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EIGENVALUES OF NONLINEAR HAMMERSTEIN EQUATIONS WITH KERNEL OF VARIABLE SIGN

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Abstract

The eigenvalues and eigenvectors for a class of nonlinear Hammerstein integral equations are considered in the paper, where the sign of the kernel $k(x, y)$ may be variable. Weakening the conditions of Sun and Lou [4], the same results are still obtained.

1. Introduction

In this paper, we consider the eigenvalues and eigenvectors of nonlinear

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Hammerstein integral equation

$$\lambda\varphi(x) = \int_G k(x, y) f(y, \varphi(y)) dy = A\varphi(x). \tag{1.1}$$

In [1], Guo used the theory of Leray-Schauder degree to investigate the eigenvalues and eigenvectors of some completely continuous nonlinear operators and applied the abstract results to (1.1), where $k(x, y)$ must be nonnegative. In [4], Sun and Lou improved Theorem 1 in [1] and applied the result to (1.1), but the sign of $k(x, y)$ may be variable. But in [4] some conditions are too strong. We will show that we also can obtain the same results when conditions are weakened, thus the extent of application will be expanded.

We proved that, except for at most a sequence of numbers $\{\lambda_n\}$, all other numbers are eigenvalues of (1.1).

2. Main Result and its Proof

In the following, G always denotes a bounded closed domain in Euclidean space $\mathbf{R}^n$ and the kernel $k(x, y)$ is defined on $G \times G$. Consider the nonlinear Hammerstein integral equation (1.1). In this section, the discussion is in $C(G)$ space, $\|\cdot\|$ denotes the norm of $C(G)$, $\|\cdot\|_p$ denotes the norm of $L^p(G)$.

Theorem 2.1. *Suppose that* (i) $k(x, y)$ *is continuous on* $G \times G$ *and there exists a function* $h \in L^p(G)(p > 1)$ *such that* $\int_G h(x)k(x, y)dx > 0$ *for all* $y \in G$; (ii) $f(x, u)$ *is continuous on* $G \times \mathbf{R}$, $f(x, 0) \equiv 0$, $f'_u(x, u)$ *exists and is continuous for sufficiently small* $|u|$; *and* (iii) $f(x, u)$ *has the lower bound*, $\lim_{|u| \to +\infty} f(x, u)/|u| = +\infty$ *uniformly on* $x \in G_1 = G \backslash \{x \in G : h(x) = 0\}$. *Then*

(i) *for any* $\lambda \neq 0, \lambda \neq \lambda_n$, $n = 1, 2, \ldots, \lambda$ *is an eigenvalue of A*, *where* $\{\lambda_n\}$ *is the sequence of eigenvalues of the linear integral operator* $K_1 : C(G) \to C(G)$, *defined by*

$$K_1\varphi(x) = \int_G k(x, y) f'_u(y, 0)\varphi(y)dy; \tag{2.1}$$

(ii) $\lim_{|\lambda| \to +\infty} \|\varphi_\lambda\| = +\infty$, *where* φ_λ *is the eigenvector of A with respect to* λ;

(iii) *for any* $\lambda \neq 0, \lambda \neq \lambda_n$, $n = 1, 2, \ldots,$ *there exist* $\sigma = \sigma(\lambda) > 0$ *and* $R = R(\lambda) > 0$ *such that, for every* $\phi \in C(G)$ *satisfying* $0 < \|\phi\| < \sigma$, *the equation* $\lambda\varphi(x) = A\varphi(x) + \phi(x)$ *has at least two continuous solutions satisfying* $\varphi \not\equiv 0$ *and* $\|\varphi\| < R$.

Proof. The proof of conclusions (ii) and (iii) is similar to the proof of [1, Theorem 1]. We only need to prove the conclusion (i).

Let $\lambda, \lambda \neq \lambda_n$, $n = 1, 2, 3, \ldots,$ be fixed. Then by the Leray-Schauder theorem (see [3] or [2, Chapter 2, Theorem 2.6]), we can choose a sufficiently small number $r = r(\lambda) > 0$ such that

$$\deg(I - \lambda^{-1}A, T_r, \theta) = \pm 1, \tag{2.2}$$

where $T_r = \{\varphi \in C(G) : \|\varphi\| < r\}$, θ is the zero element of $C(G)$. Next we will show that there exist $\overline{\varphi} \in C(G)$, $\overline{\varphi} \neq \theta$ and a sufficiently large number $R > 0$ such that

$$\varphi_1 \neq \lambda^{-1}A\varphi_1 + \mu\overline{\varphi}, \tag{2.3}$$

for all $\varphi_1 \in C(G)$ satisfying $\|\varphi_1\| = R$ and a real number $\mu \geq 0$. If this is true, by the homotopy invariance property of the topological degree, we will obtain

$$\operatorname{ind}(\lambda^{-1}A, \infty) = 0. \tag{2.4}$$

It follows from (2.2) and (2.4) that there exists $\varphi_\lambda \in C(G)$, $r < \|\varphi_\lambda\| < R$ such that $\lambda^{-1}A\varphi_\lambda = \varphi_\lambda$, i.e., λ is an eigenvalue of A.

