatrix},$$

where all $a_i \in A$, $m_i \in M$, $k_i \in \Bbbk$, $i = 1, 2$.

Proposition 1.1. *Using the above notations, we have the following statements*:

(1) E_M^A *is a positively graded* $\Bbbk$*-algebra under the grading*: $(E_M^A)_0 = \begin{pmatrix} A_0 & 0 \\ 0 & \Bbbk \end{pmatrix}$ *and* $(E_M^A)_i = \begin{pmatrix} A_i & M_i \\ 0 & 0 \end{pmatrix}$ *for all integers* $i \geq 1$ *if and only if A is a positively graded* $\Bbbk$*-algebra and M is a positively graded A-module with* $\dim_{\Bbbk} M_1 < \infty$.

(2) E_M^A *is locally finite if and only if A is locally finite and each* $\dim_{\Bbbk} M_i < \infty$.

(3) E_M^A *is standard if and only if A is standard and M is a graded A-module generated in degree* 1.

Proof. (1) is immediate from the definition of the multiplication of the extension algebra E_M^A. For all $i \geq 1$, it is obvious that $\dim_{\Bbbk}(E_M^A)_i < \infty$ if and only if $\dim_{\Bbbk}\begin{pmatrix} A_i & M_i \\ 0 & 0 \end{pmatrix} < \infty$ if and only if both $\dim_{\Bbbk} A_i < \infty$ and $\dim_{\Bbbk} M_i < \infty$. Now we complete the proof of (2) since degree 0 part is obvious. For the last statement, note that $(E_M^A)_i \cdot (E_M^A)_j = (E_M^A)_{i+j}$ if and only if

$$\begin{pmatrix} A_i & M_i \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} A_j & M_j \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_i A_j & A_i M_j \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{i+j} & M_{i+j} \\ 0 & 0 \end{pmatrix}$$

if and only if $A_i A_j = A_{i+j}$ and $A_i M_j = M_{i+j}$. Now (3) is proved by combining (1) and (2). □

Proposition 1.2. *Let* E_M^A *be defined as above* (*A and M are not necessary graded*) *and* $\mathbf{I} := \begin{pmatrix} I_1 & N \\ 0 & I_2 \end{pmatrix}$. *Then* **I** *is an ideal of* E_M^A *if and only if* I_1 *and* I_2 *are ideals of A and* $\Bbbk$, *respectively*; *and N is a submodule of M such that* $IM \subseteq N$. *Moreover, the ideals of* E_M^A *only have two forms*: $\begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} I & N \\ 0 & \Bbbk \end{pmatrix}$, *where I is an ideal of A and N is a submodule of M such that* $IM \subseteq N$.

Proof. The first assertion is a routine check by using the definition of an ideal. The last statement is immediate from the first assertion and by noting that $\Bbbk$ is a simple ring. □

Now it is the time to recall the definition of λ-Koszul algebra.

Let $\lambda : \mathbb{N} \to \mathbb{N}$ be a periodic function; write the smallest positive period as $|\lambda|$. Assume that $\lambda(1) \geq 2$ and that λ is strictly increasing on the interval $[1, |\lambda|]$.

Introduce a function

$$\delta_\lambda : \mathbb{N} \to \mathbb{N}$$

with the following properties:

(a) $\delta_\lambda(0) = 0$, $\delta_\lambda(1) = 1$, $\delta_\lambda(2) = d$, where $d = \lambda(1) + 1$, a fixed integer;

(b) $\delta_\lambda(2n+1) - \delta_\lambda(2n) = 1$ for all $n \geq 0$;

(c) $\delta_\lambda(2n) - \delta_\lambda(2n-1) = \lambda(n)$ for all $n \geq 1$.

Definition 1.3. A positively graded algebra $A = \oplus_{i \geq 0} A_i$ is called a λ-*Koszul algebra* if the trivial *A*-module $\Bbbk$ admits a minimal graded projective resolution

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to A_0 \to 0$$

such that each P_n is generated in degree $\delta_\lambda(n)$ for all $n \geq 0$.

Throughout, let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}^*$ denote the set of integers, natural numbers and positive integers, and $\Bbbk$ denote an arbitrary ground field.