Now we will prove that (2.3) holds. Since $f(x, u)$ has the lower bound, there exists $b \geq 0$ such that

$$f(x, u) + b \geq 0 \quad \text{for all} \quad (x, u) \in G \times \mathbf{R}. \tag{2.5}$$

Set

$$M = \max_{(x, y) \in G \times G} |k(x, y)|, \quad \beta = \inf_{y \in G} \int_G h(x)k(x, y)dx > 0.$$

For any $N > 0$, by the condition (iii), there exists $u^* > 0$ such that $f(x, u) \geq$

$N|u|$ for any $|u| \geq u^*$, $x \in G_1$. Choose a number $a \geq 0$ such that

$$\inf_{x \in G_1, |u| \leq u^*} [f(x, u) - N|u|] \geq -a.$$

Then

$$f(x, u) \geq N|u| - a \text{ for all } x \in G_1, -\infty < u < +\infty. \tag{2.6}$$

Define a linear operator K and a nonlinear operator F as

$$K\varphi(x) = \int_G k(x, y)\varphi(y)dy, \quad F\varphi(x) = f(x, \varphi(x)),$$

then $A = KF$. Define a linear functional $(h, \varphi) = \int_G h(x)\varphi(x)dx$, $\varphi \in C(G)$.

Choose a number $R > 0$ and let $T_R = \{\varphi \in C(G) : \|\varphi\| < R\}$. Choose $\overline{\varphi}(x) = \operatorname{sgn}\lambda \cdot \int_G k(x, y)dy = \operatorname{sgn}\lambda \cdot (K1)(x)$. Suppose there exist $\varphi_1 \in \partial T_R$ and $\mu_1 \geq 0$ such that

$$\varphi_1 = \lambda^{-1}KF\varphi_1 + \mu_1\overline{\varphi}, \tag{2.7}$$

i.e.,

$$\lambda\varphi_1 = KF\varphi_1 + \mu_1\lambda \operatorname{sgn}\lambda \cdot K1 = K(F\varphi_1 + \mu_1|\lambda|).$$

Then

$$(h, \lambda\varphi_1) = \int_G h(x)\int_G k(x, y)[f(y, \varphi_1(y)) + \mu_1|\lambda|]dydx$$

$$= \int_G h(x)\int_G k(x, y)[f(y, \varphi_1(y)) + \mu_1|\lambda| + b - b]dydx$$

$$= \int_G [f(y, \varphi_1(y)) + b + \mu_1|\lambda|]\int_G h(x)k(x, y)dxdy - b\int_G h(x)\int_G k(x, y)dydx$$

$$\geq \beta\int_G [f(y, \varphi_1(y)) + b + \mu_1|\lambda|]dy - (h, Kb)$$

$$\geq \beta M^{-1}\int_G |k(x, y)|[f(y, \varphi_1(y)) + b + \mu_1|\lambda|]dy - (h, Kb)$$

$$\geq \beta M^{-1}\left|\int_G k(x, y)[f(y, \varphi_1(y)) + b + \mu_1|\lambda|]dy\right| - (h, Kb)$$

$$= \beta M^{-1}|(K(F\varphi_1 + b + \mu_1|\lambda|))(x)| - (h, Kb) \text{ for all } x \in G.$$

Therefore,

$$\begin{aligned}(h, \lambda\varphi_1) &\geq \beta M^{-1}\| K(F\varphi_1 + b + \mu_1|\lambda|)\| - (h, Kb)\\ &= \beta M^{-1}\| \lambda\varphi_1 + Kb\| - (h, Kb)\\ &\geq \beta M^{-1}|\lambda|\|\varphi_1\| - \beta M^{-1}\| Kb\| - (h, Kb)\\ &= \beta M^{-1}|\lambda| R - C_1, \qquad (2.8)\end{aligned}$$

where $C_1 = \beta M^{-1}\| Kb\| + (h, Kb)$. Since

$$\begin{aligned}(h, \lambda\varphi_1) &= \int_G \lambda h(x)\varphi_1(x)dx = \int_{G_1} \lambda h(x)\varphi_1(x)dx\\ &\leq |\lambda|\| h\|_{L^p(G_1)}\|\varphi_1\|_{L^q(G_1)} = |\lambda|\| h\|_p\|\varphi_1\|_{L^q(G_1)},\end{aligned}$$

where $1/p + 1/q = 1$, and

$$\|\varphi_1\|^q_{L^q(G_1)} = \int_{G_1}|\varphi_1(x)|^q dx = \int_{G_1}|\varphi_1(x)||\varphi_1(x)|^{q-1}dx \leq R^{q-1}\|\varphi_1\|_{L^1(G_1)},$$

we have

$$(h, \lambda\varphi_1) \leq |\lambda|\| h\|_p R^{1-1/q}\|\varphi_1\|^{1/q}_{L^1(G_1)}. \qquad (2.9)$$