2. The λ-Koszulity of E_M^A

Define a functor

$$\mathbf{G} : gr(A) \to gr(E_M^A)$$

by

$$\mathbf{G}(X(A)) = (X(A), 0, 0)$$

and

$$\mathbf{G}(X(A) \xrightarrow{\quad f \quad} Y(A)) = ((X(A), 0, 0) \xrightarrow{\quad (f,0) \quad} (Y(A), 0, 0)),$$

where $X(A)$ and $Y(A)$ are finitely generated A-modules. From the definition of $\mathbf{G}$, it is clear that $\mathbf{G}$ is an exact faithful functor and if $P \in gr(A)$ is a graded A-projective module, then $\mathbf{G}(P)$ is a graded E_M^A -projective module.

Lemma 2.1. *Let* $e_1 = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ *and* $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\Bbbk} \end{pmatrix}$ *be a complete set of orthogonal idempotents of* E_M^A. *Then the category* $gr(A)$ *can be viewed as a full subcategory of* $gr(E_M^A)$ *consisting of the* E_M^A *-modules such that* $e_2 \cdot X = 0$, *where* $X \in gr(E_M^A)$.

Proof. It is trivial that for all $\begin{pmatrix} a & m \\ 0 & k \end{pmatrix} \in E_M^A$, we have $e_2 \begin{pmatrix} a & m \\ 0 & c \end{pmatrix} e_1 = 0$. That is to say, $e_2 E_M^A e_1 = 0$. Let $X = (X(A), V, \alpha) \in gr(E_M^A)$ such that $e_2 \cdot X = 0$. Since $e_2 \cdot X = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\Bbbk} \end{pmatrix} \cdot \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix} = 0$ for all $v \in V$, we get that $V = 0$ and $\alpha = 0$. It is easy to see that such an E_M^A -module X can be viewed as a graded A-module. That is $X \in gr(A)$. Conversely, for all $M \in gr(A)$, we have $(M, 0, 0) \in gr(E_M^A)$. Therefore we complete the proof. □

Lemma 2.2. $\begin{pmatrix} A \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} M \\ \Bbbk \end{pmatrix}$ *are the all non-isomorphic projective modules in the category* $gr(E_M^A)$. $\begin{pmatrix} A_0 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ \Bbbk \end{pmatrix}$ *are the all non-isomorphic simple modules in the category* $gr(E_M^A)$.

Proof. Note that $e_1 = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\Bbbk} \end{pmatrix}$ are a complete set of orthogonal idempotents of E_M^A. The results are proved by

$$E_M^A e_1 = \begin{pmatrix} A \\ 0 \end{pmatrix}, \quad E_M^A e_2 = \begin{pmatrix} M \\ \Bbbk \end{pmatrix}, \quad (E_M^A)_0 e_1 = \begin{pmatrix} A_0 \\ 0 \end{pmatrix} \quad \text{and} \quad (E_M^A)_0 e_2 = \begin{pmatrix} 0 \\ \Bbbk \end{pmatrix}. \quad \square$$

Lemma 2.3. *We can get a minimal graded projective resolution of* $(E_M^A)_0 = \begin{pmatrix} A_0 & 0 \\ 0 & \Bbbk \end{pmatrix}$ *from the minimal graded projective resolutions of* $\begin{pmatrix} A_0 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ \Bbbk \end{pmatrix}$.

Proof. Let

$$\mathcal{P} : \cdots \to \begin{pmatrix} P_n \\ 0 \end{pmatrix} \to \cdots \to \begin{pmatrix} P_1 \\ 0 \end{pmatrix} \to \begin{pmatrix} P_0 \\ 0 \end{pmatrix} \to \begin{pmatrix} A_0 \\ 0 \end{pmatrix} \to 0$$

and

$$\mathcal{Q} : \cdots \to \begin{pmatrix} Q_n \\ 0 \end{pmatrix} \to \cdots \to \begin{pmatrix} Q_1 \\ 0 \end{pmatrix} \to \begin{pmatrix} Q_0 \\ 0 \end{pmatrix} \to \begin{pmatrix} M \\ \Bbbk \end{pmatrix} \to \begin{pmatrix} 0 \\ \Bbbk \end{pmatrix} \to 0$$

be minimal graded projective resolutions of the related simple E_M^A -modules. Then it is easy to see that the following is the minimal graded projective resolutions of $(E_M^A)_0$:

$$\mathcal{L} : \cdots \to \begin{pmatrix} P_n \\ 0 \end{pmatrix} \oplus \begin{pmatrix} Q_{n-1} \\ 0 \end{pmatrix} \to \cdots \to \begin{pmatrix} P_0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \\ \Bbbk \end{pmatrix} \to (E_M^A)_0 \to 0. \qquad \square$$

Lemma 2.4. *Using the notations of Lemma* 2.3, *for all* $n \geq 1$,

$$L_n := \begin{pmatrix} P_n \\ 0 \end{pmatrix} \oplus \begin{pmatrix} Q_{n-1} \\ 0 \end{pmatrix}$$

is generated in degree s as a graded E_M^A *-module if and only if* P_n *and* Q_{n-1} *are generated in degree s as graded A-modules*.

Proof. Let $s \in \mathbb{N}$. Then $\begin{pmatrix} P_n \\ 0 \end{pmatrix} \oplus \begin{pmatrix} Q_{n-1} \\ 0 \end{pmatrix}$ is generated in degree s if and only if $\begin{pmatrix} P_n \\ 0 \end{pmatrix}$ and $\begin{pmatrix} Q_{n-1} \\ 0 \end{pmatrix}$ are generated in degree s as graded E_M^A-modules, which is equivalent to

$$\begin{pmatrix} P_n \\ 0 \end{pmatrix}_{s+1} = \begin{pmatrix} A & M \\ 0 & \Bbbk \end{pmatrix}_1 \cdot \begin{pmatrix} P_n \\ 0 \end{pmatrix}_s = \begin{pmatrix} A_1 & M_1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} (P_n)_s \\ 0 \end{pmatrix} = \begin{pmatrix} A_1 \cdot (P_n)_s \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} Q_{n-1} \\ 0 \end{pmatrix}_{s+1} = \begin{pmatrix} A & M \\ 0 & \Bbbk \end{pmatrix}_1 \cdot \begin{pmatrix} Q_{n-1} \\ 0 \end{pmatrix}_s = \begin{pmatrix} A_1 & M_1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} (Q_{n-1})_s \\ 0 \end{pmatrix} = \begin{pmatrix} A_1 \cdot (Q_{n-1})_s \\ 0 \end{pmatrix}.$$

$\square$

Theorem 2.5. *Let A be a standard graded algebra and M be a graded A-module generated in degree* 1. *Let* E_M^A *be the one-point extension algebra*. *Then using the notations of Lemma* 2.3, *the following are equivalent*:

(1) E_M^A *is a* λ*-Koszul algebra*;

(2) *A is a* λ*-Koszul algebra*, $\Omega^{2|\lambda|-1}(M)[-\delta_\lambda(2|\lambda|)]$ *is a* λ*-Koszul module and* Q_i *is generated in degree* $\delta_\lambda(i+1)$ *for all* $(i = 1, 2, \ldots, 2|\lambda|-2)$;

(3) *For all* $i \geq 0$, P_{i+1} *and* Q_i *are generated in degree* $\delta_\lambda(i+1)$.

Proof. E_M^A is a λ-Koszul algebra if and only if $(E_M^A)_0$ possesses a minimal graded projective resolution

$$\mathcal{L} : \cdots \to \begin{pmatrix} P_n \\ 0 \end{pmatrix} \oplus \begin{pmatrix} Q_{n-1} \\ 0 \end{pmatrix} \to \cdots \to \begin{pmatrix} P_0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \\ \Bbbk \end{pmatrix} \to (E_M^A)_0 \to 0$$

such that for all $n \geq 0$, $L_{n+1} := \begin{pmatrix} P_{n+1} \\ 0 \end{pmatrix} \oplus \begin{pmatrix} Q_n \\ 0 \end{pmatrix}$ is generated in degree $\delta_\lambda(n+1)$, which is equivalent to that of P_{n+1} and Q_n are generated in degree $\delta_\lambda(n+1)$ as graded *A*-modules by Lemma 2.4. Hence $(1) \Leftrightarrow (3)$. But (2) is obviously equivalent to that of P_{n+1} and Q_n are generated in degree $\delta_\lambda(n+1)$ as graded *A*-modules for all $n \geq 0$, which implies that $(2) \Leftrightarrow (3)$. □

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