Hence, from (2.8) as R is sufficiently large,

$$\| h\|_p R^{1-1/q}\|\varphi_1\|^{1/q}_{L^1(G_1)} \geq \beta M^{-1}R - C_1|\lambda|^{-1},$$

i.e.,

$$\begin{aligned}\|\varphi_1\|_{L^1(G_1)} &\geq (\beta M^{-1}R^{1/q}\| h\|_p^{-1} - C_1|\lambda|^{-1}R^{1/q-1}\| h\|_p^{-1})^q\\ &= (C_2R^{1/q} - C_3R^{1/q-1})^q\\ &= (C_2 - C_3R^{-1})^q R, \qquad (2.10)\end{aligned}$$

where $C_2 = \beta M^{-1} \| h \|_p^{-1}$, $C_3 = C_1 | \lambda |^{-1} \| h \|_p^{-1}$. Therefore, $\| \varphi_1 \|_{L^1(G_1)} > 0$ as R is sufficiently large.

On the other hand, since $\varphi_1 = \lambda^{-1} A\varphi_1 + \mu_1 \overline{\varphi} = \lambda^{-1} KF\varphi_1 + \operatorname{sgn} \lambda \cdot K\mu_1$ and

$$(h, K\mu_1) = \mu_1 \int_G h(x) \int_G k(x, y)\, dy dx$$

$$= \mu_1 \int_G \int_G h(x) k(x, y)\, dx dy \geq \beta\mu_1 \operatorname{mes} G \geq 0,$$

by (2.6) we have

$$(h, \lambda\varphi_1) = \int_G h(x) \int_G k(x, y)[f(y, \varphi_1(y)) + \mu_1 | \lambda |] dy dx$$

$$= \int_G h(x) \int_G k(x, y)[f(y, \varphi_1(y)) + b] dy dx + | \lambda | (h, K\mu_1) - (h, Kb)$$

$$\geq \int_G [f(y, \varphi_1(y)) + b] \int_G h(x) k(x, y)\, dx dy - (h, Kb)$$

$$\geq \beta \int_{G_1} [f(y, \varphi_1(y)) + b] dy - (h, Kb)$$

$$\geq \beta \int_{G_1} (N | \varphi_1(y) | - a + b) dy - (h, Kb)$$

$$= \beta N \int_{G_1} | \varphi_1(y) | dy + \beta(b - a) \operatorname{mes} G_1 - (h, Kb)$$

$$\geq \beta N \| \varphi_1 \|_{L^1(G_1)} - C_4, \tag{2.11}$$

where $C_4 = | (h, Kb) - \beta(b - a) \operatorname{mes} G_1 | \geq 0$. By (2.9) and (2.11), we have

$$\beta N \| \varphi_1 \|_{L^1(G_1)} - C_4 \leq | \lambda | \| h \|_p R^{1-1/q} \| \varphi_1 \|_{L^1(G_1)}^{1/q}.$$

Therefore, noticing that $\| \varphi_1 \|_{L^1(G_1)} > 0$, we have

$$N \leq \beta^{-1} | \lambda | \| h \|_p R^{1-1/q} \| \varphi_1 \|_{L^1(G_1)}^{1/q-1} + C_4 \beta^{-1} \| \varphi_1 \|_{L^1(G_1)}^{-1}.$$

Noticing that $q > 1$, $C_4 \geq 0$ and (2.10), we have

$$N \leq \beta^{-1}|\lambda|\|h\|_p R^{1-1/q}[(C_2 - C_3R^{-1})^q R]^{1/q-1} + C_4\beta^{-1}[(C_2 - C_3R^{-1})^q R]^{-1}$$

$$= \beta^{-1}|\lambda|\|h\|_p (C_2 - C_3R^{-1})^{1-q} + C_4\beta^{-1}(C_2 - C_3R^{-1})^{-q} R^{-1}.$$

Letting $R \to +\infty$ (i.e., $\|\varphi_1\| \to +\infty$), we obtain

$$N \leq \beta^{-1}|\lambda|\|h\|_p C_2^{1-q}. \tag{2.12}$$

Since N is arbitrary, (2.12) is impossible. Thus (2.7) cannot hold and (2.3) holds. Hence the conclusion (i) is true and our proof is complete. □

Remark 2.1. Theorem 1 is an improvement of Theorem 2 in [4], replacing the condition "$h(x)$ is a bounded measurable function" by the weaker one "$h \in L^p(G)(p > 1)$". This improvement enables the theorem to be applied to more areas.

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